Mathematics 1, Homework 01, Solutions
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1. Prove by induction that for any $n \in \mathbb{N}$ the number $3^{3n-1} - 2^{n+1}$ is divisible by 5.

   Proof.

   Base of the induction $P(1)$. Let us verify that for $n = 1$ the statement is true. Indeed,

   $$3^2 - 2^2 = 9 - 5 = 5 \text{ — is divisible by } 5$$

   Inductional step $P(n) \Rightarrow P(n+1)$. Let us take some arbitrary $n \in \mathbb{N}$ and let us assume

   $$3^{3n+1} - 2^{n+1} \text{ is divisible by } 5 \quad \text{— we } \textbf{assume} \text{ that this is true}$$

   Based on this information, let us try to derive that

   $$3^{3(n+1)-1} - 2^{(n+1)+1} \text{ is divisible by } 5 \quad \text{— we } \textbf{want} \text{ to prove this}$$

   Indeed, we have

   $$3^{3(n+1)-1} - 2^{(n+1)+1} = 27 \cdot 3^{3n-1} - 2 \cdot 2^{n+1} = \underbrace{25 \cdot 3^{3n-1}}_{\text{is divisible by 5}} + \underbrace{2(3^{3n+1} - 2^{n+1})}_{\text{due to assumption divisible by 5}}$$

   So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n \in \mathbb{N}$. □

2. Prove by induction that for any $n \in \mathbb{N}$ $\quad 1 + 3 + 5 + \ldots + (2n-1) = n^2$.

   Proof.

   Base of the induction $P(1)$. Let us verify that for $n = 1$ the statement is true. Indeed,

   $$1 = 1^2 \text{ — this is actually true}$$

   Inductional step $P(n) \Rightarrow P(n+1)$. Let ustake some arbitrary $n \in \mathbb{N}$ and assume that

   $$1 + 3 + 5 + \ldots + (2n-1) = n^2 \quad \text{— we } \textbf{assume} \text{ that this is true}$$

   Based on this information, let us try to derive that

   $$1 + 3 + 5 + \ldots + (2n-1) + (2(n+1) - 1) = (n+1)^2 \quad \text{— we } \textbf{want} \text{ to prove this}$$

   Using inductional assumption we transform

   $$1+3+\ldots+(2n-1)+(2(n+1)-1) = (1 + 3 + \ldots + (2n-1)) + (2n+1) = n^2 + 2n + 1 = (n+1)^2.$$

   So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n \in \mathbb{N}$. □

3. Prove by induction that for any $n \in \mathbb{N}$ $\quad 2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}$.

Some manipulation of quadratic inequalities or the second binomial formula could prove useful.

Proof.

We will use the 2nd suggestion: as in the difference of the most right term a $2(\sqrt{n+1}-\sqrt{n})$ pops up in the induction step, but in the middle the increment is $\frac{1}{\sqrt{n+1}}$ we have to relate this terms to each other, in particular simplifying the more complicated one. Here the 2nd binomial formula helps a lot

$$(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = (n+1) - n = 1 \text{ , so } \frac{1}{\sqrt{n+1} + \sqrt{n}} = \sqrt{n+1} - \sqrt{n}.$$

therefore we can estimate for $k \in \mathbb{N}$ and $1/\sqrt{k} = 2/(2\sqrt{k})$

$$2(\sqrt{k+1} - \sqrt{k}) = \frac{2}{\sqrt{k+1} + \sqrt{k}} < \frac{2}{2\sqrt{k}} < \frac{2}{\sqrt{k} + \sqrt{k-1}} = 2(\sqrt{k} - \sqrt{k-1}). \qquad (0.1)$$

After these preparations we run the usual induction scheme.

Base of the induction $P(1)$. Let us verify that for $n = 1$ the statement is true. Indeed,

$$2\sqrt{2} - 2 < 2\sqrt{\frac{9}{4}} = 2\frac{3}{2} - 2 = \frac{1}{\sqrt{1}} = 1 < 2\sqrt{1} \text{ — so, this is actually true}$$

Inductional step $P(n) \Rightarrow P(n+1)$. Let us take some arbitrary $n \in \mathbb{N}$ and assume that

$$P(n): \qquad 2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} \qquad \text{— we } \textbf{assume} \text{ that this is true}$$

Based on this information, let us try to derive

$$P(n+1): 2\sqrt{n+2} - 2 < \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1} \qquad \text{— we } \textbf{want} \text{ to prove this}$$

Using this inductional assumption and (0.1) we see

$$2\sqrt{n+2} - 2 = 2(\sqrt{n+1} - 2) + 2(\sqrt{n+2} - \sqrt{n+1}) < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} = \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}}$$

$$= \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n} + 2(\sqrt{n+1} - \sqrt{n}) = 2\sqrt{n+1}.$$

So, we see that if the statement $P(n)$ is true then the statement $P(n+1)$ is also true. Hence by induction all statements $P(n)$ are true for all $n \in \mathbb{N}$. $\square$

4. We mentioned the result: each nonempty set of natural numbers has a smallest element. It has to be used with care, however.

   Is the following correct proof correct? If you believe no, briefly explain why.

   I claim that every natural number is an interesting number. Because otherwise the set M of uninteresting natural numbers is nonempty, so there is a minimal element m_u. But then m_u has the interesting property that it is the smallest uninteresting number, and this makes it interesting. So $m_u \notin M$, contradiction.

   ANSWER The statement "n is an interesting number" is not a mathematical statement, as its truth can change with the context - as it happened in our argument. Therefore, our set M is not well defined and so it can not produce a contradiction.

5. Let A, B, C be any sets, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (*A clever distinction of two cases gives a quite short argument.*)

   PROOF We have to show $\forall x : \quad (x \in A \cup (B \cap C)) \Leftrightarrow (x \in (A \cup B) \cap (A \cup C))$. So let us fix an arbitrary x. There are the follwing to possible cases for x.

   - $x \in A$, the x is in the left side, but also in $A \cup B$ and $A \cup C$ so also in the right side.
   - $x \notin A$. Then x is in the left side iff $(x \in B \ \& \ x \in C)$ iff $(x \in (A \cup B) \& x \in (A \cup C))$ iff x is in the right side

   As one of these cases must occur, x is in the LS iff it is in the right side, so the sets are equal.

6. Prove that $\{a\} = \{b\}$ iff (i.e. if and only) $a = b$. Show now, that for pairs as defined in the lecture we have $(a, b) = (b, a)$ iff $a = b$.

   PROOF As equal sets include each other, we have $\{a\} \subset \{b\}$, so $a \in \{b\}$. By the definition of $\{b\}$ this is equivalent to $a = b$. Conversely if $a = b$ then both sets have the same unique element $a(= b)$.

   If $a = b$ we have also $\{a, b\} = \{a\} = \{b\} = \{b, a\}$, so that

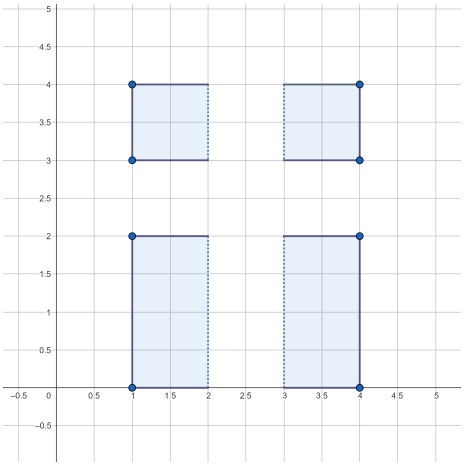
   $$(a, b) = \{\{a\}, \{a, b\}\} = \{\{a\}\} = \{\{b\}\} = \{\{\underline{\ }\}, \{b, a\}\} = (b, a).$$

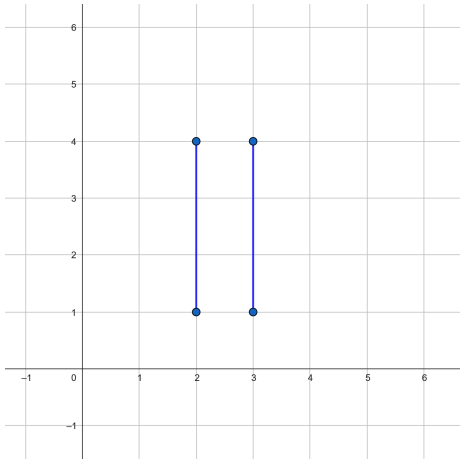
   For the necessity: if $(a, b) = \{\{a\}, \{a, b\}\} = \{\{b\}, \{b, a\}\} = (b, a)$ then we see that $\{a\} \in \{\{b\}, \{b, a\}\}$. So $\{a\} = \{b\}$ which implies by the above $a = b$ or $\{b, a\} = \{a\}$, so $b \in \{a\}$ which again gives $a = b$. In any case, we see that $a = b$. $\square$

7. Given a subsets of $\mathbb{R}$ $A = [0,2] \cup [3,4]$, $B = [1,4]$, $C = [2,3]$, shade the corresponding set E_1 and E_2 on the plane $\mathbb{R}^2$:

   1) $E_1 = (B \setminus C) \times A$
   2) $E_2 = (A \cap C) \times B$

   Solution.

1) 

2) 

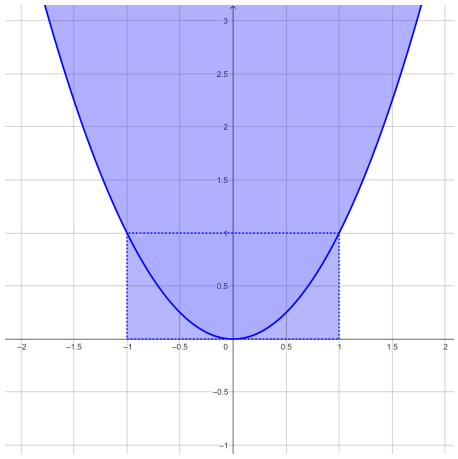
8. Prove that there are no sets $A, B \subset \mathbb{R}$ such that the disk $D = \{(x,y) \ : \ x,y \in \mathbb{R} \text{ and } x^2 + y^2 \leq 1\}$ equals to $A \times B$.

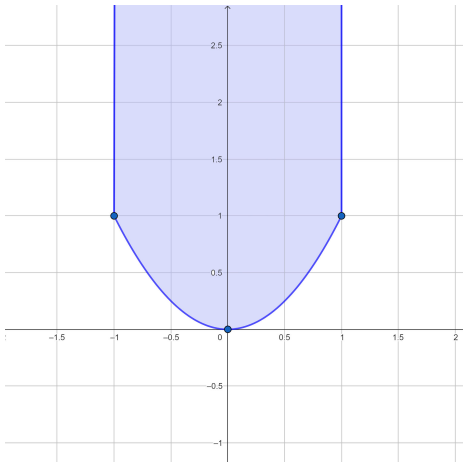
   Proof:. We have $(0,1) \in D$ and $(1,0) \in D$. If $D = A \times B$ than $(1,0) \in A \times B$ implies $1 \in A$ and $(0,1) \in A \times B \Rightarrow 1 \in B$. But then $(1,1) \in A \times B$ but not in D, contradiction.

9. Given an infinite family of sets $A_k = \{ \ (x,y) \in \mathbb{R}^2 \mid y \geq x^{2k} \ \}$ shade the sets

   1) $\bigcup_{k=1}^{\infty} A_k$ 2) $\bigcap_{k=1}^{\infty} A_k$ on the plane $\mathbb{R}^2$

   Solution.

1) 

2) 

10. Find the domain (the admissible set) of the functions $f : \mathbb{R} \to \mathbb{R}$ defined by the following formulas:

1) $f(x) = \frac{\sqrt{7-x}}{x^2-3x+2}$
2) $f(x) = \sqrt{\frac{x^2-4}{x+1}}$
3) $f(x) = \sqrt[3]{x} + \sqrt[4]{x+2} + \sqrt{3-x}$
4) $f(x) = \log_{7-x} \frac{x^2-1}{x-2}$

Solution.

1) Problems: denominator **and** square root
   Denominator: $x^2 - 3x + 2 \neq 0 \iff (x-2)(x-1) \neq 0 \iff x \neq 1$ and $x \neq 2$
   Square root: $7 - x \geq 0 \iff x \in (-\infty, 7]$
   Answer: $x \in (-\infty, 1) \cup (1, 2) \cup (2, 7]$
2) Problems: denominator **and** square root
   Denominator: $x + 1 \neq 0 \iff x \neq -1$
   Square root: $\frac{x^2-4}{x+1} \geq 0 \iff \frac{(x-2)(x+2)}{x+1} \geq 0 \iff$ use Wavy Curve Method
   WCM: no sign change on any interval not containing the dangerous point 1 nor the zero points $-2, 2$ in its interior. So decide for each of $(-\infty, -2)$ $(-2, 1), (1, 2)$ and $(2, \infty)$ individually. As each of the factors changes sign at a different point, admissible and non-admissible intervals should follow each other. The admissible ones will also contain $-2, 2$ but not 1! Indeed
   Wavy Curve Method $\implies x \in [-2, -1) \cup [2, +\infty)$
   Answer: $x \in [-2, -1) \cup [2, +\infty)$
3) Problems: square root **and** the root of order 4
   Square root: $3 - x \geq 0 \iff x \in (-\infty, 3]$
   Root of order 4: $x + 2 \geq 0 \iff x \in [-2, -\infty)$
   Answer: $x \in (-\infty, 3] \cap [-2, -\infty) = [-2, 3]$
4) Problems: argument of log **and** base of log **and** base of log again (not equal to 1)
   Argument of log: $\frac{x^2-1}{x-2} > 0 \iff \frac{(x-1)(x+1)}{x-2} > 0 \iff$ use Wavy Curve Method
   Wavy Curve Method $\implies x \in (-1, 1) \cup (2, +\infty)$
   Base of log: $7 - x > 0 \implies x \in (-\infty, 7)$
   Base of log is not equal to 1: $7 - x \neq 1 \implies x \neq 6$
   Answer: $x \in (-\infty, 7) \cap \Big((-1, 1) \cup (2, +\infty)\Big) \setminus \{6\} = (-1, 1) \cup (2, 6) \cup (6, 7)$

Answers:

1) $D(f) = (-\infty, 1) \cup (1, 2) \cup (2, 7]$

2) $D(f) = [-2, -1) \cup [2, +\infty)$

3) $D(f) = [-2, 3]$

4) $D(f) = (-1, 1) \cup (2, 6) \cup (6, 7)$