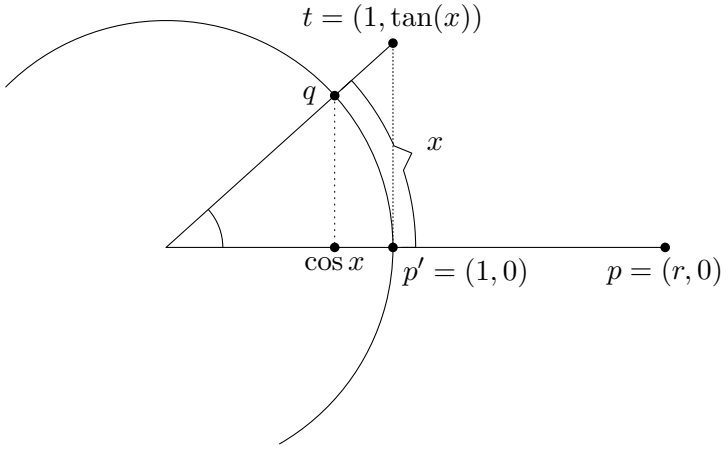
Lemma Let $q \in \mathbb{R}^2$ be a point on the unit circle (i.e. distance one to the origin) and $p = (r, 0) \in \mathbb{R}^2$ be a point outside, i.e. of distance $r > 1$ to the origin. If we denote $p' = \frac{1}{r}p$ the intersection of the ray from the origin to p with the circle, then

$$\|q - p'\| \leq \|q - p\|$$

i.e. projecting a point onto the circle moves it closer to any fixed point on this circle.

Proof We take the inspiration from the picture attached, which shows the situation after a rotation which does not change any distances. Since $\cos(x) \leq 1 \leq r$ we have $r - \cos(x) \geq 1 - \cos(x) \geq 0$ and hence

$$\|q - p\|^2 = (r - \cos(x))^2 + \sin^2(x) \geq (1 - \cos(x))^2 + \sin^2(x) = \|q - p'\|^2 \quad \square$$



Corollary If $p \neq q \in \mathbb{R}^2$ are both not in the (closed) unit disk, then for their projections onto the unit circle $\hat{p} = \frac{1}{\|p\|}p$ and $\hat{q} = \frac{1}{\|q\|}q$ we have also

$$\|\hat{p} - \hat{q}\| \leq \frac{1}{\min(\|p\|, \|q\|)}\|p - q\| < \|p - q\|.$$

Proof: Without loss of generality we assume $1 < \|q\| \leq \|p\|$ and so considering $\tilde{p} = \frac{1}{\|q\|}p$ not inside the unit disk we have $(\tilde{p} - \hat{q}) = \frac{1}{\|q\|}(p - q)$ and hence $\|\tilde{p} - \hat{q}\| \leq \frac{1}{\|q\|}\|p - q\|$. Now we can apply the Lemma to $\tilde{p}$ and q. Note that $\hat{p} = \frac{1}{\|\tilde{p}\|}\tilde{p}$ and so $|\hat{p} - \hat{q}\| \leq \|\tilde{p} - \hat{q}\| \leq \frac{1}{\|q\|}\|p - q\|$, as needed.

Proposition. $0 < x < \frac{\pi}{2}$ then $x < \tan(x)$.

Proof: We look again at the picture, here $q = p(x)$ in the notation from 1.5.b) of the script. We pick any chain of chords in the arc $\overset{\frown}{p(0)p(x)}$ using the points

$$p' = p(0), p(x_1), \ldots, p(x_k) \ldots p(x) = q \text{ and consider } \hat{p}(x_k) = (1, p(x_k)_2/p(x_k)_1) = (1, \tan(x_k))$$

which are is the intersection point of the vertical segment $\overline{p(0)t}$ with the rays from the origin through $p(x_k) = (\cos(x_k), \sin(x_k))$. Using the Corollary we get

$$\sum_{k=1}^{n} \|p(x_{k-1}) - p(x_k)\| \leq \sum_{j=1}^{n} \|\hat{p}(x_k) - \hat{p}(x_{k-1})\| \leq \tan(x),$$

so as the supremum over the left sums is the lenght x of the arc we conclude $x \leq \tan(x)$. Actually the inequality is sharp, repeating the argument for the upper halfarc $\overset{\frown}{p(\frac{x}{2})p(x)}$ we obtain, using the full strength of the Corollary, that $\frac{x}{2} \leq \frac{1}{\sqrt{1+(\tan(x/2))^2}}(\tan(x) - \tan(\frac{x}{2})) < \tan(x) - \tan(\frac{x}{2})$. Adding this with the inequality $\frac{x}{2} \leq \tan(\frac{x}{2})$ we get $x < \tan(x)$ as wanted. $\square$