Mathematics 1, Homework 01
Leipzig University, WiSe 2026/27, Bernd Kirchheim
Due Date: 22.10.26 until 23:59 on-line in the moodle

This homework will be estimated from 10 points (each problem is one point). Problems which contain subitems will be estimated by one point only if at least 75% of answers are correct. Try to work on the problems in small groups, 2-4 people proved to be the most most productive size and **use AI only after first attempts and modestly** - as mastering this kind of question on your own is crucial for passing the exam! As already explained, the marking of your problems will serve as a feedback only on your own progress, not on AI's ability to solve the questions. Enjoy the work on the questions and your success!

For problems №7 and №9 e.g., consisting of only two subitems, 75% implies, that areas in both subitems are shaded correctly but the boundary of not more than one area is marked incorrectly. (Please, indicate the boundary of an area by a **dash line** if you do not include it in a shaded area and by a **solid line** if you include it in the shaded area).

1. Prove by induction that for any $n \in \mathbb{N}$ the number $3^{3n-1} - 2^{n+1}$ is divisible by 5.

2. Prove by induction that for any $n \in \mathbb{N}$ $\quad 1 + 3 + 5 + \ldots + (2n-1) = n^2$.

3. Prove by induction that for any $n \in \mathbb{N}$ $\quad 2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}$.

   Some manipulation of quadratic inequalities or the second binomial formula could prove useful.

4. We mentioned the result: each nonempty set of natural numbers has a smallest element. It has to be used with care, however.

   Is the following correct proof correct? If you believe no, briefly explain why.

   I claim that every natural number is an interesting number. Because otherwise the set M of uninteresting natural numbers is nonempty, so there is a minimal element m_u. But then m_u has the interesting property that it is the smallest uninteresting number, and this makes it interesting. So $m_u \notin M$, contradiction.

5. Let A, B, C be any sets, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (*A clever distinction of two cases gives a quite short argument.*)

6. Prove that $\{a\} = \{b\}$ iff (i.e. if and only) $a = b$. Show now, that for pairs as defined in the lecture we have $(a, b) = (b, a)$ iff $a = b$.

7. Given a subsets of $\mathbb{R}$ $\quad A = [0, 2] \cup [3, 4]$, $\quad B = [1, 4]$, $\quad C = [2, 3]$, shade the corresponding set E_1 and E_2 on the plane $\mathbb{R}^2$:

   1) $\quad E_1 = (B \setminus C) \times A$

2) $E_2 = (A \cap C) \times B$

8. Prove that there are no sets $A, B \subset \mathbb{R}$ such that the disk $D = \{(x, y) \ : \ x, y \in \mathbb{R} \text{ and } x^2 + y^2 \leq 1\}$ equals to $A \times B$.

9. Given an infinite family of sets $A_k = \{ \ (x, y) \in \mathbb{R}^2 \mid y \geq x^{2k} \ \}$ shade the sets

   1) $\bigcup_{k=1}^{\infty} A_k$ 2) $\bigcap_{k=1}^{\infty} A_k$ on the plane $\mathbb{R}^2$.

   Here an infinite union of sets is the set of those point which are in (at least) one of the sets, and infinite intersection of sets consists of those point, which are in each of these sets.

10. Find the maximal domain $D(f)$ of the functions $f : \mathbb{R} \to \mathbb{R}$ defined by the following formulae - so even without knowing yet what a general function precislely is, decide for which real x expression on the right hand side is well defined.

    1) $f(x) = \frac{\sqrt{7-x}}{x^2-3x+2}$
    2) $f(x) = \sqrt{\frac{x^2-4}{x+1}}$
    3) $f(x) = \sqrt[3]{x} + \sqrt[4]{x+2} + \sqrt{3-x}$
    4) $f(x) = \log_{7-x} \frac{x^2-1}{x-2}$

Blank of answers for problem 10

use any paper/format for problems 1-9

Student's Name/matricle number: _____

(10) Write your answers here:

1) $D(f) =$ _____

2) $D(f) =$ _____

3) $D(f) =$ _____

4) $D(f) =$ _____