Asymptotic properties of heavy diffusion particles system

Vitalii Konarovskyi

konarovskiy@gmail.com

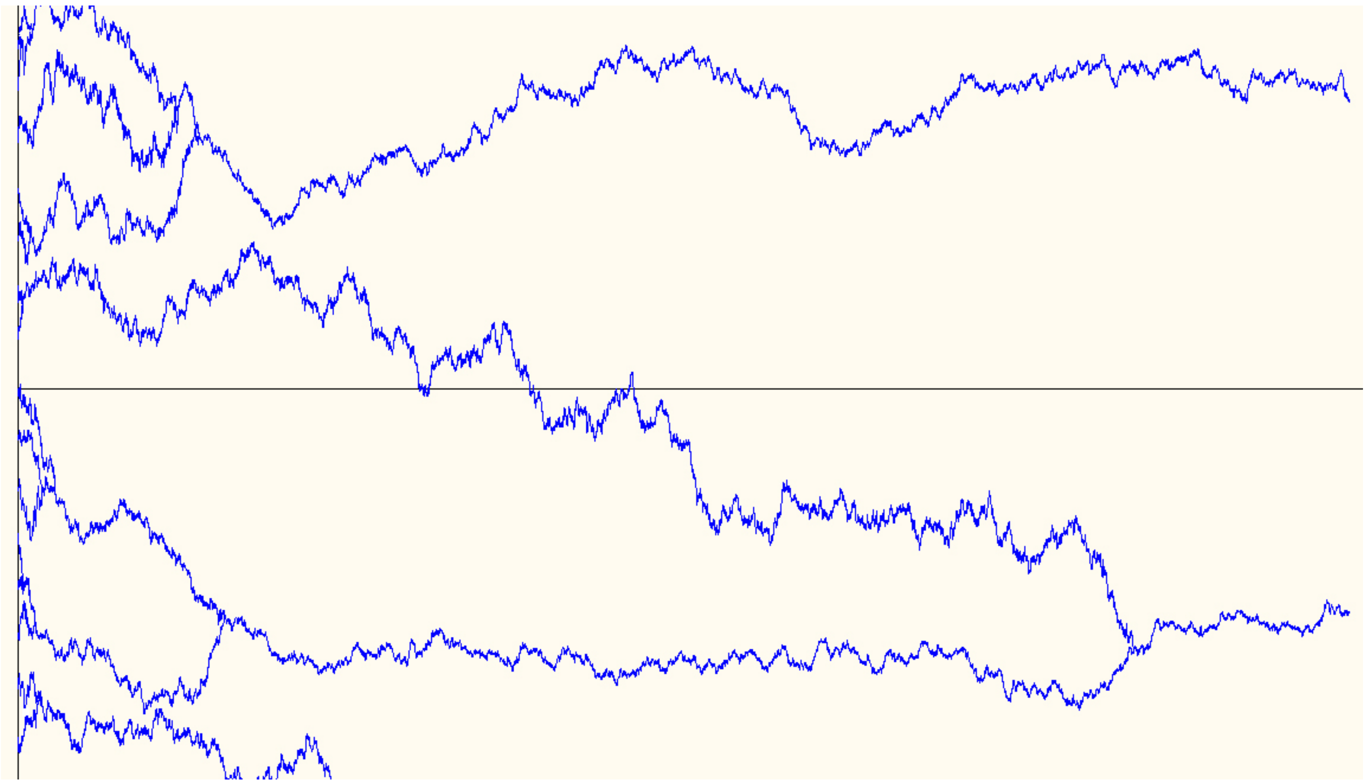
Yuriy Fedkovych Chernivtsi National University, 2, Kotsyubyns'kyi Str., Chernivtsi 58012, Ukraine

Friedrich Schiller University Jena, 2, Ernst-Abbe-Platz Str., Jena 07737, Germany

1. Main object of investigation

We consider a system of diffusion particles on the real line that

1. start from some finite or countable set of points with masses;
2. move independently up to the moment of the meeting;
3. coalesce;
4. have their mass adding after sticking;
5. have their diffusion changed correspondingly to the changing of the mass ($\sigma^2 = \frac{1}{m}$).



2. Examples and known facts

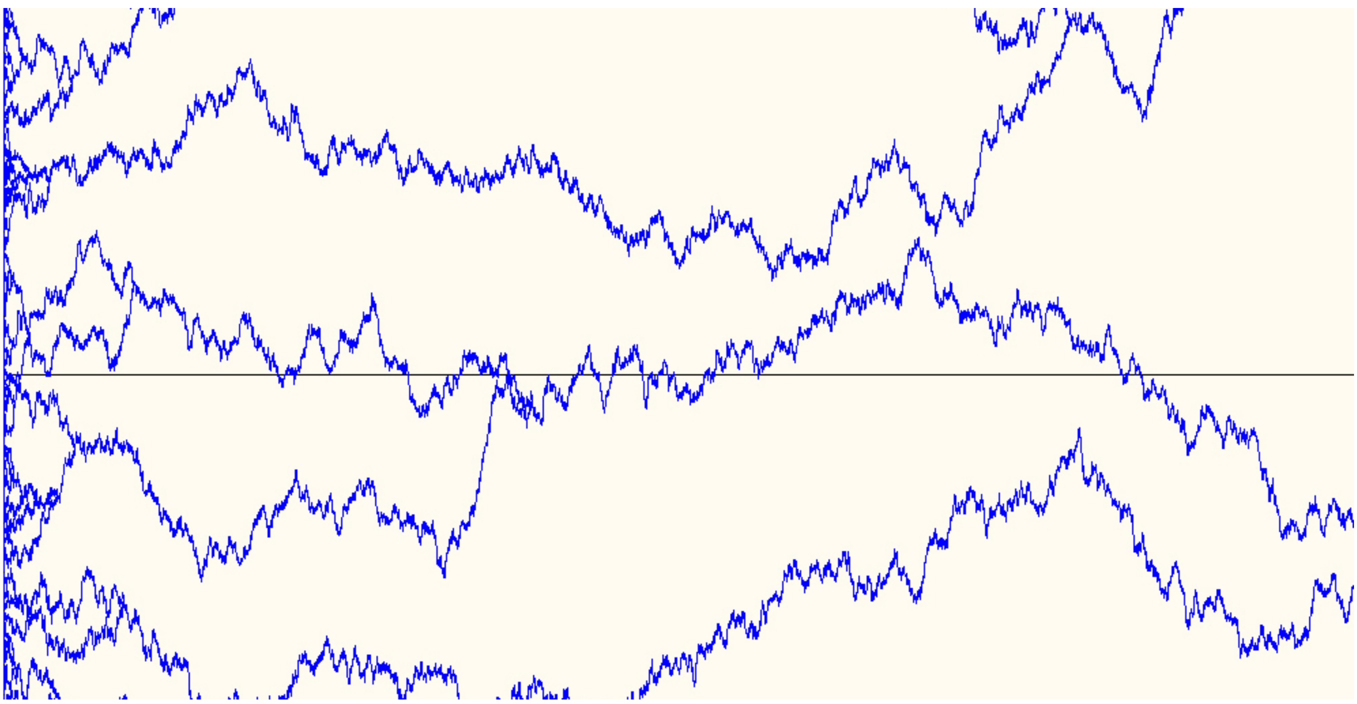
- Arratia flow

It is the system of Brownian particles which move independently up to the moment of the meeting. Then they coalesce and move together as a Brownian motion (diffusion coefficient of every particle equals one).

Mathematical description

Such flows one can describe by a system of processes $\{x(u,t),\ t \geq 0, u \in \mathbb{R}\}$ such that

1. $x(u,\cdot)$ is a Brownian motion;
2. $x(u,0) = u, \quad u \in \mathbb{R}$;
3. $x(u,t) \leq x(v,t), \quad u < v, \ t \geq 0$;
4. $\langle x(u,\cdot), x(v,\cdot)\rangle_t = 0, \quad t < \tau_{u,v}$, where $\tau_{u,v} = \inf\{t : x(u,t) = x(v,t)\}$.



3. Asymptotic behaviour of Arratia flow

Law of the iterated logarithm for a Wiener process

$$\overline{\lim_{t\to 0}} \frac{|X(u,t)-u|}{\sqrt{2t \ln\ln t^{-1}}} = 1 \quad \text{a.s.}$$

Asymptotic in the sup-norm (*Shamov A. '10*)

$$\overline{\lim_{t\to 0}} \sup_{u\in[0,1]} \frac{|X(u,t)-u|}{\sqrt{t \ln t^{-1}}} = 1 \quad \text{a.s.}$$

Asymptotic of cluster size (*Dorogovtsev A. A., Vovchanskii M. B. '13*)

Let $\nu(t) = \lambda\{u : X(u,t) = X(0,t)\},\ t \geq 0$. Then a.s.

$$\overline{\lim_{t\to 0}} \frac{\nu(t)}{\sqrt{2t \ln\ln t^{-1}}} \geq 1,$$

$$\overline{\lim_{t\to 0}} \frac{\nu(t)}{2\sqrt{t \ln\ln t^{-1}}} \leq 1.$$

4. Existence of a heavy diffusion particles system

Theorem (Konarovskyi)

For every non-decreasing sequence of real numbers $\{x_i,\ i \in \mathbb{Z}\}$ *and sequence of strictly positive real numbers* $\{b_i,\ i \in \mathbb{Z}\}$ *such that*

$$\overline{\lim_{n\to\pm\infty}}\{(x_{n+1}-x_n)\wedge b_{n+1}\wedge b_n\} > 0,$$

there exists a set of processes $\{X_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$, *satisfying*

1°) $X_k(\cdot)$ *is a continuous square integrable martingale with respect to the filtration*

$$\mathcal{F}_t = \sigma(X_i(s),\ s \leq t,\ i \in \mathbb{Z});$$

2°) $X_k(0) = x_k$, $k \in \mathbb{Z}$;

3°) $X_k(t) \leq X_l(t)$, $k < l$, $t \geq 0$;

4°) $\langle X_k(\cdot)\rangle_t = \int\limits_0^t \frac{1}{m_k(s)}ds$, $t \geq 0$,
where $m_k(t) = \sum\limits_{i\in A_k(t)} b_i$,
$A_k(t) = \{i \in \mathbb{Z}:\ \exists\, s \leq t,\ X_k(s) = X_i(s)\}$;

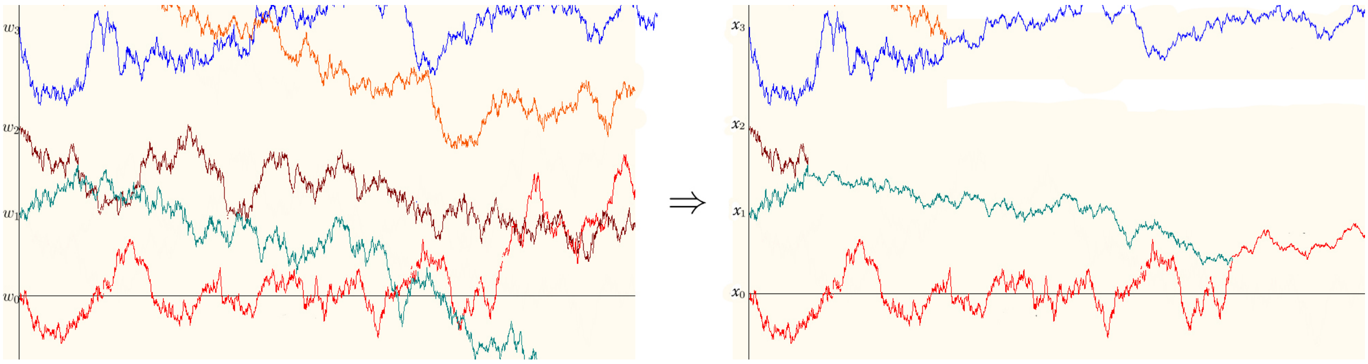
5°) $\langle X_k(\cdot), X_l(\cdot)\rangle_t = 0$,
$t < \tau_{k,l} = \inf\{t:\ X_k(t) = X_l(t)\}$.

Moreover, the conditions 1°)-5°) *uniquely determine the distribution of the process in the space* $(\mathrm{C}([0,\infty)))^Z$.

5. Idea of construction $(x_k(0) = k, \quad m_k(0) = 1, \quad k \in \mathbb{Z})$

Construction of finite system

Let $\{w_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$ be a system of independent Wiener processes. For fixed $n \in \mathbb{N}$ construct a process $X^n(t),\ t \geq 0$ in $\mathbb{R}^{2n+1}$ by coalescing and rescaling of the trajectories of $\{w_k,\ k = -n, \ldots, n\}$.



Proposition

The process $X^n(t),\ t \geq 0$ satisfies the conditions

1°) $X_k^n(\cdot)$ *is a continuous square integrable martingale;*

2°) $X_k^n(0) = k,\ k = -n, \ldots, n;$

3°) $X_k^n(t) \leq X_l^n(t),\ k < l,\ t \geq 0;$

4°) $\langle X_k^n(\cdot)\rangle_t = \int\limits_0^t \frac{1}{m_k^n(s)} ds,\ t \geq 0,$

where $m_k^n(t) = |\{i :\ \exists s \leq t\ X_i^n(s) = X_k^n(s)\}|;$

5°) $\langle X_k^n(\cdot), X_l^n(\cdot)\rangle_t = 0, \quad t < \tau_{k,l}^n,$

where $\tau_{k,l} = \inf\{t :\ X_k^n(t) = X_l^n(t)\}.$

5. Idea of construction
$(x_k(0) = k, \quad m_k(0) = 1, \quad k \in \mathbb{Z})$

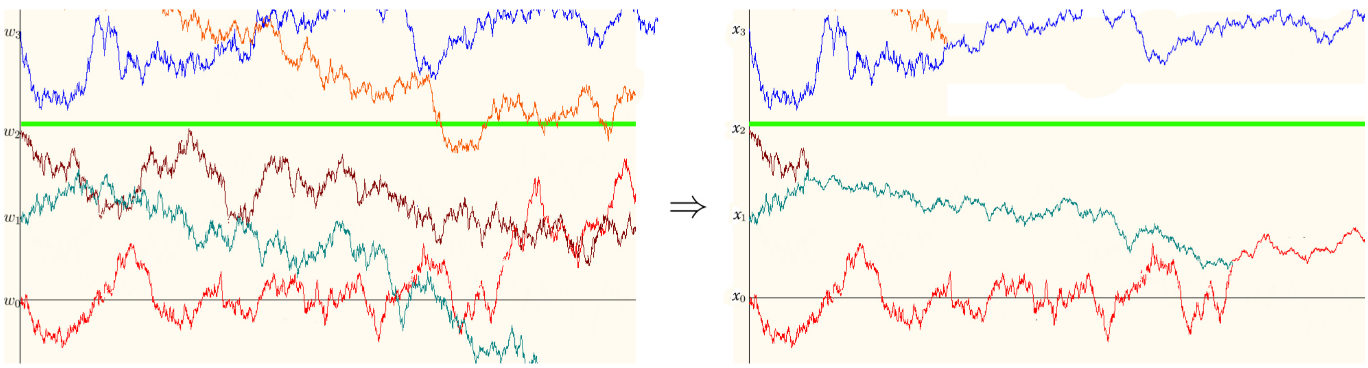
Passing to the limit

We can pass to the limit as the number of particles tend to infinity by the following lemma

Lemma (Konarovskyi)

Let $\{w_k(t),\ t \geq 0,\ k = 0, 1, \ldots\}$ *be a system of independent Wiener processes. Define*
$\xi_k^T = \max\limits_{t \in [0,T]} \{k + w_k(t)\}$, $\eta_k^T = \min\limits_{t \in [0,T]} \{k + w_k(t)\}$.
Then for every $T > 0$ *and* $\delta \in (0, 1)$

$$\mathbb{P}\left\{ \overline{\lim_{n\to\infty}} \left\{ \max_{k=1,\ldots,n} \xi_k^T < n + \delta,\ \eta_{n+1}^T > n + \delta \right\} \right\} = 1.$$



We have
$\mathbb{P}\left\{\exists N\, \forall n \geq N\, \forall t \in [0, T]\ X_k^n(t) = X_k^N(t)\right\} = 1,$
for all $k \in \mathbb{Z}$. Hence the lemma holds

Lemma

The sequence $\{X_k^n(\cdot),\ n \geq k\}$ *converges almost surely in* $\mathrm{C}([0, \infty))$.

6. Asymptotic behaviour

Estimation of asymptotic growth of mass

Let $w_1(\cdot)$, $w_2(\cdot)$ be independent Wiener processes and

$$\sigma_{k,l} = \inf\{t : \; k + w_1(t) = l + w_2(t)\}.$$

1. $\mathbb{P}\{\tau_{k,l} \le t\} \le \mathbb{P}\{\sigma_{k,l} \le t\}$;
2. $\mathbb{P}\left\{ \overline{\lim_{t\to+\infty}} \frac{m_k(t)}{4\sqrt{t \ln\ln t}} \le 1 \right\} = 1$;
3. $\mathbb{P}\{\tau_{k,k+p} < +\infty\} = 1$.

Asymptotic behaviour of one particle

Using the estimation of asymptotic growth of the mass and the law of the iterated logarithm for a Wiener process we have

1. $\mathbb{P}\left\{ \lim_{t\to+\infty} \frac{|X_k(t)|}{\sqrt{2t \ln\ln t}} = 0 \right\} = 1$;
2. $\mathbb{P}\left\{ \overline{\lim_{t\to+\infty}} \frac{|X_k(t)|}{\sqrt[4]{t^{1-\varepsilon}}} = \infty \right\} = 1$, for all $\varepsilon \in (0,1)$.

Open problem: What types of functions φ and ψ are for which

$$\overline{\lim_{t\to\infty}} \frac{m_k(t)}{\varphi(t)} = 1, \qquad \overline{\lim_{t\to\infty}} \frac{|X_k(t)|}{\psi(t)} = 1.$$

7. Large deviations principle (finite case)

Let $\widetilde{\mathrm{C}} = \{f \in \mathrm{C}([0,1];\mathbb{R}^n) : \ f_i(t) \leq f_j(t), \ i < j\}$ and let H be a set of absolutely continuous functions $g \in \mathrm{C}([0,1];\mathbb{R}^n)$ such that $\dot{g}_k \in L_2[0,1]$, $k = 1, \ldots, n$. Denote

$$I_x(g) = \begin{cases} \frac{1}{2}\int\limits_0^1 \|\dot{g}(t)\|^2 dt, \ g(0) = x, \ g \in H \cap \widetilde{\mathrm{C}}, \\ +\infty \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Theorem (Konarovskyi)

Let $X(t), \ t \in [0,1]$, satisfy conditions 1°)-5°) with $X(0) = x$ and $b_k = 1, \ k = 1, \ldots, n$. Then the family $\{X(\varepsilon\cdot), \ \varepsilon > 0\}$ satisfies LDP in the space $\mathrm{C}([0,1];\mathbb{R}^n)$ with the good rate function I_x.

Proposition

Let $k, l = 1, \ldots, n$ be fixed. Define $r = \min\{j : x_j = x_k \wedge x_l\}$, $R = \max\{j : x_j = x_k \vee x_l\}$,

$$S_x^2 = \frac{1}{R - r + 1}\sum_{j=r}^{R}\left(\sum_{i=r}^{R}\frac{x_i}{R - r + 1} - x_j\right)^2.$$

Then

$$\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{\tau_{k,l} \leq \varepsilon\} = -\frac{A_{k,l}}{2} S^2,$$

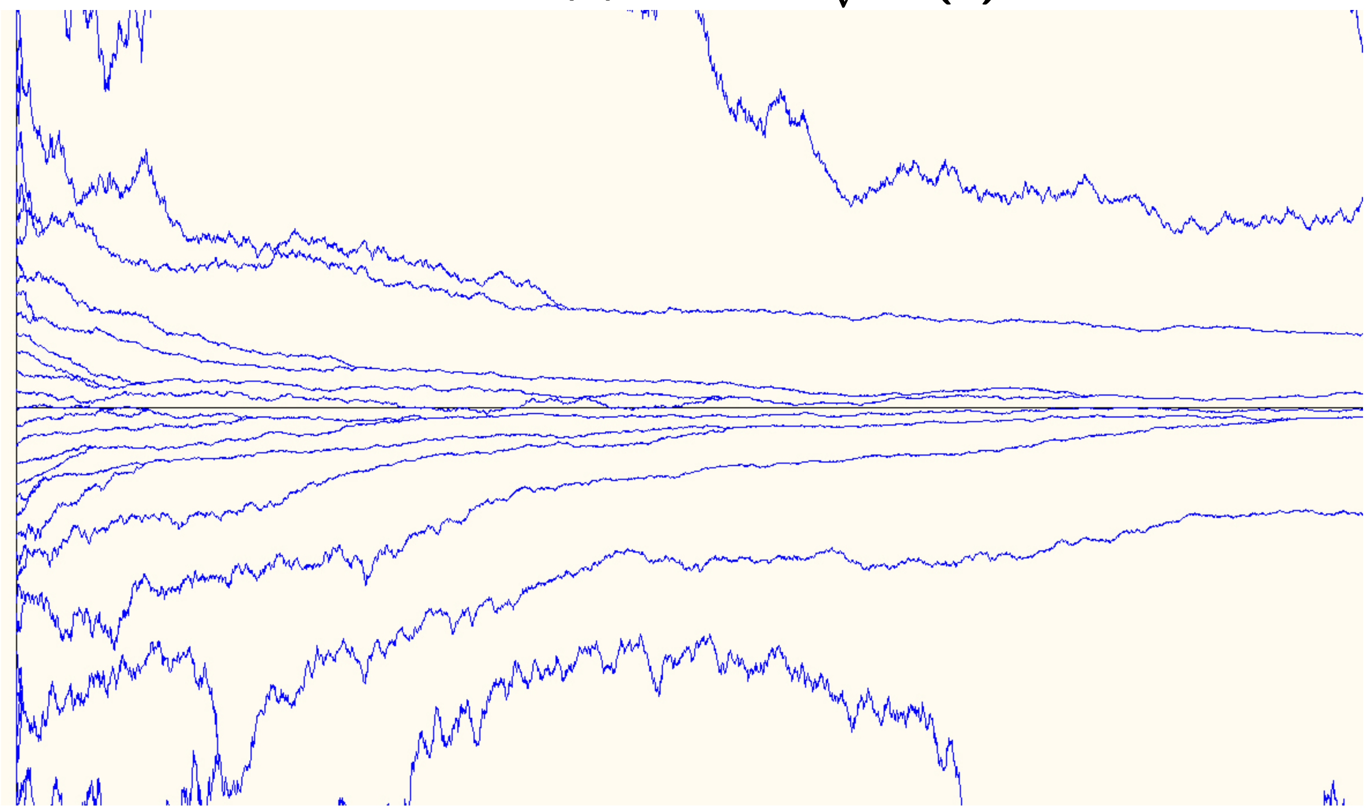
where $A_{k,l}$ is the number of elements of $\{x_r, \ldots, x_R\}$.

8. Heavy diffusion particles system with drift

System of diffusion particles on the real line that

1. start from some set of points with masses;
2. move independently up to the moment of the meeting;
3. coalesce;
4. have their mass adding after sticking;
5. have their diffusion changed correspondingly to the changing of the mass;
6. have evolution of particle described by SDE

$$dx(t) = \frac{a(x(t))}{m(t)}dt + \frac{\sigma(x(t))}{\sqrt{m(t)}}dw(t).$$



$a(x) = -x,\ |x| < 1, a(x) = sign(x),\ |x| \geq 1, \sigma(x) = (x^2 \vee 0, 1) \wedge 1$

9. Existence of the mathematical model

Theorem (Konarovskyi)

Let a, σ *be bounded Lipschitz continuity functions and* $\inf\limits_{x\in\mathbb{R}} \sigma(x) > 0$. *Then for every non-decreasing sequence of real numbers* $\{x_i,\ i \in \mathbb{Z}\}$ *and sequence of strictly positive real numbers* $\{b_i,\ i \in \mathbb{Z}\}$ *such that*

$$\overline{\lim_{n\to\pm\infty}} \{(x_{n+1} - x_n) \wedge b_{n+1} \wedge b_n\} > 0,$$

there exists $\zeta_i(t),\ t \geq 0,\ i \in \mathbb{Z}$, *satisfying*

1°) $\mathfrak{M}_i = \zeta_i(\cdot) - \int\limits_0^{\cdot} \frac{a(\zeta_i(s))}{m_i(s)} ds$ *is a continuous square integrable martingale with respect to the filtration* $\mathcal{F}_t^{\zeta} = \sigma(\zeta_i(s),\ s \leq t,\ i \in \mathbb{Z})$, *where* $m_i(t) = \sum\limits_{j\in A_i(t)} b_j$, $A_i(t) = \{j:\ \exists s \leq t\ \zeta_j(s) = \zeta_i(s)\}$;

2°) $\zeta_i(0) = x_i$, $i \in \mathbb{Z}$;

3°) $\zeta_i(t) \leq \zeta_j(t)$, $i < j$, $t \geq 0$;

4°) $\langle \mathfrak{M}_i \rangle_t = \int\limits_0^t \frac{\sigma^2(\zeta_i(s))}{m_i(s)} ds$, $t \geq 0$;

5°) $\langle \mathfrak{M}_i, \mathfrak{M}_j \rangle_t = 0, \quad t < \tau_{i,j} = \inf\{t : \zeta_i(t) = \zeta_j(t)\}$.

The conditions 1°)-5°) *uniquely determine the distribution of the process in the space* $(\mathrm{C}([0,\infty)))^{\mathbb{Z}}$

Estimation of asymptotic growth of the mass ($b_k = 1$, $x_{k+1} - x_k > \delta$, $k \in \mathbb{Z}$)

$$\mathbb{P}\left\{ \overline{\lim_{t\to+\infty}} \frac{\delta m_k(t)}{8\|\sigma\| \sqrt{t \ln\ln t}} \leq 1 \right\} = 1.$$

10. Start from all points

Question: Whether there exists a system of processes $\{X(u,t),\ t \geq 0,\ u \in \mathbb{R}\}$ such that

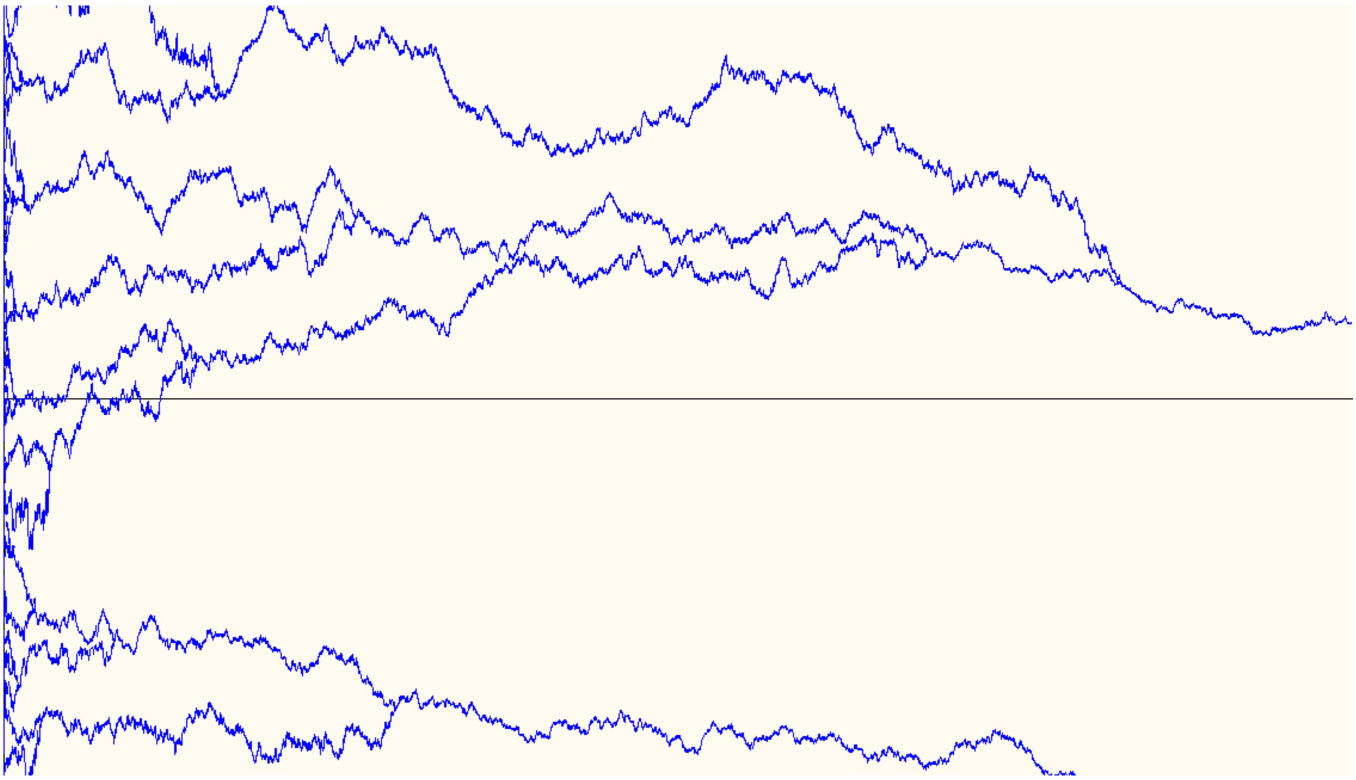
1°) $X(u,\cdot)$ is a continuous square integrable martingale with respect to the filtration $\mathcal{F}_t = \sigma(x(u,s),\ s \leq t,\ u \in \mathbb{R})$;

2°) $X(u,0) = u$, $u \in \mathbb{R}$;

3°) $X(u,t) \leq X(v,t)$, $u < v$, $t \geq 0$;

4°) $\langle X(u,\cdot)\rangle_t = \int\limits_0^t \frac{1}{m(u,s)} ds$, $t \geq 0$,
$m(u,t) = \lambda\{v :\ X(v,t) = X(u,t)\}$;

5°) $\langle X(u,\cdot), X(v,\cdot)\rangle_t = 0$,
$t < \tau_{u,v} = \inf\{t :\ X(u,t) = X(v,t)\}$.



REFERENCES

- *Arratia R. A.* Brownian motion on the line (1979), PhD dissertation, Univ. Wiskonsin, Madison.
- *Dorogovtsev A. A.* One Brownian stochastic flow (2004), Theory of Stochastic Processes, 10(26), no. 3–4, 21–25.
- *Dorogovtsev A. A.* Measure-valued processes and stochastic flows (2007), Institute of mathematic NAS of Ukraine, Kiev. (in Russion)
- *Shamov A.* On short-time asymptotics of one-dimensional Harris flows arXiv:1010.5349v1.
- *Dorogovtsev A. A., Gnedin A. V., Vovchanskii M. B.* Iterated logarithm law for size of clusters in Arratia flow (2012), Theory of Stochastic Processes, 18(34), no. 2, 1–7.
- *Konarovskyi V. V.* On infinite system of diffusion particles with coalescing(2011), Theory of Probability and its Application, 55, no 1, 134–144.
- *Konarovskyi V. V.* System of sticking diffusion particles of variable mass (2012), Ukrainian Mathematical Jornal, 62, no 1, 97–113.
- *Konarovskyi V. V.* The martingale problem for a measurevalued process with heavy diffusion particles (2011), Theory of stochastic processes, 17, no. 1, 50-60.
- *Konarovskyi V. V.* Mathematical model of heavy diffusion particles system with drift, Communications on stochastic analysis. (accepted for publication)
- *Konarovskyi V. V.* Large deviations principle for finite system of heavy diffusion particles, Theory of stochastic processes. (accepted for publication)