IMPORTANT LIMITS USING EULERS NUMBER

Throughout, $a_n = \left(1+\frac{1}{n}\right)^n$ and $e = \lim_{n\to\infty} a_n$.

3) Claim: $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$**.** Let $x_n \to \infty$. Choose the unique $k_n \in \mathbb{N}$ such that

$$k_n \le x_n < k_n + 1 \qquad (k_n = \lfloor x_n \rfloor \text{ the integer part of } x_n),$$

so $k_n > x_n - 1$, and hence $k_n \to \infty$. (Since $x_n \to \infty$, we may assume $x_n \ge 1$, so $k_n \ge 1$.) Therefore

$$\lim_{n\to\infty}\left(1+\frac{1}{k_n}\right)^{k_n} = e,$$

as $(a_{k_n})_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$, up to repeated terms.[1]

Moreover, for all n,

$$1+\frac{1}{k_n+1} < 1+\frac{1}{x_n} \le 1+\frac{1}{k_n},$$

and all these bases are strictly larger than 1, so with $k_n \le x_n < k_n+1$ we get

$$y_n := \left(1+\frac{1}{k_n+1}\right)^{k_n} \le \left(1+\frac{1}{x_n}\right)^{x_n} \le \left(1+\frac{1}{k_n}\right)^{k_n+1} =: z_n.$$

Of course

$$\lim_{n\to\infty} z_n = e \text{ as stated in the Definition in 1.2.11}$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(1+\frac{1}{k_n+1}\right)^{k_n+1} \Big/ \lim_{n\to\infty}\left(1+\frac{1}{k_n+1}\right) = e/1 = e.$$

So by the sandwich theorem

$$\lim_{n\to\infty}\left(1+\frac{1}{x_n}\right)^{x_n} = e,$$

and since (x_n) was an arbitrary sequence tending to ∞, by Heine's (sequential) definition of the limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e.$$

Calculating the same limit for $x \to -\infty$ and substituting $y = -x$ we get

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{y\to\infty}\left(1+\frac{1}{-y}\right)^{-y} = \lim_{y\to\infty}\frac{1}{\left(1-\frac{1}{y}\right)^{y}} = \frac{1}{1/e} = e$$

Finally, if $t \searrow 0$ then for $x = \frac{1}{t} \nearrow \infty$ and so get for the onesided limit

$$\lim_{t\to 0+0}(1+t)^{\frac{1}{t}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e.$$

Analogously

$$\lim_{t\to 0-0}(1+t)^{\frac{1}{t}} = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = e,$$

and so $\lim_{t\to 0}(1+t)^{\frac{1}{t}} = e$ by 1.4.e).

4). By 1.5.(c) property of the logarithm and the Heine definition of the limit, we have

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1+x)}{x} &= \lim_{x\to 0}\ln\left((1+x)^{\frac{1}{x}}\right)\\ &= \ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right) \overset{3)}{=} \ln e = 1.\end{aligned}$$

[1]The k_n need not be strictly increasing, but $a_{k_n} \to e$ still holds: given $\varepsilon > 0$, pick N with $|a_m - e| < \varepsilon$ for $m \ge N$; and we have $k_n \ge N$ if $x_n > N$. The same argument gives $a_{k_n+1} \to e$.

5) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=?$**.** What is a^x? We have

$$a=e^{\ln a}\implies a^x=e^{x\ln a}.$$

If $a=1$ the limit is $0=\ln 1$, so let $a\neq 1$. We define

$$f(x)=\frac{a^x-1}{x}.$$

Then $f(x)\neq 0$ for all $x\neq 0$, and

$$xf(x)=a^x-1\longrightarrow 0\quad\text{when } x\to 0,$$

so we may use 4) with $u=xf(x)\neq 0$, $u\to 0$. Since

$$x\ln a=\ln(a^x)=\ln\big(1+xf(x)\big),$$

it follows that

$$1=\lim_{x\to 0}\frac{\ln\big(1+xf(x)\big)}{xf(x)}=\lim_{x\to 0}\frac{x\ln a}{xf(x)}=\lim_{x\to 0}\frac{\ln a}{f(x)}\implies\lim_{x\to 0}\frac{f(x)}{\ln(a)}=\frac{1}{1}\implies\lim_{x\to 0}f(x)=\ln a.$$

6) $\lim\limits_{x\to 0}\dfrac{(1+x)^\mu-1}{x}=\mu$**.** Indeed, this a consequence of the previous. Wlog $\mu\neq 0$ and we start as in 5)

$$\lim_{x\to 0}\frac{(1+x)^\mu-1}{x}=\lim_{x\to 0}\frac{e^{\mu\ln(1+x)}-1}{\mu\ln(1+x)}\mu\frac{\ln(1+x)}{x}=1\cdot\mu\cdot 1,$$

the third fraction has limit 1 by result 4). Moreover, the first fraction has limit 1 if $x\to 0$, since then $y=\mu\ln(1+x)\to 0$ and by 5) with $a=e$ we get $\lim_{y\to 0}\frac{e^y-1}{y}=\ln(e)=1$. □

It is important to have that $y=\mu\ln(x+1)\neq 0$ if $x\in(-1,1)\setminus\{0\}$, but this can and has to checked!