Maths for International Students of the Faculty of Physics Leipzig (IPSP)
Mathematics 1
Linear Algebra and Calculus of Functions of One Variable

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Leipzig University, Winter Semester 2026/27

November 9, 2026

References

The programm of the course "Mathematics 1. Linear Algebra and Calculus of Functions of One Variable" is standard and the related material can be found practically in any book devoted to the calculus of functions of one variable and the linear algebra.

Books recommended in the official syllabus:

https://www.physgeo.uni-leipzig.de/en/studying/courses-of-study/

- Serge Lang: Linear Algebra, Springer
- Serge Lang: A First Course in Calculus, Springer
- Kenneth A. Ross: Elementary Analysis, Springer
- Stephen Abbott: Understanding Calculus, Springer

Books I frequently use myself to prepare my lectures and homework (to be completed):

Logic, sets, functions

- Keith Devlin, Sets, Functions, and Logic. An Introduction to Abstract Mathematics, CRC.
- Steven Galovich, Introduction to Mathematical Structures, HBJ, Academic Press.

Calculus of functions of one real variable

- Vladimir A. Zorich, Mathematical Analysis I, Springer.

Linear algebra

- Steven J. Leon, Linear Algebra with Applications, Prentice Hall.

Problem books

- One-dimensional calculus:
  B. Demidovich, Problems in Math Analysis (edition in English).
- Matrices and determinants:
  Steven J. Leon, Linear Algebra with Applications, Prentice Hall. (I have 8-th edition)
  (numerous exercises after each chapter).

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1. **Notation**

$\forall$ means "for all"

$\exists$ means "there exists"

$\exists!$ means "there exists a unique"

: means "such that"

:= means "denote by"

$\mathbb{N}$ is a set of all natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{Z}$ is a set of all integer numbers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Q}$ is a set of all rational numbers, $\mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z},\ n \in \mathbb{N} \right\}$

$\mathbb{R}$ is a set of all real numbers

an interval $(a, b) \subset \mathbb{R}$ is the set of all real x such that $a < x < b$

an interval $[a, b] \subset \mathbb{R}$ is the set of all real x such that $a \leq x \leq b$

$\operatorname{tg} x := \tan x$ is the tangent of x, $\quad \operatorname{ctg} x := \cot x$ is the cotangent of x

$\operatorname{arctg} y := \arctan y$ is the arctangent of y, $\quad \operatorname{arcctg} y := \operatorname{arccot} y$ is the arccotangent of y

1 One-dimensional calculus

1.1 Basic concepts

1. **Logic**

   - Logical operations: and &, or $\vee$, negation $\neg$, equivalence $\Leftrightarrow$.
     Most challenging is the implication $\Rightarrow$, also know as "if A then B". It has the following truth table

A	B	A $\Rightarrow$ B
T	T	T
T	F	F
F	T	T
F	T	T

     From this we see that $A \Rightarrow B$ is equivalent to $(\neg A) \vee B$
   - Negation of "and" and "or". Negation of an implication.
     Using truth tables we can easily verify that $\neg(A\&B) \Leftrightarrow ((\neg A) \vee (\neg B))$ and that $\neg(A \vee B) \Leftrightarrow ((\neg A)\&(\neg B))$. In fact: as always $\neg(\neg C) \Leftrightarrow C$ and since $(C \Leftrightarrow D) \Leftrightarrow (\neg C \Leftrightarrow \neg D)$ both of the previous equivalences follows from the other one.
   - Necessary and sufficient condition
     If an implication $A \Rightarrow B$ holds, then we say that A is sufficient for B (to be true), and that B is necessary for A (to be true).
   - Equivalence of the implication to the contrapositive, proof by contradiction
     In general, if we can prove the implication $(\neg A) \Rightarrow F$ to hold, then the truth table shows that we have to be in the fourth row and hence $\neg A$ is false, so A is true. Now we give an example of a

2. **Proof by contradiction**

   Example. (Euclid) There are infinitely many primes (A prime p is a natural number with precisely two divisors, in other words $p > 1$ and only 1 and p are dividing p).

   Proof. By contradiction, assume there are only finitely many primes $p_1 < p_2 < \ldots < p_n$. We multiply all of them and add a number one, i.e. we consider

   $$N = p_1 \cdot \ldots \cdot p_n + 1 = 1 + \prod_{j=1}^{n} p_j \in \mathbb{N}.$$

   First note that if p is any prime then there is by assumption a $k \leq n$ such that $p = p_k$ and hence

   $$\frac{N}{p} = \frac{1}{p_k} + \prod_{\substack{j=1 \\ j \neq k}}^{n} p_j \notin \mathbb{N},$$

so no prime can divide N.

In particular N is not a prime itself, so it has divisors which are bigger than one but less than N. We choose D to be the smallest among them (the existence of the minimum of each nonempty set of natural numbers will be discussed/established in the sequel.) Because D divides N, we know from the previous that D cannot be a prime. So there must be a divisor d of D such that $1 < d < D$. But then $\frac{N}{d} = \frac{N}{D} \cdot \frac{D}{d} \in \mathbb{N}$, so $d > 1$ is a divisor of N contradicting the minimality of D. *qed*

3. **Quantifiers**

- sentences(statements) and predicates: a (mathematical statement) has a well defined truth value (true **T** or wrong F), even if we do not now this value. A predicate P depends on a variable, and for each admissible choice of the variable x is $P(x)$ a statement, eg. $P(x) : x^2 > 2026$ where x is real of complex.
- $\forall xl$ forall x, $\exists$ there is an x, $\exists\, y!$ there exists a unique y. The variable is usually running through a well defined set, in the we obtain from the predicate a statement. For example
$$\exists x \in \mathbb{Q} : x^2 = 2 \text{ is false}, \forall_{x>0} : x + \frac{1}{x} \geq 2 \text{ is true .}$$
- Negation of the statement with quantifiers

$$(\neg(\forall_{x\in A} : P(x)) \Leftrightarrow \exists_{x\in A} : \neg(P(x)) \text{ and the analog } (\neg(\exists_{x\in A} : P(x)) \Leftrightarrow \forall_{x\in A} : \neg(P(x))$$

Example. Construct the explicit negation of the statement

$$\Big(\forall\, n \in \mathbb{N} \;\; \exists\, x \in (1, +\infty) \;\; \forall\, y \in [0, 1] \quad x + y > n^2\Big)$$

Determine which of the statements (the original one or its negation) is true.

4. **Proof by induction**

$$\boxed{\Big(\, P(1) \text{ is true } \Big) \wedge \Big(\forall\, n \in \mathbb{N} \quad P(n) \implies P(n+1)\Big) \quad \implies \quad \forall\, n \in \mathbb{N} \quad P(n) \text{ is true}}$$

Example.

1) For any $n \in \mathbb{N}$ the number $5^{n+2} + 6^{2n+1}$ is divisible by 31
2) Prove by induction that for any $n \in \mathbb{N} \quad 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$
3) If M is a non-empty set of natural numbers, then M has a smallest element.
This is the basis of the so called "Proof by strong induction", we only need to show that $P(n)$ is true provided $\forall k \in \mathbb{N} : k < n \Rightarrow P(k)$, we can use a much stronger assumption for many more predecessors of $n(+1)$ which should simplify the proof itself.
DIY: Show that every number natural is a product of prime numbers.

Proof.

1) Base of the induction $P(1)$. Let us verify that for $n = 1$ the statement is true. Indeed,

$$5^{1+2} + 6^{2\cdot 1+1} = 5^3 + 6^3 = 125 + 216 = 341 \text{ — is divisible by 31} \quad (341 : 31 = 11)$$

Inductional step $P(n) \Rightarrow P(n+1)$. Let us take some arbitrary $n \in \mathbb{N}$ and let us assume

$$5^{n+2} + 6^{2n+1} \text{ is devisable by 31} \qquad \text{— we } \textbf{assume} \text{ that this is true}$$

Based on this information, let us try to derive that

$$5^{(n+1)+2} + 6^{2(n+1)+1} = 5^{n+3} + 6^{2n+3} \text{ is divisible by 31} \qquad \text{— we } \textbf{want} \text{ to prove this}$$

Indeed, we have

$$5^{n+3} + 6^{2n+3} = 5 \cdot 5^{n+2} + 36 \cdot 6^{2n+1} = \underbrace{5 \cdot 5^{n+2} + 6 \cdot 6^{2n+1}}_{\text{is divisible by 31}} + \underbrace{31 \cdot 6^{2n+1}}_{\text{is divisible by 31}}$$

So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n \in \mathbb{N}$. $\square$

2) Base of the induction $P(1)$. Let us verify that for $n = 1$ the statement is true. Indeed,

$$1 = \frac{1 \cdot (1+1)}{2} \qquad \text{— is true}$$

Inductional step $P(n) \Rightarrow P(n+1)$. Let us assume that for some $n \in \mathbb{N}$

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \qquad \text{— we } \textbf{assume} \text{ that this is true}$$

and basing on this information let us try to derive that

$$1 + 2 + \ldots + n + (n+1) = \frac{(n+1)\big((n+1)+1\big)}{2} \qquad \text{— we } \textbf{want} \text{ to prove this}$$

Indeed, using the inductional assumption we can transform

$$\underbrace{1 + 2 + \ldots + n}_{= \frac{n(n+1)}{2}} + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)\Big(\frac{n}{2} + 1\Big) = \frac{(n+1)(n+2)}{2}$$

So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n \in \mathbb{N}$. $\square$

3) We only need to: for all n holds $P(n)$: If $M \subset \mathbb{N}$ and $\exists m : (m \in M \& m \leq n)$ then there is a smallest element in M.

Base of the induction $P(1)$. Let us verify that for $n = 1$ the statement is true. Indeed if $m \in M$ then m is a natural number and if $m \leq 1$ then $m = 1$ which is the smalles element of $\mathbb{N}$ and also M.

Inductional step $P(n) \Rightarrow P(n+1)$. If M contains some $m \leq n$ then M has by $P(n)$ a smallest element. Otherwise, since we can in the proof of $P(n+1)$ the existence of $m \in M$ with $m \leq n+1$ we see that because $\neg(m \leq)$ necessarily $m = n+1 \in M$ and is the smallest element of M. *As before we conclude $P(n)$ holds for all $n \in \mathbb{N}$.* $\square$

5. **Sets**

- $x \in A$ and $B \subset A$ — elements and subsets.
  Example $\{a, b\}$, note $\{a, a\} = \{a\}$ and $\{a, b\} = \{b, a\}$
- Barber paradox, universal set X
- Equality of sets: $A = B \Leftrightarrow (A \subset B) \wedge (B \subset A)$, proofs of set identities
- Operations with sets: $A \cup B$, $A \cap B$, $A \setminus B$, $A' = cA = A^c s = X \setminus A$ complement
- Infinite union and intersection
- Cartesian product consists of pairs $(a, b) = \{\{a\}, \{a, b\}\}$ where $a \in A$ and $b \in B$

EXAMPLE. For any an indexed family of subsets $\{A_i\}_{i \in I}$ of some universal set X the following equalities hold:

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \setminus A_i) \qquad X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i)$$

6. **Relations**

- $a, b \in A$, $a \overset{\mathcal{R}}{\sim} b$
- Examples: $a \leq b$ (real numbers), $a|b$ (divides, integers), $a \perp b$ (lines on the plane) etc
- Reflexive, symmetric, transitive relations
- Equivalence relation is a relation which is reflexive, symmetric, transitive
- Formally: relation $\mathcal{R}$ is some subset of $A \times A$. Then $a \overset{\mathcal{R}}{\sim} b \Leftrightarrow (a, b) \in \mathcal{R}$

7. **Functions**

DEFINITION. Let A and B be any non-empty sets.

- A *function* (or a *map*) $f : A \to B$ is a rule which associate with each member of A a unique element of B.
- A is called the *domain* of f (we denote the domain $D(f)$), and B is called *codomain*.
- Functions f and g are called *equal* if $D(f) = D(g)$ and $\forall\, a \in D(f)\ f(a) = g(a)$.
- g is an *extension* of f if $D(f) \subset D(g)$ and $\forall\, a \in D(f)\ f(a) = g(a)$.

8. **Domain and range of a function**

DEFINITION. If $f : A \to B$ is a function then

$$D(f) := A \quad \text{is the } \textit{domain} \text{ of } f$$
$$R(f) := \left\{ b \in B \;\middle|\; \exists\, a \in A :\ b = f(a) \right\} \quad \text{is the } \textit{range} \text{ of } f$$

If $f(a) = b$ then we say b is the *image* of a and a is a *pre-image* of b.

9. **Injection, surjection and bijection functions**

DEFINITION. Let $f : A \to B$ be a function.

- f is an *injection* $\iff$ f is one-to-one $\iff$ $\forall\, a_1,\, a_2 \in A$ if $f(a_1) = f(a_2)$ then $a_1 = a_2$.
- f is a *surjection* $\iff$ f is onto $\iff$ $R(f) = B$ $\iff$ $\forall\, b \in B$ $\exists\, a \in A$: $f(a) = b$
- f is a *bijection* $\iff$ f is both an injection and a surjection

10. **Countable sets**

- The set A is *countable* $\iff$ $\exists$ a bijection $A \leftrightarrow \mathbb{N}$
- The set of all rational numbers $\mathbb{Q}$ is countable The direct construction of the required bijection is harder than expected, as different pairs of integers can represent the same fraction. We proceed in 3 (or 4) steps

  a) First we define a bijection from $i : \mathbb{Z} \to \mathbb{N}$ by

$$i(k) = \begin{cases} 1 & \text{if } k = 0 \\ 2k & \text{if } k \in \mathbb{N} \\ -2k+1 & \text{if } -k \in \mathbb{N} \end{cases}.$$

  Now we can define a bijection (verify this yourself!) $B : \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$ by

$$Bi(k, n) = 2^{i(k)-1}(2n - 1),$$

  we could of course also use a (more symmetric) variant of Cantor's curve for $\mathbb{N} \times \mathbb{N}$, which was drawn on the blackboard.
  After this, we find a surjective map $f : \mathbb{N} \to \mathbb{Q}$ by

$$f(k) = \frac{m}{n} \text{ if } Bi(m, n) = k.$$

  b) A fact, more general and more **important**: If M is an infinite set for which a surjective map $f : \mathbb{N} \to M$ exists, then there is also a bijection $b : \mathbb{N} \to M$. This b can be defined recursively. We set $b(1) = f(1)$ and, having defined already $b(1), \ldots, b(n)$, we set $b(n+1) = f(\min(\{k\,;\ f(k) \notin \{b(1), \ldots, b(n)\}))$. Verify that $b(n+1)$ is always well-defined and (harder, but strong induction makes it easy) that b is surjective! ("No element left behind")

  For our particular f defined in a) we get the required bijection $b : \mathbb{N} \to \mathbb{Q}$ by b).

  We now formalize two techniques used in the proof, having them we could write $f = ((m, n) \mapsto \frac{m}{n}) \circ Bi^{-1}$.

11. **Inverse function**

DEFINITION. A function $f : A \to B$ is *invertible* on A if f is an injection on A. If f is invertible on $D(f)$ then the *inverse function* to f is defined by

$$g := f^{-1}, \quad D(g) = R(f), \quad g(b) = a \quad \iff \quad b = f(a), \quad \forall\, a \in D(f), \quad \forall\, b \in R(f)$$

12. **Composition of functions**

DEFINITION. Assume functions $f : A \to B$ and $g : B \to C$ satisfy $R(f) \subset B$. A *composition* of f and g is the function $h : A \to C$ defined by

$$h := g \circ f, \qquad D(h) = D(f) = A, \qquad h(a) \,=\, g(f(a)), \qquad a \in A$$

13. **Real numbers**

- Common sense concept: infinite decimal fractions
- More formal: the real number can be associated with certain sequences of rational numbers $\{r_n\}$ which "approximate" (in a certain sense) this real number.
- The set of all real number $\mathbb{R}$ is non-countable (we will prove this once the infinite decimal fractions got clarified), difficult prove.
- Axioms of $\mathbb{R}$: operations $x + y$ and $x \cdot y$ possess commutativity, associativity, existence on neutral element, existence of inverse, distributivity, relation $x \leq y$
- Intervals (a, b), $[a, b)$, $[a, b]$, $(-\infty, b)$, $[a, +\infty)$ etc

14. **Bounded sets, lower and upper bounds**

DEFINITION.

- $M \in \mathbb{R}$ is an *upper bound* of a set $A \subset \mathbb{R} \iff \forall\, x \in A \quad x \leq M$
- $m \in \mathbb{R}$ is a *lower bound* of a set $A \subset \mathbb{R} \iff \forall\, x \in A \quad x \geq m$
- $A \subset \mathbb{R}$ is *bounded from above* $\iff$ $\exists$ at least one upper bound of A
- $A \subset \mathbb{R}$ is *bounded from below* $\iff$ $\exists$ at least one lower bound of A
- $A \subset \mathbb{R}$ is *bounded* $\iff$ A is bounded both from above and from below

15. Supremum and infimum

DEFINITION.

$M \in \mathbb{R}$ is the *supremum* of a set $A \subset \mathbb{R} \iff$

1) M is an upper bound of A

2) if $M' \in \mathbb{R}$ is some other upper bound of A then $M \leq M'$

$m \in \mathbb{R}$ is the *infimum* of a set $A \subset \mathbb{R} \iff$

1) m is a lower bound of A

2) if $m' \in \mathbb{R}$ is some other lower bound of A then $m \geq m'$

NOTATION.

$\sup A$ is the supremum of A, $\sup A = +\infty \iff A$ is not bounded from above

$\inf A$ is the infimum of A, $\inf A = -\infty \iff A$ is not bounded from below

16. Characterization of supremum and infimum using quantifiers

THEOREM.

$M \in \mathbb{R}$ is the supremum of $A \subset \mathbb{R} \iff$

1) $\forall\, x \in A \quad x \leq M$

2) $\forall\, \varepsilon > 0 \quad \exists\, x_\varepsilon \in A\colon \quad M - \varepsilon < x_\varepsilon$

$m \in \mathbb{R}$ is the infimum of $A \subset \mathbb{R} \iff$

1) $\forall\, x \in A \quad m \leq x$

2) $\forall\, \varepsilon > 0 \quad \exists\, x_\varepsilon \in A\colon \quad x_\varepsilon < m + \varepsilon$

PROBLEM. Find the supremum and the infimum of the set $\left\{ \frac{2nm}{n^2+m^2} \;\middle|\; n,\, m \in \mathbb{N} \right\} \subset \mathbb{R}$

17. Least upper bound axiom

AXIOM OF THE SET $\mathbb{R}$.

$A \subset \mathbb{R}$ is bounded from above $\implies \exists$ the number $M \in \mathbb{R}$ which is the supremum of A

$A \subset \mathbb{R}$ is bounded from below $\implies \exists$ the number $m \in \mathbb{R}$ which is the infimum of A

REMARK. The least upper bound axiom does not hold for the set of rational numbers $\mathbb{Q}$:

$$A = \{\, r \in \mathbb{Q} :\ r^2 < 2 \,\}, \quad A \text{ is bounded from above, BUT } \quad \nexists\, q \in \mathbb{Q} :\ q = \sup A$$

18. Functions of a real variable

DEFINITION. $f : D(f) \subset \mathbb{R} \to \mathbb{R}$

- Admissible set (maximal domain) of an algebraic expression
- Graph of a function of a real variable
- Range of a function of a real variable

19. **Monotone functions**

DEFINITION. Assume $f : [a, b] \to \mathbb{R}$.

- f — $\nearrow$ on $[a, b] \iff \forall x_1,\, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) \leq f(x_2)$
- f — $\uparrow$ on $[a, b] \iff \forall x_1,\, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) < f(x_2)$
- f — $\searrow$ on $[a, b] \iff \forall x_1,\, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) \geq f(x_2)$
- f — $\downarrow$ on $[a, b] \iff \forall x_1,\, x_2 \in [a, b] \;\; x_1 < x_2 \;\; \Rightarrow \;\; f(x_1) > f(x_2)$

20. **Invertibility of strictly monotone functions**

THEOREM. A strictly monotone function $f : D(f) \subset \mathbb{R} \to \mathbb{R}$ in invertible on the interval of its monotonicity and the inverse function f^{-1} is strictly monotone in the same sense to f, i.e.

- f — $\uparrow$ on $D(f) \quad \Longrightarrow \quad f^{-1}$ — $\uparrow$ on $R(f)$
- f — $\downarrow$ on $D(f) \quad \Longrightarrow \quad f^{-1}$ — $\downarrow$ on $R(f)$

1.2 Limit of a sequence

1. **Definition of a sequence**

DEFINITION. A sequence $\{x_n\}_{n=1}^{\infty}$ is a map (a function) $f : \mathbb{N} \to \mathbb{R}$, $x_n := f(n)$

2. **Informal notion helping to understand quantifiers in the definition of a limit**

We say an interval (α, β) is a "feeder" for $\{x_n\}_{n=1}^{\infty} \iff \forall\, N \in \mathbb{N}\ \exists\, n \geq N$: $x_n \in (\alpha, \beta)$
(i.e. the sequence $\{x_n\}_{n=1}^{\infty}$ "returns" to the interval (α, β) infinitely many times).

We say an interval (α, β) is a "catcher" for $\{x_n\}_{n=1}^{\infty} \iff \exists\, N \in \mathbb{N}$: $\forall\, n \geq N$: $x_n \in (\alpha, \beta)$
(i.e. once entering the interval (α, β), the sequence $\{x_n\}_{n=1}^{\infty}$ can never get out of it).

We say the sequence $\{x_n\}_{n=1}^{\infty}$ is *bounded* $\iff$ $\exists\ M \geq 0$: $|x_n| \leq M$ $\forall\, n \in \mathbb{N}$.
(i.e. there exists at least one "catcher" for a sequence $\{x_n\}_{n=1}^{\infty}$).

3. **Limit of a sequence**

DEFINITION. $a \in \mathbb{R}$ is the *limit* of a sequence $\{x_n\}_{n=1}^{\infty}$ (equivalently, the sequence $\{x_n\}_{n=1}^{\infty}$ *converges* to $a \in \mathbb{R}$) iff

$$\forall\, \varepsilon > 0 \quad \exists\, N = N(\varepsilon) : \quad \forall\, n \geq N \qquad |x_n - a| < \varepsilon$$

i.e. for any $\varepsilon > 0$ the interval $(a - \varepsilon, a + \varepsilon)$ is a "catcher" for a sequence $\{x_n\}_{n=1}^{\infty}$.

NOTATION. $\quad a = \lim\limits_{n\to\infty} x_n \quad$ or $\quad x_n \underset{n\to\infty}{\longrightarrow} a.$

4. **Infinitesimal sequences**

DEFINITION.

A sequence $\{\alpha_n\}_{n=1}^{\infty}$ is called *infinitesimal* $\iff$ $\lim\limits_{n\to\infty} \alpha_n = 0$

A sequence $\{x_n\}_{n=1}^{\infty}$ is called *infinitely large* $\iff$ $\lim\limits_{n\to\infty} |x_n| = +\infty$ $\iff$

$$\forall M > 0 \quad \exists N = N(M) : \quad \forall n \geq N \quad |x_n| \geq M$$

5. **Characterization of a limit in terms of infinitesimal**

THEOREM. Assume $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $a \in \mathbb{R}$. Then

$$\exists \lim_{n\to\infty} x_n = a \quad \iff \quad \exists \text{ an infinitesimal } \{\alpha_n\}_{n=1}^{\infty} \text{ such that } \forall\, n \in \mathbb{N} \quad x_n = a + \alpha_n$$

6. **Properties of infinitesimal sequences**

1) $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are infinitesimal $\implies$ $\{\alpha_n + \beta_n\}_{n=1}^{\infty}$ is infinitesimal

2) $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal, $\{x_n\}_{n=1}^{\infty}$ is bounded $\implies$ $\{\alpha_n x_n\}_{n=1}^{\infty}$ is infinitesimal

3) $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal, $\alpha_n \neq 0$, $\forall\, n \in \mathbb{N}$ $\iff$ $\{\frac{1}{\alpha_n}\}_{n=1}^{\infty}$ is infinitely large

7. **Basic properties of limits**

THEOREM 1. The limit of a sequence, if exists, is unique.

THEOREM 2. $\exists \lim\limits_{n\to\infty} x_n = a$ (a is a finite real number) $\implies$ $\{x_n\}_{n=1}^{\infty}$ is bounded.

THEOREM 3. $\exists \lim\limits_{n\to\infty} x_n = a$, $\exists \lim\limits_{n\to\infty} y_n = b$, $\forall n \in \mathbb{N}$ $x_n \leq y_n$ $\implies$ $a \leq b$

THEOREM 4. $\forall n \in \mathbb{N}$ $x_n \leq y_n \leq z_n$, $\exists \lim\limits_{n\to\infty} x_n = a$, $\exists \lim\limits_{n\to\infty} z_n = a$ $\implies$ $\exists \lim\limits_{n\to\infty} y_n = a$

COROLLARY. $\{\beta_n\}_{n=1}^{\infty}$ is infinitesimal, $|\alpha_n| \leq |\beta_n|$ $\implies$ $\{\alpha_n\}_{n=1}^{\infty}$ is infinitesimal.

8. **Arithmetic operations with limits**

THEOREM 1. $\exists \lim\limits_{n\to\infty} x_n = a$ $\exists \lim\limits_{n\to\infty} y_n = b$ $\implies$ $\exists \lim\limits_{n\to\infty} (x_n \pm y_n) = a \pm b$

THEOREM 2. $\exists \lim\limits_{n\to\infty} x_n = a$ $\exists \lim\limits_{n\to\infty} y_n = b$ $\implies$ $\exists \lim\limits_{n\to\infty} (x_n y_n) = ab$

LEMMA. $\exists \lim\limits_{n\to\infty} y_n = b$ and $y_n \neq 0$, $b \neq 0$ $\implies$ $\{\frac{1}{y_n}\}_{n=1}^{\infty}$ is bounded.

THEOREM 3. $\exists \lim\limits_{n\to\infty} x_n = a$ $\exists \lim\limits_{n\to\infty} y_n = b$ and $y_n \neq 0$, $b \neq 0 \implies \exists \lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{a}{b}$

9. **Monotone sequences**

DEFINITION.

- $\{x_n\}_{n=1}^{\infty}$ — $\nearrow$ $\iff$ $\forall\, n \in \mathbb{N}$ $x_n \leq x_{n+1}$
- $\{x_n\}_{n=1}^{\infty}$ — $\searrow$ $\iff$ $\forall\, n \in \mathbb{N}$ $x_n \geq x_{n+1}$

THEOREM.

1) $\{x_n\}_{n=1}^{\infty}$ — $\nearrow$ $\exists\, M \in \mathbb{R}$: $\forall\, n \in \mathbb{N}$ $x_n \leq M$ $\implies$ $\exists \lim\limits_{n\to\infty} x_n \leq M$

2) $\{x_n\}_{n=1}^{\infty}$ — $\searrow$ $\exists\, m \in \mathbb{R}$: $\forall\, n \in \mathbb{N}$ $x_n \geq m$ $\implies$ $\exists \lim\limits_{n\to\infty} x_n \geq m$

10. **Important examples**

1) $q > 1 \implies \lim_{n\to\infty} \frac{n}{q^n} = 0$, first show $\forall n \geq 4 : a_n = \frac{n^2}{2^n} > a_{n+1}$ hence $\frac{n}{2^n} \leq \frac{1}{n}$ so $\lim_{n\to\infty} \frac{n}{2^n} = 0$ Bernoullis inequality, easy from induction, $\forall h > -1\ \forall k \in \mathbb{N} : (1+h)^k \geq 1 + hk$, hence if $M(= M_q) > 1/(q-1)$ then $q^M > 2$ and, $(m-1)M < n \leq mM \implies \frac{n}{q^n} < \frac{Mm}{2^{m-1}} = 2M\frac{m}{2^m}$, now finish easily.

2) $q \in \mathbb{R} \implies \lim_{n\to\infty} \frac{q^n}{n!} = 0$. Indeed $a_n = q^n/q!$ satifies $a_{n+1} < \frac{1}{2}a_n$ if $n > 2q$. So for $n_0 > 2q$ fixed we have (by induction in k) $a_{n_0+k} \leq a_{n_0}/2^k$, now finish easily with 1).

3) $\lim_{n\to\infty} \sqrt[n]{n} = 1$, for $a_n = \sqrt[n]{n}$ we obviuously have $1 < a_n$ and using 1) that forall $\varepsilon > 0$ and $q = 1 + \varepsilon$ there is N such that $\forall n \geq N : n/q^n < 1$. Now finish easily.

4) $a > 0 \implies \lim_{n\to\infty} \sqrt[n]{a} = 1$ Simply notethat $1/\sqrt[n]{n} < \sqrt[n]{a} < \sqrt[n]{n}$ if $\frac{1}{n} < a < n$ and finish easily using 3).

11. **Euler's number e**

Theorem For all $n \geq 1$, $x_1, \ldots, x_n \in [0, \infty)$ we have the AGM (arithmetic-geometric means) inequality

$$x_1 \cdot \ldots \cdot x_n \leq \Big(\frac{x_1 + \ldots + x_n}{n}\Big)^n,$$

with equality if and only if $x_1 = \cdots = x_n$.

Proof:

1) The case $n = 2$

$$x_1 x_2 \leq \frac{(x_1 + x_2)^2}{4} \quad \text{and equality iff } x_1 = x_2, \tag{1}$$

as (1) $\iff 0 \leq (x_1 - x_2)^2$.

2) Binary upwards induction

Suppose the inequality holds for n. Then

$$\begin{aligned} x_1 \cdot \ldots \cdot x_n \cdot x_{n+1} \cdot \ldots \cdot x_{2n} &\underset{(*)}{\overset{\text{IA}}{\leq}} \Big(\frac{x_1 + \ldots + x_n}{n}\Big)^n \Big(\frac{x_{n+1} + \ldots + x_{2n}}{n}\Big)^n \\ &= \frac{1}{n^{2n}}\Big((x_1 + \ldots + x_n)(x_{n+1} + \ldots + x_{2n})\Big)^n \\ &\underset{(**)}{\leq} \frac{1}{n^{2n}}\Big(\frac{(x_1 + \ldots + x_n + x_{n+1} + \ldots + x_{2n})^2}{4}\Big)^n = \Big(\frac{x_1 + \ldots + x_{2n}}{2n}\Big)^{2n}, \end{aligned}$$

where $(**)$ is (1) applied to $x_1 + \cdots + x_n$ and $x_{n+1} + \cdots + x_{2n}$.

Equality in $(*)$ implies $x_1 = \cdots = x_n$ and $x_{n+1} = \cdots = x_{2n}$; equality in $(**)$ then implies $x_1 = x_{n+1}$.

3) Induction 1 down

Suppose the inequality holds for $n+1$, and let $\frac{1}{n}(x_1+\cdots+x_n) = (\text{aver}) > 0$. Then $x_1+\cdots+x_n+(\text{aver}) = (n+1)(\text{aver})$, so

$$(\text{aver})^n \cdot (\text{aver}) = \Big(\frac{x_1+\cdots+x_n+(\text{aver})}{n+1}\Big)^{n+1} \underset{(***)}{\overset{\text{IA}}{\geq}} x_1\cdots x_n \cdot (\text{aver}),$$

and dividing by (aver) gives

$$(\text{aver})^n \geq x_1\cdots x_n.$$

Equality in $(***)$ implies $x_1 = \cdots = x_n$ $(= (\text{aver}))$.

Together, 1)–3) prove the theorem for every n: steps 1) and 2) give all powers of 2, and 3) fills in every n between them. □

Corollary 1 If $a, b > 0$ are real then $a \cdot b^n \leq \left(\frac{a+n\cdot b}{n+1}\right)^{n+1}$, with equality iff $a = b$.
Proof: Apply the above Theorem to $x_1 = a$ and $x_j = b$ if $2 \leq j \leq n+1$. □

Lemma For $n \in \mathbb{N}$ we define $x_n = \left(1+\frac{1}{n}\right)^n$, $y_n = \left(1+\frac{1}{n}\right)^{n+1}$ and $z_n = \left(1-\frac{1}{n}\right)^n$.
Then for all $n \in \mathbb{N}$: $x_n < x_{n+1}, z_n < z_{n+1}$ and $y_n > y_{n+1}$.
Proof: For the x_n use Corollary 2 with $a = 1, b = 1+\frac{1}{n}$, for z_n also Corollary 2 with $a = 1$ but $b = 1-\frac{1}{n}$. Finally the inequality for the y_n follows since $y_n = 1/z_{n+1}$. □.

Corollary 2 There is a real number $e = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$ and for all $n \in \mathbb{N} : x_n < e < y_n$.
Proof: For all $n \in \mathbb{N} : x_n < y_n \leq y_1$ and $\{x_n\}_{n=1}^{\infty} \uparrow$ implies $\exists \ell_x = \lim\limits_{n\to\infty} x_n$. Analogous exists $\exists \ell_y = \lim\limits_{n\to\infty} y_n$. Since $y_n = (1+\frac{1}{n})x_n$ we get for $n \to \infty$ that $\ell_y = 1 \cdot \ell_x$. For each n and $m > n+1$ we have $x_m \geq x_{n+1}$, so if $m \to \infty$ we conclude $\ell_x \geq x_{n+1} > x_n$. Similarly $\ell_y \leq y_{n+1} < y_n$. □

Definition. The number just found is called Eulers number e defined by.

$$e := \lim_{n\to\infty}\Big(1+\tfrac{1}{n}\Big)^n, \qquad (e \approx 2,718281828459045\ldots)$$

1.3 Subsequences

(a) **Definition of a subsequence**

DEFINITION. Assume $f : \mathbb{N} \to \mathbb{R}$ is a sequence and assume $\varphi : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Denote by $h := f \circ \varphi$ the composition $h(k) = f(\varphi(k))$. Then the sequence $h : \mathbb{N} \to \mathbb{R}$ is called a *subsequence* of the sequence $f : \mathbb{N} \to \mathbb{R}$.

$$x_n := f(n), \qquad x_{n_k} = h(k), \qquad n_k := \varphi(k), \qquad n_1 < n_2 < \ldots < n_k < \ldots$$

NOTATION. $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty} \iff \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$

REMARK: for most applications, in fact all in this course, we will only use that $n_k \to \infty$ if $k \to \infty$, a much more flexible but slighty more confusing condition than $k_n < k_{n+1}$.

(b) **Any subsequence of a convergent sequence converges to the same limit**

THEOREM. $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, $\exists \lim\limits_{n\to\infty} x_n = a \implies \exists \lim\limits_{k\to\infty} x_{n_k} = a$

(c) **Nested interval theorem**

THEOREM. Assume a sequence of closed intervals $[a_n, b_n] \subset \mathbb{R}$ satisfies the properties

1) $\forall\, n \in \mathbb{N} \quad [a_{n+1}, b_{n+1}] \subset [a_n, b_n]$
2) $b_n - a_n \to 0 \quad \text{as} \quad n \to \infty$

Then $\exists!\ c \in \mathbb{R}$: $c \in \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$. IDEA: Based on Theorem in 1.2.9. look at $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$, show they are equal and do the job for c.

(d) **Bolzano–Weierstrass theorem**

THEOREM. Each bounded sequence has a convergent subsequence:

$\{x_n\}_{n=1}^{\infty}$ is bounded $\implies$ $\exists\, \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that $\exists\, a \in \mathbb{R} :\ a = \lim\limits_{k\to\infty} x_{n_k}$

IDEA After preparation c) easy, we use: for a sequence $\{x_k\}_{k=1}^{n}$,a feeder $[a, b]$ and $c = \frac{1}{2}(a + b)$ is $[a, c]$ a feeder or $[c, b]$ is a feeder. Now built a nested sequence of feeders and show they (all) catch a subsequence.

REMARK The british lecture approach is differs. It says: if $\{x_n\}_{n_1}^{\infty}$ is any (real) sequence then it contains a monotone subsequence. The (mountaineers) proof will be given in the lecture (only!).

(e) **Cauchy sequence**

DEFINITION. $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is a *Cauchy sequence* (or a *fundamental sequence*) if

$$\forall\, \varepsilon > 0 \quad \exists\, N(\varepsilon) \in \mathbb{N} : \quad \forall\, n, m \geq N(\varepsilon) \qquad |x_n - x_m| < \varepsilon$$

skip

THEOREM. Every fundamental sequence is convergent:

$$\{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \text{ is a Cauchy sequence} \iff \exists\, a \in \mathbb{R} : \quad x_n \underset{n\to\infty}{\longrightarrow} a$$

In fact: a Cauchy sequence is always bounded, and convergent iff it has a convergent subsequence.

1.4 Limit of a function

(a) **Definition of the limit of a function according to Cauchy**

DEFINITION (C). Assume $x_0 \in (a,b)$, $f : (a,b) \setminus \{x_0\} \to \mathbb{R}$, $y_0 \in \mathbb{R}$. We write

$$\lim_{x\to x_0} f(x) = y_0 \qquad \text{or} \qquad f(x) \underset{x\to x_0}{\to} y_0$$

and say y_0 is the *limit* of $f(x)$ as x tends to x_0 iff

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta = \delta(\varepsilon) > 0 : \;\; \forall\, x \in (a,b) \quad (x \neq x_0) \;\&\; |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - y_0| < \varepsilon$$

The set $\overset{\circ}{U}_\delta(x_0) := (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ is called a *punctured neighbourhood* of x_0.

(b) **Definition of the limit of a function according to Heine**

DEFINITION (H). Assume $x_0 \in (a,b)$, $f : (a,b) \setminus \{x_0\} \to \mathbb{R}$ $y_0 \in \mathbb{R}$. We write

$$\lim_{x\to x_0} f(x) = y_0 \qquad \text{or} \qquad f(x) \underset{x\to x_0}{\to} y_0$$

and say y_0 is the *limit* of $f(x)$ as x tends to x_0 iff

$$\forall\, \{x_n\}_{n=1}^{\infty} \subset (a,b) \setminus \{x_0\} : \qquad \lim_{n\to\infty} x_n = x_0 \quad \Longrightarrow \quad \lim_{n\to\infty} f(x_n) = y_0$$

(c) **Equivalence of definitions**

THEOREM. Definition (C) $\iff$ Definition (H)

(d) **Arithmetic operations with limits of functions**

THEOREM 1. $\exists \lim_{x\to x_0} f(x),\ \exists \lim_{x\to x_0} g(x) \quad \Longrightarrow \quad \exists \lim_{x\to x_0} (f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$

THEOREM 2. $\exists \lim_{x\to x_0} f(x), \exists \lim_{x\to x_0} g(x) \quad \Longrightarrow \quad \exists \lim_{x\to x_0} (f(x)\cdot g(x)) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$

LEMMA. $\lim_{x\to x_0} g(x) > 0 \quad \Longrightarrow \quad \exists\, \delta > 0\text{:} \quad |x - x_0| < \delta \;\Rightarrow\; g(x) > 0$

THEOREM 3. $\exists \lim_{x\to x_0} f(x), \exists \lim_{x\to x_0} g(x) \neq 0 \quad \Longrightarrow \quad \exists \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$

REMARK Using Heine's Definition (almost) all in (d) follows from 1.2.8, arithmetic of limits of sequences. Soon we will also have the powerful technique of substitution - which can be applied in more ways to functions than to sequences.

(e) **One-sided limits**

DEFINITION. Assume $f : [a,b] \to \mathbb{R}$, $x_0 \in [a,b]$.

$$\exists \lim_{x\to x_0-0} f(x) = y_0 \quad \Longleftrightarrow \quad \forall\, \varepsilon > 0 \;\; \exists\, \delta > 0\text{:}\; x \in (x_0 - \delta, x_0) \;\Rightarrow\; |f(x) - y_0| < \varepsilon$$

$$\exists \lim_{x\to x_0+0} f(x) = y_0 \quad \Longleftrightarrow \quad \forall\, \varepsilon > 0 \;\; \exists\, \delta > 0\text{:}\; x \in (x_0, x_0 + \delta) \;\Rightarrow\; |f(x) - y_0| < \varepsilon$$

THEOREM. $\exists \lim_{x\to x_0} f(x) = y_0 \quad \Longleftrightarrow \quad \exists \lim_{x\to x_0-0} f(x) = \lim_{x\to x_0+0} f(x) = y_0$

1.5 Indeterminate forms and important examples

(a) **Types of indeterminate forms**

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty \quad 1^\infty, \quad \infty^0, \quad 0^0$$

not well defined as they allow different outcome. Look at $\frac{x^3+1}{x^2+1}$ versus $\frac{x+1}{x^2+1}$ for $x \to \infty$. We define limits at infinity and infinite limits, eg if $f : (a, \infty) \to \mathbb{R}$ then

- $\lim\limits_{x\to\infty} f = y \in \mathbb{R}$ if $\forall \varepsilon > 0 \exists\, T \in \mathbb{R} : x > T \Rightarrow |f(x) - y| < \varepsilon$
- $\lim\limits_{x\to\infty} f = -\infty$ if $\forall Y \in \mathbb{R} \exists T \in \mathbb{R} : x > t \Rightarrow f(x) < Y$, and similarly.

(b) **Auxiliary inequalities**

THEOREM.

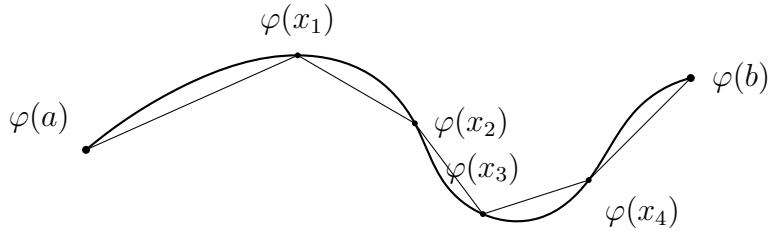
1) $x_1, x_2 \in \mathbb{R} \implies |\sin x_1 - \sin x_2| \le |x_1 - x_2|, \quad |\cos x_1 - \cos x_2| \le |x_1 - x_2|$
2) $\lim\limits_{x\to 0} \sin x = 0, \quad \lim\limits_{x\to 0} \cos x = 1$
3) $0 < x < \frac{\pi}{2} \implies \sin x < x < \tan x, \quad x \in \mathbb{R} \implies |\sin x| \le |x|$

All these statements require a (reasonably) solid definition of sin and cos. It is possible (in certain limits) with concepts from school mathematics, we repeat them here

i) **distance of two points in the plane:** if $p = (p_1, p_2), q = (q_1, q_2) \in \mathbb{R}^2$ the distance between them is

$\|p - q\| = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$ in accordance with Pythagoras' theorem.

ii) **length of a curve:** if $\varphi : [a, b] \to \mathbb{R}^2$ a map whose image is what is considered a curve, we define its length to be the supremum the length of all chains of segements/chords that can be inscribed into the curve, see the picture below where we used beside begin and start also $x_1 < x_2 < x_3 < x_4$ in (a, b) to pick more points on the curve.



$$\ell(\varphi) = \sup(\{\sum_{k=1}^{n} \|\varphi(x_k) - \varphi(x_{k-1})\| \ : \ n \in \mathbb{N}, a = x_0 < x_1 < x_2 < \ldots < x_n = b\}).$$

If we want to apply this for a circular arc, which would require the not yet defined $\sin, \cos$ for parametrisation, we can instead work fully geometrical, replacing $\varphi(x_k)$ by y_k in the arc such that y_{k+1} is not in the subarc from y_0 to y_k. So no going backwards with the points! In both cases, the length is obviously at least the distance of the endpoints.

Now, all you have to believe is that for any $x \in [0, 2\pi)$ there is a unique circular arc on the unit circle $\{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = 1\}$ which starts at $(1, 0)$ in positive (i.e. anti clockwise direction) and has length x. Then denoting the endpoint of this arc by $p(x)$ we can define $\cos(x) = p(x)_1$ and $\sin(x) = p(x)_2$. Obviously $\|p(0) - (1, 0)\| \leq 0$ so $p(0) = (1, 0)$ and denote the arc as $\overset{\frown}{p(0)p(x)}$. If $x = 2\pi$, which the old greeks discovered to be the length of the whole unit circle, the arc is closing an $p(2\pi) = (1, 0) = p(0)$. This allows to complete the definition of sin and cos by requesting them to be 2π periodic on $\mathbb{R}$.
As the lengths of touching arcs add up, each halfcircle has length π and so we have $p(x + \pi) = -p(x)$ which says $\sin(x + \pi) = -\sin(x)$ and $\cos(x + \pi) = -\cos(x)$ if $x \in [0, \pi)$, and by 2π periodicity also for all real x. One more important comes looking at the complementary arc to $\overset{\frown}{p(0)p(2\pi - x)}$ which is $\overset{\frown}{p(2\pi - x)p(2\pi)}$ of length x and there the image of $\overset{\frown}{(p(0)p(x))}$ under reflection on the horizontal axis. Thus we get, first for $x \in [0, 2\pi)$ and then by periodicity for all $x \in \mathbb{R}$

$$\sin(-x) = \sin(2\pi - x) = -\sin(x) \& \cos(-x) = \cos(2\pi - x) = \cos(x)$$

Next: relax, the rest is simple(r), we now proof the Theorem above

1)-Proof: Since $|y| = |-y|$ we can change from x_1, x_2 to $x_1 + k\pi, x_2 + k\pi + m2\pi$, where $k, m \in \mathbb{Z}$ without changing the left hand sides of the inequatilies. So wlog $x_1 \in [0, \pi)$ and $x_2 \in [0, 2\pi)$ and our geometric definition relying on $p(x_1), p(x_2)$ can be used. We can assume $x_1 < x_2$ since swapping them does not change the inequalities. Now

$$x_2 = \ell(\overset{\frown}{p(0)p(x_2)}) = \ell(\overset{\frown}{p(0)p(x_1)}) + \ell(\overset{\frown}{p(x_1)p(x_2)}),$$

so $\ell(\overset{\frown}{p(x_1)p(x_2)}) = x_2 - x_1$, draw a picture! This shows

$$\begin{aligned} |x_2 - x_1| = x_2 - x_1 = \ell(\overset{\frown}{p(x_1)p(x_2)}) &\geq \|p(x_2) - p(x_1)\| \\ &= \sqrt{(\cos(x_2) - \cos(x_1))^2 + (\sin(x_2) - \sin(x_1))^2} \end{aligned}$$

and now use $\sqrt{a^2 + b^2} \geq \max(a, b)$ to finish.

2)-Proof: is now trivial, as $(1, 0) = p(0) = (\cos(0), \sin(0))$, so we have from 1) with $x = x_2$ and $0_x = 1$ that $|\sin(x)| \leq |x|$ and $|\cos(x) - 1| \leq |x|$, which implies both limits.

3)-Proof: It remains only to show that $0 < \sin(x) < x < \tan(x) = \frac{\sin(x)}{\cos(x)}$ if $0 < x < \frac{\pi}{2}$. Halfing the halfcircle $\overset{\frown}{p(0)p(\pi)}$ again, we have $p(\frac{\pi}{2}) = (0, 1)$ and so $p(x)$ is in the first quadrant where all coodinates and hence $\sin(x), \cos(x)$ are positive. Moreover, since any line intersects the circle in at most 2 points, $p(\frac{x}{2})$ is not on the line $\overline{p(0)p(x)}$ and hence, e.g using Pythagoras again

$$\|p(0) - p(\frac{x}{2})\| + \|p(\frac{x}{2}) - p(x)\| > \|p(0) - p(x)\|, \text{ hence}$$

$$x = \ell(\overset{\frown}{p(0)p(x)}) > \|p(0) - p(x)\| \geq |\sin(x) - \sin(0)| = \sin(x).$$

The remaining inequality $x < \tan(x)$ if $0 < x < \frac{\pi}{2}$ requires a more careful comparison of the length of two arcs, the details are quite elementary but require a bit more time. They can be found at the MATHIPSP1 webseite in the link "tangent(x) bigger than x"

(c) **Properties of a^x and $\log_a x$ which are postulated for now**

We postulate the properties of the exponential and the logarithmic functions which actually come together with the definition of the exponential function and its inverse. We will return to discussion of these properties in the section "Continuity". These properties also allowed to use without further explanation when you solve homework problems.

THEOREM. Assume $a > 0$ and $a \neq 1$. Then

1) $\forall \{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \quad x_n \to x_0 \quad \Longrightarrow \quad a^{x_n} \to a^{x_0}$
2) $\forall \{x_n\}_{n=1}^{\infty} \subset \mathbb{R} \quad x_n \to x_0, \quad x_0 > 0 \quad \Longrightarrow \quad \log_a x_n \to \log_a x_0$

(d) **Important examples**

1) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

2) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

3) $\lim\limits_{x\to +\infty} (1+\frac{1}{x})^x = e \qquad \lim\limits_{x\to -\infty} (1+\frac{1}{x})^x = e \qquad \lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

4) $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$

5) $\lim\limits_{x\to 0} \frac{a^x-1}{x} = \ln a$

6) $\lim\limits_{x\to 0} \frac{(1+x)^\mu-1}{x} = \mu$

Proof: We can quickly show results 1) and 2), for 3) till 6) look at the link "IMPORTANT LIMITS USING EULERS NUMBER" on the course website.

From 1.5.(b)3) we have that if $0 < x < \frac{\pi}{2}$ then $1 > \frac{\sin x}{x} > \cos(x)$, by the properties of sin and cos this is true also if $0 < -x < \frac{\pi}{2}$, so for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$. Hence 1) follows now from the sandwich theorem and 1.5.(b) about cos.

To show 2), we link it to 1) and the most simple relation between cos and sin is $1 - \cos^2(x) = \sin^2(x)$. Thus the second binomial formula might prove useful.

1.6 Infinite limits and asymptotes

(a) Infinite limits, vertical asymptotes

DEFINITION.

$$\lim_{x\to x_0-0} f(x) = +\infty \iff \forall\, M>0\ \ \exists\, \delta>0:\ \ x\in(x_0-\delta, x_0)\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to x_0+0} f(x) = +\infty \iff \forall\, M>0\ \ \exists\, \delta>0:\ \ x\in(x_0, x_0+\delta)\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to x_0-0} f(x) = -\infty \iff \forall\, M<0\ \ \exists\, \delta>0:\ \ x\in(x_0-\delta, x_0)\ \Rightarrow\ f(x)<M$$
$$\lim_{x\to x_0+0} f(x) = -\infty \iff \forall\, M<0\ \ \exists\, \delta>0:\ \ x\in(x_0, x_0+\delta)\ \Rightarrow\ f(x)<M$$

DEFINITION. The line $x = x_0$ is a *vertical asymptote* of the function $f : \mathbb{R} \to \mathbb{R}$ if at least one of the four statements is true:

$$\lim_{x\to x_0+0} f(x) = +\infty, \quad \lim_{x\to x_0-0} f(x) = +\infty, \quad \lim_{x\to x_0-0} f(x) = +\infty, \quad \lim_{x\to x_0-0} f(x) = -\infty$$

(b) Behavior of functions at infinity

DEFINITION.

$$\lim_{x\to+\infty} f(x) = +\infty \iff \forall\, M>0\ \ \exists\, R>0:\ \ x>R\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to-\infty} f(x) = +\infty \iff \forall\, M>0\ \ \exists\, R<0:\ \ x<R\ \Rightarrow\ f(x)>M$$
$$\lim_{x\to+\infty} f(x) = -\infty \iff \forall\, M<0\ \ \exists\, R>0:\ \ x>R\ \Rightarrow\ f(x)<M$$
$$\lim_{x\to-\infty} f(x) = -\infty \iff \forall\, M<0\ \ \exists\, R<0:\ \ x<R\ \Rightarrow\ f(x)<M$$

(c) Limits at infinity, horizontal asymptotes

DEFINITION.

$$\lim_{x\to+\infty} f(x) = y_0 \iff \forall\, \varepsilon>0\ \ \exists\, R>0:\ \ x>R\ \Rightarrow\ |f(x)-y_0|<\varepsilon$$
$$\lim_{x\to-\infty} f(x) = y_0 \iff \forall\, \varepsilon>0\ \ \exists\, R<0:\ \ x<R\ \Rightarrow\ |f(x)-y_0|<\varepsilon$$

DEFINITION. The line $y = y_0$ is a *horizontal asymptote* of the function $f : \mathbb{R} \to \mathbb{R}$ if at least one of the two statements is true:

$$\lim_{x\to+\infty} f(x) = y_0 \quad \text{or} \quad \lim_{x\to-\infty} f(x) = y_0$$

(d) **How to remember definitions of finite/infinite limits?**

$$y_0 = \lim_{x \to x_0} f(x)$$

x_0 is finite or infinite
y_0 is finite or infinite

	Values of $f(x)$	Values of x	When is used?
Small number	$\forall\, \varepsilon > 0$	$\exists\, \delta > 0$	For a finite value
Large number	$\forall\, M > 0$	$\exists\, R > 0$	For an infinite value

(e) **Oblique asymptotes**

DEFINITION. The line $y = kx + b$ is an *oblique asymptote* of the function $f : \mathbb{R} \to \mathbb{R}$ if at least one of the two statements is true:

$$\lim_{x \to +\infty} \Big(f(x) - kx - b \Big) = 0 \quad \text{or} \quad \lim_{x \to -\infty} \Big(f(x) - kx - b \Big) = 0$$

REMARK. A horizontal asymptote is a particular case of an oblique asymptote with $k = 0$.

THEOREM. $y = kx + b$ is a horizontal or an oblique asymptote of $f : \mathbb{R} \to \mathbb{R}$ as $x \to +\infty$ iff the following two limits exist:

1) $\exists \lim\limits_{x \to +\infty} \frac{f(x)}{x} = k$
2) $\exists \lim\limits_{x \to +\infty} (f(x) - kx) = b$

The similar statements hold as $x \to -\infty$.

1.7 Continuity

(a) **Continuity of a function at a point**

DEFINITION. $f : (a, b) \to \mathbb{R}$ is *continuous* at a point $x_0 \in (a, b) \iff \lim\limits_{x \to x_0} f(x) = f(x_0)$

REMARK. The following statements are equivalent:

1) $f : (a, b) \to \mathbb{R}$ is continuous at x_0
2) $\forall\, \varepsilon > 0 \;\; \exists\, \delta(\varepsilon) > 0$: $\quad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$
3) $\forall\, \{x_n\}_{n=1}^{\infty} \subset (a, b)$: $\quad x_n \to x_0 \implies f(x_n) \to f(x_0)$

DEFINITION. $f : (a, b) \to \mathbb{R}$ is continuous *on* $(a, b) \iff \forall\, x_0 \in (a, b)$ f is continuous at x_0

(b) **Stability of sign theorem**

THEOREM. f is continuous at x_0, $f(x_0) > 0 \implies \exists\, \delta > 0$: $\forall\, x \in (x_0 - \delta, x_0 + \delta)$ $f(x) > 0$

(c) **Arithmetic operations with continuous functions**

THEOREM.

1) f, g are continuous at $x_0 \implies f \pm g$ are continuous at x_0
2) f, g are continuous at $x_0 \implies f \cdot g$ is continuous at x_0
3) f, g are continuous at x_0, $g(x_0) \neq 0 \implies f/g$ is continuous at x_0

(d) **Left and right continuity**

DEFINITION.

$f : [a, b] \to \mathbb{R}$ is *left continuous* at $x_0 \iff \lim\limits_{x \to x_0 - 0} f(x) = f(x_0)$

$f : [a, b] \to \mathbb{R}$ is *right continuous* at $x_0 \iff \lim\limits_{x \to x_0 + 0} f(x) = f(x_0)$

$$f : [a, b] \to \mathbb{R} \text{ is continuous } \textit{on } [a, b] \iff \begin{cases} f \text{ is continuous on } (a, b) \\ f \text{ is left continuous at } a \\ f \text{ is right continuous at } b \end{cases}$$

THEOREM. f is continuous at $x_0 \in (a, b) \iff f$ is left and right continuous at x_0.

12. **Types of discontinuity**

DEFINITION. Assume $x_0 \in (a, b)$, $f : (a, b) \setminus \{x_0\} \to \mathbb{R}$

- x_0 is a *removable discontinuity* $\iff \exists$ finite $\lim\limits_{x \to x_0 \pm 0} f(x)$, $\lim\limits_{x \to x_0 - 0} f(x) = \lim\limits_{x \to x_0 + 0} f(x)$
- x_0 is a *jump discontinuity* $\iff \exists$ both finite $\lim\limits_{x \to x_0 \pm 0} f(x)$, $\lim\limits_{x \to x_0 - 0} f(x) \neq \lim\limits_{x \to x_0 + 0} f(x)$

- $x = x_0$ is a *vertical asymptote* $\iff$ $\exists \lim\limits_{x \to x_0 \pm 0} f(x) = \pm\infty$ (at least one of limits)
- x_0 may be a discontinuity of no above type $\iff$ $\nexists \lim\limits_{x \to x_0 \pm 0} f(x)$ $\quad (f(x) = \sin \frac{1}{x})$

13. **Continuity of some elementary functions**

- $f(x) = x^n$ is continuous on $\mathbb{R}$, $n \in \mathbb{N}$
- $f(x) = \frac{1}{x^n}$ is continuous on $\mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$
- $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is continuous on $\mathbb{R}$
- $f(x) = \sin x$, $g(x) = \cos x$ are continuous on $\mathbb{R}$
- $f(x) = \tan x$ is continuous at $x \neq \frac{\pi}{2} + \pi k$, $g(x) = \operatorname{ctg} x$ is continuous at $x \neq \pi k$, $k \in \mathbb{Z}$
- $f(x) = x^p$, $p \in (0, +\infty)$ is continuous on $[0, +\infty)$
- $f(x) = a^x$, $a > 0$, is continuous on $\mathbb{R}$

14. **Continuity of a composition of functions**. Most important

THEOREM. $f : (a, b) \to \mathbb{R}$, $g : (c, d) \to \mathbb{R}$, $x_0 \in (a, b)$, $f(x_0) \in (c, d)$

f is continuous at x_0, $\quad g$ is continuous at $f(x_0) \quad \Longrightarrow \quad g \circ f$ is continuous at x_0

15. **Two intermediate value theorems (Bolzano–Cauchy theorems)**

THEOREM 1. f is continuous on $[a, b]$, $f(a) \leq 0$, $f(b) \geq 0 \implies \exists\, c \in [a, b]$: $f(c) = 0$

COROLLARY 2. f is continuous on $[a, b]$, $f(a) = y_1$, $f(b) = y_2$, $y_1 \leq y_2 \implies [y_1, y_2] \subset R(f)$

16. **Continuity of the inverse function**

THEOREM. Assume $f : (a, b) \to \mathbb{R}$ is continuous and strictly monotone on (a, b). Then f is invertible on (a, b) and f^{-1} is continuous on $R(f)$.

17. **Continuity of some elementary inverse functions**

- $f(x) = \sqrt[n]{x}$ is continuous on $D(f)$, $n \in \mathbb{N}$
- $f(x) = \log_a x$, $a > 0$, $a \neq 1$, is continuous on $(0, +\infty)$
- $f(x) = \arcsin x$, $g(x) = \arccos x$ are continuous on $[-1, 1]$
- $f(x) = \operatorname{arctg} x$, $g(x) = \operatorname{arcctg} x$ are continuous on $\mathbb{R}$

18. **Two extreme value theorems (Weierstrass theorems)**

THEOREM 1. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\implies$ f is bounded on $[a,b]$, i.e.

$$\exists\, M > 0 : \qquad \forall\, x \in [a,b] \qquad |f(x)| \leq M$$

THEOREM 2. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\implies$ f achieves on $[a,b]$ its maximum and minimal values, i.e. $\exists\ c_1,\ c_2 \in [a,b]$ such that

$$f(c_1) = \inf_{x\in[a,b]} f(x) \qquad \text{and} \qquad f(c_2) = \sup_{x\in[a,b]} f(x)$$

19. **Uniform continuity**

DEFINITION. A function $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* on $D(f)$ iff

$$\forall\, \varepsilon > 0 \quad \exists\, \delta(\varepsilon) > 0 : \quad \forall\, x_1, x_2 \in D(f) \qquad |x_1 - x_2| < \delta \quad \Longrightarrow \quad |f(x_1) - f(x_2)| < \varepsilon$$

EXAMPLE both functions $f : x \mapsto \frac{1}{x}$ and $g = \sin \circ f$ are continuous on $(0,1]$, g is even bounded. But none of them is unformly continuous on this intervall $(0,1]$ of definition.

20. **Heine–Cantor theorem**

THEOREM. $[a,b] \subset \mathbb{R}$ is a closed bounded interval, $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\implies$ f is uniformly continuous on $[a,b]$.

Proof: Idea - converge with Bolzano-Weierstrass for sequences to a contradiction!

21. COROLLARY. Let $f : (a,b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous $\Leftrightarrow$ $\exists \lim_{x\to a+0} f(x) \Leftrightarrow \exists F : [a,b] \to \mathbb{R}$ continuous which extends f, i.e. $F(x) = f(x)$ if $x \in (a,b]$.

1.8 Differentiability and derivatives

1. **Main definition**

   DEFINITION. Assume $f : (a, b) \to \mathbb{R}$, $x_0 \in (a, b)$. A function f is *differentiable at* a point x_0 iff there exists a finite limit

   $$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

   The number $f'(x_0)$ is called the *derivative* of a function f at x_0. Another notation (due to Leibniz, visit him at Augustusplatz!)

   $$\frac{df}{dx}(x_0) := f'(x_0)$$

   - We say f is *differentiable on* (a, b) if for any $x_0 \in (a, b)$ the function f is differentiable at x_0.
   - We say f is *continuously differentiable* on (a, b) if f is differentiable on (a, b) and the function f' is continuous on (a, b).

2. **Physical and geometric meaning of the derivative**

   - From a physical point of view the derivative of a function is the instantaneous speed (or the rate of change) of the function.
   - From a geometrical point of view the derivative of a function f at a point x_0 is equal to the slope of the tangent line to the graph of the function at the point x_0.