AGM INEQUALITY WITHOUT ROOTS AND EULER'S NUMBER e

We give a treatment of 1.2.11 without using roots. All x_i, a_i, etc. are positive reals.

Theorem 1. *For all $n \geq 1$,*

$$x_1 \cdot \ldots \cdot x_n \leq \Big(\frac{x_1 + \ldots + x_n}{n}\Big)^n,$$

with equality if and only if $x_1 = \cdots = x_n$.

Proof:

1) The case $n = 2$.

$$x_1 x_2 \leq \frac{(x_1 + x_2)^2}{4} \quad \text{and equality iff } x_1 = x_2, \tag{1}$$

as (1) $\iff 0 \leq (x_1 - x_2)^2$.

2) Binary upwards induction. Suppose the inequality holds for n. Then

$$\begin{aligned} x_1 \cdot \ldots \cdot x_n \cdot x_{n+1} \cdot \ldots \cdot x_{2n} &\underset{(*)}{\overset{\text{IA}}{\leq}} \Big(\frac{x_1 + \ldots + x_n}{n}\Big)^n \Big(\frac{x_{n+1} + \ldots + x_{2n}}{n}\Big)^n \\ &= \frac{1}{n^{2n}} \Big((x_1 + \ldots + x_n)(x_{n+1} + \ldots + x_{2n})\Big)^n \\ &\underset{(**)}{\leq} \frac{1}{n^{2n}} \Big(\frac{(x_1 + \ldots + x_n + x_{n+1} + \ldots + x_{2n})^2}{4}\Big)^n = \Big(\frac{x_1 + \ldots + x_{2n}}{2n}\Big)^{2n}, \end{aligned}$$

where $(**)$ is (1) applied to $x_1 + \cdots + x_n$ and $x_{n+1} + \cdots + x_{2n}$.

Equality in $(*)$ implies $x_1 = \cdots = x_n$ and $x_{n+1} = \cdots = x_{2n}$; equality in $(**)$ then implies $x_1 = x_{n+1}$.

3) Induction 1 down. Suppose the inequality holds for $n+1$, and let $\frac{1}{n}(x_1 + \cdots + x_n) = (\text{av}) > 0$. Then $x_1 + \cdots + x_n + (\text{av}) = (n+1)(\text{av})$, so

$$(\text{av})^n \cdot (\text{av}) = \Big(\frac{x_1 + \cdots + x_n + (\text{av})}{n+1}\Big)^{n+1} \underset{(***)}{\overset{\text{IA}}{\geq}} x_1 \cdots x_n \cdot (\text{av}),$$

and dividing by (av) gives

$$(\text{av})^n \geq x_1 \cdots x_n.$$

Equality in $(***)$ implies $x_1 = \cdots = x_n$ $(= (\text{av}))$.

Together, 1)–3) prove the theorem for every n: steps 1) and 2) give all powers of 2, and 3) fills in every n between them. □

Corollary 2. *If $a, b > 0$ are real then $a \cdot b^n \leq \left(\dfrac{a + n \cdot b}{n+1}\right)^{n+1}$, with equality iff $a = b$.*

Proof: Apply Theorem 1 to $x_1 = a$ and $x_j = b$ if $2 \leq j \leq n+1$. □

Lemma 3. *. For $n \in \mathbb{N}$ we define $x_n = \big(1 + \frac{1}{n}\big)^n$, $y_n = \big(1 + \frac{1}{n}\big)^{n+1}$ and $z_n = \big(1 - \frac{1}{n}\big)^n$. Then for all $n \in \mathbb{N}$: $x_n < x_{n+1}, z_n < z_{n+1}$ and $y_n > y_{n+1}$.*

Proof: For the x_n use Corollary 2 with $a = 1, b = 1 + \frac{1}{n}$, for z_n also Corollary 2 with $a = 1$ but $b = 1 - \frac{1}{n}$. Finally the inequality for the y_n follows since $y_n = 1/z_{n+1}$. □.

Corollary 4. *There is a real number $e = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$ and for all $n \in \mathbb{N} : x_n < e < y_n$.*

Proof: For all $n \in \mathbb{N} : x_n < y_n \leq y_1$ and $\{x_n\}_{n=1}^{\infty} \uparrow$ implies $\exists \ell_x = \lim\limits_{n\to\infty} x_n$. Analogous exists $\exists \ell_y = \lim\limits_{n\to\infty} y_n$. Since $y_n = (1 + \frac{1}{n}) x_n$ we get for $n \to \infty$ that $\ell_y = 1 \cdot \ell_x$. For each n and $m > n+1$ we have $x_m \geq x_{n+1}$, so if $m \to \infty$ we conclude $\ell_x \geq x_{n+1} > x_n$. Similarly $\ell_y \leq y_{n+1} < y_n$. □