Mathematics 1, Problemsheet 3
Leipzig University, WiSe 2026/27, Bernd Kirchheim
Due Date: 5.11.26 until 23:59 on-line in the moodle

This homework will be estimated from 10 points (each problem is one point). Try to work on the problems in small groups, 2-4 people proved to be the most most productive size and **use AI only after first attempts and modestly** - as mastering this kind of question on your own is crucial for passing the exam! As already explained, the marking of your problems will serve as a feedback only on your own progress, not on AI's ability to solve the questions. Enjoy the work on the questions and your success!

In problems №1-7 compute the limits and explain your answers.

1. $\lim\limits_{n\to\infty} \frac{\sqrt{n+2}-\sqrt{n-2}}{\sqrt{n+1}-\sqrt{n-1}} =$
   [Hint: The second binomial formula might help here.]

2. $\lim\limits_{n\to\infty} \frac{5^{2n+1}}{n!} \sin \frac{n^2+1}{2n} =$

3. $\lim\limits_{n\to\infty} \frac{\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}}{\frac{1}{3}+\frac{1}{9}+\ldots+\frac{1}{3^n}} =$
   [Hint: Compare with problem №10.]

4. $\lim\limits_{n\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!} =$

5. $\lim\limits_{n\to\infty} (n!)^{\frac{1}{2(n!)}} =$

6. $\lim\limits_{n\to\infty} \left(\frac{n+2}{n+1}\right)^{2n} =$

7. $\lim\limits_{n\to\infty} \sqrt[n]{2^n+6^n} =$

8. For $a > 0$ define a sequence recursively $x_1 = a.$ and $x_{n+1} = \sqrt{x_n + 2}$ if x_n is already known.

   i) Let $a \in (0, 2)$, show that $\{x_n\}_{n=1}^{\infty}$ is an increasing and bounded sequence.

   ii) From the lectures you now know, that there exists $\ell = \lim\limits_{n\to\infty} x_n$. To find ℓ first show that $x_{n+1}^2 = x_n + 2$ and explain what happens if you send now n to ∞. Can you now determine the limit ℓ uniquely? How does it depend on a?

9. What happens in the previous problem if $a > 2$? Can you draw a picture of the situation?

10. **Zeno's paradox of Achilles and the tortoise** Achilles and the tortoise start a race. Achilles is running with speed 1 und the turtle with speed $q \in (0, 1)$, therefore the slower tortoise gets an advance of length 1 (since the old greek did not know the metric system, we drop the units here). Zeno argues that Achilles can not catch the tortoise: whenever he gets to the point where the tortoise was before it already moved further, and this repeats on and on.

    Model the development of the situation quantitatively, and explain how it is represented by summing infinitely many numbers (which?). As a physicists, you should have no problem to

determine the time and location when Achilles will finally catch the tortoise, explain which famous summation formula you just derived.