Mathematics 1, Problemsheet 2
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Due Date: 29.10.26 until 23:59 on-line in the moodle

This homework will be estimated from 10 points (each problem is one point). Problems which contain subitems will be estimated by one point only if at least 75% of answers are correct. Try to work on the problems in small groups, 2-4 people proved to be the most most productive size and **use AI only after first attempts and modestly** - as mastering this kind of question on your own is crucial for passing the exam! As already explained, the marking of your problems will serve as a feedback only on your own progress, not on AI's ability to solve the questions. Enjoy the work on the questions and your success!

1. Which of the following relations determine the variable y as a function of the variable x?

   1) $y^2 + 2 = (x+1)^2$
   2) $\frac{y}{x} = x - y$
   3) $|y-1| = |x+2|$
   4) $\sin y = \cos x$
   5) $3y - 2 = 4(x-1)$
   6) $x^2 \le y \le x + 2$
   7) $y^3 = x^2 - 1$
   8) $|y| = \sin(x+1)$

2. **How (not) to order the real plane** We introduce the lexicographical order: if

   $$p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \text{ then } p_1 < p_2 \text{ if } x_1 < x_2 \vee (x_1 = x_2 \,\&\, y_1 < y_2).$$

   a) Prove that for any $p_1, p_2 \in \mathbb{R}^2$ precisely one of the following statements hold

   $$p_1 < p_2 \qquad p_1 = p_2 \qquad p_2 < p_1.$$

   b) Shade a general point $p \in \mathbb{R}^2$ and shade the region of all $q \in \mathbb{R}^2$ such that $p < q$.

   c) Suppose that $p_1, p_2, q_1, q_2 \in \mathbb{R}^2$ satisfy $p_1 < p_2$ and $q_1 < q_2$. show that $p_1 + q_1 < p_2 + q_2$.

   d) Now suppose even that $0 < p_1 < p_2$ and $0 < q_1 < q_2$, where $0 = (0,0)$. Does this imply that $p_1 \cdot q_2 < p_2 \cdot q_2$?

   Both in c) and d) the operations "+" and "$\cdot$" are defined coordinatewise.

3. Determine the range of the following functions $f : \mathbb{R} \to \mathbb{R}$

   1) $f(x) = x^2 + 4x - 1$
   2) $f(x) = \frac{2x}{x^2+1}$
   3) $f(x) = 3 - |x+1|$
   4) $f(x) = \begin{cases} 1, & x \in [1, +\infty) \\ 5 + 2x, & x \in (-1, 1) \\ -3 + \sin x & x \in (-\infty, -1] \end{cases}$

4. Given the formulas for the functions f and g write the formulas for the function $f \circ f$, $g \circ f$, $f \circ g$, $g \circ g$:

1) $f(x) = 2x + 1, \quad g(x) = 3x - 2$
2) $f(x) = x^2 + 1, \quad g(x) = 2\sin x$
3) $f(x) = x + \frac{1}{x}, \quad g(x) = 2^x$
4) $f(x) = \frac{1}{x}, \quad g(x) = \frac{2x+1}{2x-1}$

5. Determine which of the functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ are a) injective b) surjective c) bijective?

   1) $f(x, y) = (3x, 2y)$
   2) $f(x, y) = (x - y, y)$
   3) $f(x, y) = (x^2 + 1, y^3 - 1)$
   4) $f(x, y) = (x + y, x + 2y)$
   5) $f(x, y) = (\sin y, \cos x)$
   6) $f(x, y) = (y^3, x + 1)$

6. Prove that if $0 < a < b$ then $\sqrt{ab} < \frac{a+b}{2}$. Now show also the general inequalities between the harmonic, geometric, arithmetic and square mean:

$$a < \frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}} < b.$$

7. We recall that for $S \subset \mathbb{R}$ we have $M = \max S$ iff $M \in S$ and M is an upper bound

   a) Show that the maximum of $S \subset \mathbb{R}$ exists if and only if $\sup S \in S$.
   b) Prove: if $S_1, S_2 \subset \mathbb{R}$ both have a maximum then $\max(S_1 + S_2) = \max S_1 + \max S_2$, here $S_1 + S_2 = \{s_1 + s_2 \, : \, s_1 \in S_1 \text{ and } s_2 \in S_2$, the Minkowski sum of two sets of numbers.

8. In the situation of problem 7, show

   a) if $S \subset \mathbb{R}$ has supremum, then for $-S = \{-s, s \in S\}$ we have $\inf(-S) = -\sup(S)$, in particular $-S$ automatically has an infinimum. For the proof only use the definition of the supremum and infinimum.
   b) if $S_1, S_2 \subset \mathbb{R}$ both have a supremum, then $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$. Here the characterization of sup by quantifiers will prove useful.

9. Find the supremum and infinimum, or show that they do not exist for the following sets

$$S = \left\{ \frac{1}{n} - \frac{1}{n+m} + \frac{1}{m} \; : \; n, m \in \mathbb{N} \right\} \text{ and } T = \{ \frac{2^n}{n^2}, \; : \; n \in \mathbb{N} \}$$

   When does the maximum or minimum exist? Give enough reason.

10. Given the sequence $x_n = \sqrt{n+1} - \sqrt{n}$, $n = 1$ and an $\varepsilon > 0$ show the existence of N such that for all $n \in \mathbb{N}$ holds $n > N \Rightarrow x_n < \varepsilon$. Does the sequence $\{x_n\}_{n=1}^{\infty}$ have a limit? Justify your answer rigerously. *[The second binomial formula could be helpful.]*