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The decay of the sphaleron

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Baryon number violation in the electroweak standard model is expected to proceed through classical transitions over the sphaleron barrier connecting vacua of different Chern–Simons number. The required energy is of the order of 10 TeV. The event structure is studied in this paper by following the real-time evolution of the sphaleron field configuration. On average, the sphaleron is found to decay into 42 W-bosons and 8 Higgs particles, if the Higgs mass is close to the W-boson mass.

1. Introduction

The sphaleron is a static solution of the bosonic sector of the electroweak standard model [1–3]. This solution is in fact unstable and describes a saddle point separating two vacuum sectors inequivalent under large gauge transformations. Transitions from one topological sector to a neighbouring one are accompanied by a change in baryon number [4],

$$\Delta B = \Delta L = n_g \, \Delta n_{CS}, \tag{1}$$

with n_g the number of generations, and Δn_{CS} the change of Chern–Simons number

$$n_{CS} = \frac{g^2}{16\pi^2} \int d^3x \, \epsilon^{ijk} \left(\partial_i W_j^a W_k^a + \frac{g}{3} \epsilon^{abc} W_i^a W_j^b W_k^c \right). \tag{2}$$

Vacuum tunneling transitions through this barrier are very much suppressed due to the smallness of the electroweak coupling constant. Therefore baryon number violation in the standard model was thought to be unobservable. Later it was realized, however, that $(B+L)$-violation can also occur in connection with classical transitions over the sphaleron barrier [3]. This may happen at high temperature [5], or in high energy collisions. This possibility of observing baryon number violation at future high-energy accelerators has stimulated a lot of discussion recently, originating from a paper by Ringwald [6]. His instanton estimates show a steep increase of electroweak cross sections at small energy, and seem to reach the s-wave unitarity bound at about the sphaleron energy (of the order of 10 TeV). Corrections to the leading approximation are large, however, in this energy range. It is therefore still unclear whether electroweak cross sections do become large eventually, although calculational tools have been improved substantially in the mean time [7–13].

Instanton estimates might not be very appropriate near the sphaleron energy because tunneling becomes irrelevant there. The dominant contribution is expected to arise from classical paths across the sphaleron. The basic uncertainty in estimating the cross section is due to the unknown overlap of those states with the given initial state. The event structure on the other hand can be estimated reliably by studying the real-time evolution of classical field configurations initially close to the sphaleron. Such an analysis is carried out in the present paper.

W-boson and Higgs multiplicities and average energies have been estimated before [3,14], working directly with the sphaleron field configuration. However, the sphaleron decays into some outgoing wave, which only after some time shows free-field behavior. Only after that time a particle interpretation becomes possible. This will lead to a reduced fraction of Higgs particles. It is still higher, however, than leading-order instanton estimates would predict.

2. Field equations

We study the SU(2) Yang–Mills Higgs theory given by the lagrangian

$$\mathscr{L} = -\tfrac{1}{2}\left(F_{\mu\nu}F^{\mu\nu}\right) + |D_\mu \Phi|^2 - \lambda\left(\Phi^\dagger \Phi - \tfrac{1}{2}v^2\right)^2, \tag{3}$$

where

$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig\left[W_\mu, W_\nu\right], \tag{4}$$

$$W_\mu = \tfrac{1}{2}\tau^a W_\mu^a, \tag{5}$$

$$D_\mu = \partial_\mu - igW_\mu. \tag{6}$$

The general spherical symmetric ansatz is [15,16]

$$W_0(\boldsymbol{r}, t) = \frac{1}{2g} G(r,t)(\boldsymbol{n\tau}), \tag{7}$$

$$W(\boldsymbol{r}, t)=\frac{1}{2g}\left[\frac{1-f_{\mathrm{A}}(r, t)}{r}(\boldsymbol{n}\times\boldsymbol{\tau})+\frac{f_{\mathrm{B}}(r, t)}{r}(\boldsymbol{\tau}-(\boldsymbol{n\tau})\boldsymbol{n})+f_{\mathrm{C}}(r, t)(\boldsymbol{n\tau})\boldsymbol{n}\right], \tag{8}$$

$$\Phi(\boldsymbol{r}, t)=\frac{v}{\sqrt{2}}\left[H(r, t)+iK(r, t)(\boldsymbol{n\tau})\right]\begin{pmatrix}0\\1\end{pmatrix}, \tag{9}$$

with $\boldsymbol{n}=\boldsymbol{r}/r$. The corresponding equations of motion in the temporal gauge $W_0=0$ are

$$\ddot{f}_{\mathrm{A}}-f_{\mathrm{A}}''+\frac{1}{r^2}(f_{\mathrm{A}}^2+f_{\mathrm{B}}^2-1)f_{\mathrm{A}}+M_{\mathrm{W}}^2\left[(H^2+K^2)f_{\mathrm{A}}+K^2-H^2\right]$$
$$+f_{\mathrm{A}}f_{\mathrm{C}}^2-2f_{\mathrm{B}}'f_{\mathrm{C}}-f_{\mathrm{B}}f_{\mathrm{C}}'=0, \tag{10}$$

$$\ddot{f}_{\mathrm{B}}-f_{\mathrm{B}}''+\frac{1}{r^2}(f_{\mathrm{A}}^2+f_{\mathrm{B}}^2-1)f_{\mathrm{B}}+M_{\mathrm{W}}^2\left[(H^2+K^2)f_{\mathrm{B}}-2HK\right]$$
$$+f_{\mathrm{B}}f_{\mathrm{C}}^2+2f_{\mathrm{A}}'f_{\mathrm{C}}+f_{\mathrm{A}}f_{\mathrm{C}}'=0, \tag{11}$$

$$\ddot{f}_{\mathrm{C}}+\frac{2}{r^2}(f_{\mathrm{A}}^2+f_{\mathrm{B}}^2)f_{\mathrm{C}}+M_{\mathrm{W}}^2(H^2+K^2)f_{\mathrm{C}}+2M_{\mathrm{W}}^2(H'K-HK')$$
$$+\frac{2}{r^2}(f_{\mathrm{A}}'f_{\mathrm{B}}-f_{\mathrm{A}}f_{\mathrm{B}}')=0, \tag{12}$$

$$\ddot{H}-\frac{1}{r}(rH)''+\frac{1}{2r^2}(f_{\mathrm{A}}^2+f_{\mathrm{B}}^2+1)H-\frac{1}{r^2}(Hf_{\mathrm{A}}+Kf_{\mathrm{B}})+\tfrac{1}{2}M_{\mathrm{H}}^2(H^2+K^2-1)H$$
$$-\frac{1}{r}Kf_{\mathrm{C}}+\tfrac{1}{4}Hf_{\mathrm{C}}^2-f_{\mathrm{C}}K'-\tfrac{1}{2}Kf_{\mathrm{C}}'=0, \tag{13}$$

$$\ddot{K}-\frac{1}{r}(rK)''+\frac{1}{2r^2}(f_{\mathrm{A}}^2+f_{\mathrm{B}}^2+1)K+\frac{1}{r^2}(Kf_{\mathrm{A}}-Hf_{\mathrm{B}})+\tfrac{1}{2}M_{\mathrm{H}}^2(H^2+K^2-1)K$$
$$+\frac{1}{r}Hf_{\mathrm{C}}+\tfrac{1}{4}Kf_{\mathrm{C}}^2+f_{\mathrm{C}}H'+\tfrac{1}{2}Hf_{\mathrm{C}}'=0, \tag{14}$$

with $M_{\mathrm{W}}=\frac{1}{2}gv$, $M_{\mathrm{H}}=\sqrt{2\lambda}\,v$ the masses of the gauge bosons and the Higgs particle.

There exists [17] one unstable mode in the spectrum of small fluctuations around the static sphaleron solution of eqs. (10)–(14). We will excite this mode to initiate the decay of the sphaleron [18]. For an analysis of the final state we use the residual U(1) gauge symmetry [15]

$$\begin{pmatrix} G \\ f_C \end{pmatrix} \to \begin{pmatrix} G \\ f_C \end{pmatrix} + \begin{pmatrix} \dot{\Theta}(r,t) \\ \Theta'(r,t) \end{pmatrix}, \tag{15}$$

$$\begin{pmatrix} f_A \\ f_B \end{pmatrix} \to \begin{pmatrix} \cos\Theta(r,t) & -\sin\Theta(r,t) \\ \sin\Theta(r,t) & \cos\Theta(r,t) \end{pmatrix} \begin{pmatrix} f_A \\ f_B \end{pmatrix}, \tag{16}$$

$$\begin{pmatrix} H \\ K \end{pmatrix} \to \begin{pmatrix} \cos\frac{1}{2}\Theta(r,t) & -\sin\frac{1}{2}\Theta(r,t) \\ \sin\frac{1}{2}\Theta(r,t) & \cos\frac{1}{2}\Theta(r,t) \end{pmatrix} \begin{pmatrix} H \\ K \end{pmatrix}, \tag{17}$$

of the ansatz (7)–(9) to go to the unitary gauge $K(r, t) = 0$ introducing a nonzero $W_0(r, t)$. In this gauge the equations for small fluctuations around the vacuum are

$$\ddot{f}_A - f_A'' + M_W^2(f_A - 1) + \frac{2}{r^2}(f_A - 1) = 0, \tag{18}$$

$$\ddot{f}_B - f_B'' + M_W^2 f_B + \dot{f}_C' - \dot{G} = 0, \tag{19}$$

$$\ddot{H} - \frac{1}{r}(rH)'' + M_H^2(H - 1) = 0, \tag{20}$$

$$\ddot{f}_C + M_W^2 f_C + \frac{2}{r^2} f_C - \frac{2}{r^2}\dot{f}_B' - \dot{G}' = 0, \tag{21}$$

$$\frac{1}{r}(rG)'' - \dot{f}_C' - \frac{2}{r}\dot{f}_C + \frac{2}{r^2}\dot{f}_B - \frac{2}{r^2}G - M_W^2 G = 0. \tag{22}$$

The condition $\partial_\mu W_\mu = 0$ provides

$$\frac{2}{r^2} f_B - \frac{2}{r} f_C + \dot{G} - f_C' = 0 \tag{23}$$

which allows us to remove the mixed time derivatives and to decouple the equation of the G-field. The solutions of eqs. (18)–(23) are

$$f_A(r,t) = 1 + a_k r j_1(kr)\cos(\omega t + \alpha_0), \tag{24}$$

$$\begin{pmatrix} f_B \\ f_C \end{pmatrix} = b_k \begin{pmatrix} r j_0(kr) \\ j_0(kr) \end{pmatrix} \cos(\omega t) + c_k \begin{pmatrix} r j_2(kr) \\ -2 j_2(kr) \end{pmatrix} \cos(\omega t), \tag{25}$$

$$G = (-b_k - 2c_k)\frac{k}{\omega} j_1(kr)\sin(\omega t), \tag{26}$$

$$H = 1 + h_k j_0(kr)\cos(\Omega t + \beta_0), \tag{27}$$

with $\omega^2 = k^2 + M_W^2$, $\Omega^2 = k^2 + M_H^2$ and the spherical Bessel functions

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x). \tag{28}$$

The energy of these small fluctuations is given by

$$E = \int E(k)\,\mathrm{d}k$$

$$= \frac{2\pi^2}{g^2}\int\left(\frac{\omega^2}{k^2}a_k^2 + (b_k - c_k)^2 + \frac{3M_W^2}{2k^2}\left(b_k^2 + 2c_k^2\right) + 2M_W^2\frac{\Omega^2}{k^2}h_k^2\right)\mathrm{d}k. \tag{29}$$

3. Numerical results

To investigate the fate of a sphaleron we start with the sphaleron solution slightly shifted into the direction of the unstable model, $(\delta f_B, \delta f_C, \delta H)$ normalized to $\int_0^\infty \mathrm{d}r[(\delta f_B)^2 + 2r^2(\delta f_C)^2 + (M_W^2/2r^2)(\delta H)^2] = 0.01 M_W$ with vanishing momenta $\dot{f}_A$, $\dot{f}_B$, $\dot{f}_C$, $\dot{H}$, $\dot{K}$. We follow the time evolution using an explicit discretization of the equations of motion on a lattice with 4000 points from $r = 0$ to $r = 40\ M_W^{-1}$ in time steps of $\Delta t = 0.0025\ M_W^{-1}$.

We choose coupling constants such that $M_W = M_H = 81.5$ GeV, and $v = 244$ GeV.

Fig. 1 shows how energy spreads out in space with increasing time. The energy of free-field fluctuations (29) approaches a value of 94.2 percent of the sphaleron energy. The remaining energy corresponds to small interaction terms.

The Chern–Simons number of this radiation field (fig. 2) approaches an asymptotic value of about 0.09. This should not come as a surprise. The Chern–Simons number is integer-valued only for vacuum configurations. The relation (1) will not remain valid in the presence of a radiation field which in general has a non-zero and non-integer Chern–Simons number (see e.g. Christ [19]). The particular asymptotic value 0.09 has presumably no deeper meaning. It is close enough to zero, however, and therefore we can conclude that the Chern–Simons number remains a useful tool in detecting, in a numerical simulation, baryon number violation without studying fermions directly. This might no longer be true at still higher energies (or corresponding temperatures).

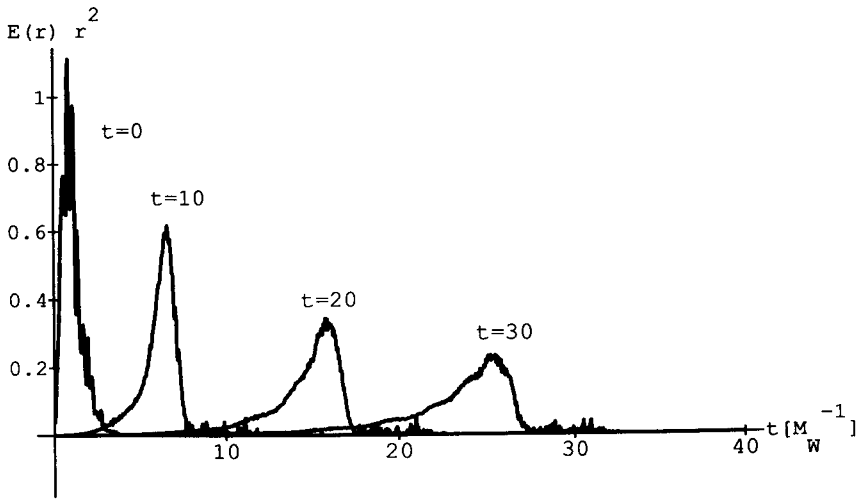

Fig. 1. Radial energy distribution (in units of M_W) of the decaying sphaleron at different times (in units of M_W^{-1}).

It takes a time of order 4.5 M_W^{-1} until the sphaleron radiation shows free-field behavior. In order to demonstrate this, we gauge transform the solution of the field equations into unitary gauge $K(r, t) = 0$, Fourier transform in r according to eqs. (24)–(27) and check the free-field time dependence of the Fourier coefficients a_k, b_k, c_k and h_k.

Fig. 3 shows this time development of the $k = 1 M_W$ component of the Fourier transformed Higgs field h_k fitted with the harmonic time dependence of a free field. It nicely satisfies the dispersion relation $\omega^2 = k^2 + M_H^2$.

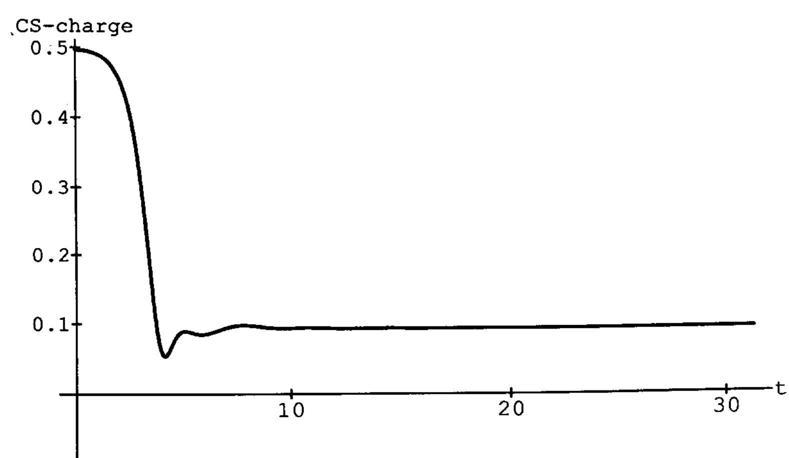

Fig. 2. Time evolution of the Chern–Simons charge.

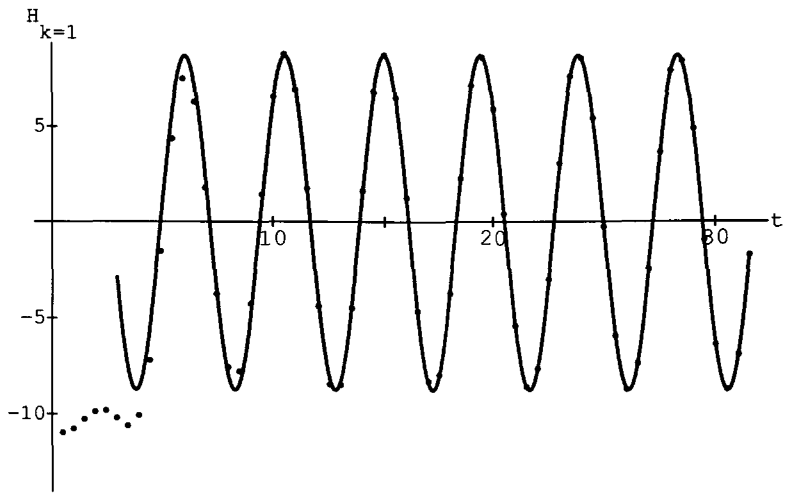

Fig. 3. Time evolution of he Fourier component of momentum $k = 1\ M_W$ of the Higgs field fitted to free-field behaviour.

The Fourier coefficients a_k, b_k, c_k and h_k provide us with the energy distribution of the outgoing wave (eq. (29)) which we interpret according to $E(k) = N(k)\omega(k)$ as multiplicity distribution of the Higgs particles and gauge bosons plotted in fig. 4. We find an average multiplicity of 42.3 W-bosons and 7.6 Higgs particles. The Higgs bosons carry about 11.5% of the available energy. The

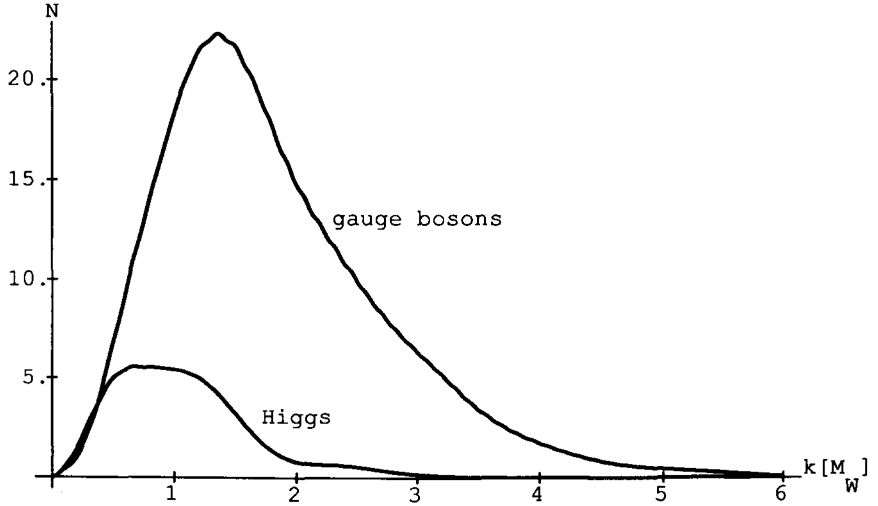

Fig. 4. Multiplicity distribution $4\pi k^2 N(k)$ of the final-state particles over the momentum k in units of M_W.

momenta are of the order of the W-mass, neither non-relativistic nor extremely relativistic.

4. Discussion

We have obtained quite precise results for the average multiplicities of W-bosons and Higgs bosons in sphaleron decay. Studying the time evolution until free-field behavior becomes manifest we find a smaller Higgs content than previously obtained [3,14] working directly on the sphaleron. The obtained ratio is just what one might have expected counting degrees of freedom: one Higgs boson compared to nine isospin and polarization states of W-bosons. Our results show however no indication of a suppression of Higgs bosons, as would be expected from the leading-order instanton estimates. This may be another indication that the instanton approximation breaks down near the sphaleron energy.

Results are expected to depend on the Higgs mass. We have only studied the case $M_H = M_W$.

We have not discussed the problem of quantization. The underlying assumption is that classical effects are dominant as long as the fields are strong. Later on quantization is of course required for a particle interpretation, but even in this regime classical evolution will correctly describe the time dependence of expectation values, because we are in a weak coupling regime.

More detailed assumptions on the quantum state would be required if we were to determine individual S-matrix elements for the decaying state which would then allow us to compute the decay width, as well as more detailed distributions. Since we start with a well-defined state, i.e. a sphaleron with no fluctuating modes excited, a coherent-state description (as opposed to a mixed state) should be adequate. We have not followed this approach any further, however.

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