MULTI-LOOP FREE ENERGY OF THE HETEROTIC STRING NEAR THE HAGEDORN TEMPERATURE

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For the heterotic string we compute the critical exponent of the multi-loop free energy F near the Hagedorn temperature T_H. With d uncompactified dimensions we find in particular the two-loop critical behaviour $F^{(2)} \sim (T_H - T)^{(d-2)/2}$, in contrast to the one-loop (free string) result $F^{(1)} \sim (T_H - T)^{d/2}$. For $d=3$, we correspondingly obtain a finite pressure but a diverging two-loop energy density at the Hagedorn temperature which would appear as a limiting temperature in this case. Cosmological consequences are studied.

The behaviour of string theories is expected to differ significantly from that of point particle theories at short distances. There are essentially two regimes where this would in principle be observable. One is in high energy scattering, where it has been shown [1] that wide angle string scattering actually does not test very short distances, no matter how large the energy is. The other relevant case is that of high temperature which has attracted much interest recently (see refs. [2–11], also for a list of further references).

The standard expectation based on the behaviour of the one-loop free energy (i.e. a gas of free strings) is a phase transition at (or below [5]) the Hagedorn temperature T_H. The canonical ensemble does not exist for $T > T_H$ but the one-loop energy density is finite as $T \to T_H$, and the Hagedorn temperature is therefore not interpreted as a limiting temperature. It should be very interesting to check whether this conclusion remains valid when higher order terms in string perturbation theory are taken into account. Some attempts [6,7] have been made to study two-loop contributions, but this question remained unanswered.

As we discuss below we actually find a multi-loop contribution to the energy density diverging like $(T_H - T)^{-1/2}$ for the relevant case of three uncompactified spatial dimensions. This implies a drastic change in opinion because now the Hagedorn temperature would have to be interpreted as limiting temperature as it was originally conceived [12].

The backreaction of the interacting string gas on the graviton $g_{\mu\nu}$ and the dilaton ϕ background field is described, to first order in sigma model perturbation theory, by the equations of motion

$$\beta^g_{\mu\nu} = R_{\mu\nu} + 2\nabla_\mu \nabla_\phi \phi = \frac{e^{2\phi}}{2\pi\beta V d} \sum_{g \geq 1} \langle \partial X_\mu \bar{\partial} X_\nu \rangle_g ,$$

$$\beta^\phi = R - 4(\nabla\phi)^2 + 2\nabla^2\phi = \frac{e^{2\phi}}{8\pi\beta V d} \sum_{g \geq 1} \langle R^{(2)} \rangle_g , \quad (1)$$

with $R^{(2)}$ being the genus-g world-sheet curvature. The antisymmetric tensor (torsion) field may be set to zero consistently.

Eq. (1) follows from the requirement of cancellation of short-distance singularities against additional singularities from modular integrations on higher genus world-sheets (Fischler–Susskind mechanism [13]), which may be rephrased as a cancellation of BRST anomalies from tadpoles and background fields [14]. For the bosonic string this has been discussed in detail by Polchinski [15], the heterotic string is discussed in refs. [16–18].

The thermal expectation values of the tadpoles appearing on the right hand side of eq. (1) may be evaluated in a euclidean path integral formulation considering the string moving on $S^1 \times R^d \times M_{compact}$ [2]. The inverse temperature β is given by the period of compactified euclidean time. This compactification leads to the appearance of new states (winding states), which are characterized at genus g by 2 g-vectors $\boldsymbol{n}$, $\boldsymbol{m}\in\mathbb{Z}^g$ representing the winding numbers around the $2g$ non-trivial cycles of the surface [3,4]. The world-sheet partition function is given by

$$Z_g = Z_g^{\rm cl} Z_g^{\rm qu} \tag{2}$$

with

$$Z_g^{\rm cl} = V_d \beta \exp\left(\frac{-\beta^2}{2\pi} (\tau\boldsymbol{n} - \boldsymbol{m}) \frac{1}{\operatorname{Im}\tau} (\bar{\tau}\boldsymbol{n} - \boldsymbol{m}) \right) \tag{3}$$

and $Z_g^{\rm qu}$ describes the quantum fluctuation around the corresponding classical solution. τ is the $g\times g$ period matrix.

Since the first-quantized formalism generates only connected diagrams, the thermal free energy F_g is essentially given by Z_g and not $\log Z_g$ [2],

$$F_g = \beta^{-1} Z_g . \tag{4}$$

The tadpoles of eq. (1) can also be expressed in terms of Z_g,

$$\langle R^{(2)} \rangle_g = 4\pi(2-2g) \exp[(2g-2)\phi]\, Z_g ,$$

$$\langle \partial X_i \bar{\partial} X_j \rangle_g = -\pi g \exp[(2g-2)\phi]\, Z_g ,$$

$$\langle \partial X_0 \bar{\partial} X_0 \rangle_g = \langle \partial X_0^{\rm qu} \bar{\partial} X_0^{\rm qu} \rangle + \langle \partial X_0^{\rm cl} \bar{\partial} X_0^{\rm cl} \rangle$$

$$= \left(-\pi g Z_g - \pi \beta^2 \frac{\partial F}{\partial \beta} \right) \exp[(2g-2)\phi] . \tag{5}$$

Thus we obtain the background field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + 2\nabla_\mu \nabla_\nu \phi + 2 g_{\mu\nu} [(\nabla\phi)^2 - \nabla^2\phi]$$

$$= \sum_g \tfrac{1}{2} \exp(2g\phi)\, t_{\mu\nu}^{(g)} ,$$

$$R - 4(\nabla\phi)^2 + 4\nabla^2\phi = -\sum_g (g-1) \exp(2g\phi) f_g \tag{6}$$

with $t_{\mu\nu}^{(g)} = \mathrm{diag}(\rho^{(g)}, p^{(g)}, ..., p^{(g)})$ the energy-momentum tensor of a perfect fluid,

$$f_g = \frac{1}{V_d} F_g , \quad \rho^{(g)} = f_g + \beta \frac{\partial f_g}{\partial \beta} , \quad p^{(g)} = -f_g . \tag{7}$$

Eqs. (6) are consistent with the effective action

$$S_{\rm eff} = \int {\rm d}^{d+1}x \sqrt{-g} \Big(\exp(-2\phi)\, [R + 4(\nabla\phi)^2]$$

$$- \sum_g \exp[(2g-2)\phi]\, f_g(\sqrt{-g_{00}}\,\beta) \Big) . \tag{8}$$

The crucial next step is the evaluation of the free energy density f_g at higher genus. Techniques for calculating higher-loop fermionic string amplitudes have made significant progress recently [19,20]. It is known how to express all necessary determinants and Green functions on a higher-genus Riemann surface by theta functions, the prime form $E(z, \omega)$ and the holomorphic differential $\sigma(z)$.

Possible singularities of the free energy near the Hagedorn temperature would have to arise from divergences of the modular integral, i.e. they are determined by the behaviour of the integrand near the boundary of the moduli space. Following ref. [20] we denote a boundary as Δ_0 or Δ_1 resp., depending on whether a nontrivial or trivial (dividing) homology cycle shrinks to zero. We consider the case $g=2$ as an example. Δ_1 corresponds to the limit $\tau\to\mathrm{diag}(\tau_1, \tau_2)$ representing two tori with moduli τ_1, τ_2 joined at a node. This limit provides a diverging contribution to the free energy (at any β) which is however cancelled by a one-loop contribution with a background field insertion. The remaining finite contribution behaves essentially like the one-loop contribution squared, i.e. leads to finite energy density and pressure for $\beta\to\beta_{\rm H}$, $d=3$.

The situation is different for the contribution coming from Δ_0. If e.g. the non-dividing homology cycle a_1 shrinks to zero one obtains a torus with modular parameter τ_2 and two marked points p_1, p_2 joined by a long thin handle. The period matrix τ behaves like

$$\tau_{11} \to {\rm i}\infty , \quad \tau_{12} \to \int_{p_1}^{p_2} {\rm d}z , \quad \tau_{22} \to \tau_2 .$$

In this limit, the behaviour of theta functions, prime form and other ingredients is well known [21]. The contribution of internal dimensions can, at least for lattice compactifications, be expressed by generalized lattice theta functions which show a regular behaviour (no poles and no zeroes) in the degeneration limit. Therefore the critical exponents will not depend on the compactification scheme, as discussed at one loop in refs. [8,9].

The most singular behaviour, and indeed the only one which leads to a divergence above $T_{\rm H}$, appears in the Neveu–Schwarz (NS) sector of the long handle. For zero winding number it corresponds to the propagation of a spurious NS tachyon along the handle which is cancelled by summation over the spin structures on the corresponding cycle. For non-zero winding numbers this cancellation no longer takes place because the mixing of the sum over winding numbers with the sum over spin structures leads to different GSO-like projections required for modular invariance [3,5]. The contribution of this corner of moduli space to the partition function Z_g is given by

$$Z_{g=2} \sim \int_0 \frac{\mathrm{d}|t|}{(\log|t|)^{d/2}} \, |t|^{-5/2+\beta^2/4\pi^2+\pi^2/4\beta^2} . \qquad (9)$$

Details of the derivation will be given elsewhere [18]. From eq. (9) we first of all find the inverse Hagedorn temperature

$$\beta_{\mathrm{H}} = \pi(1+\sqrt{2}) , \qquad (10)$$

which is the same as the value derived from the one-loop calculation. Approaching β_{H} we obtain

$$\begin{aligned} f_{g=2} &\sim (\beta-\beta_{\mathrm{H}})^{(d-2)/2} \quad d<2 , \\ &\sim \log(\beta-\beta_{\mathrm{H}}) \quad d=2 , \\ &\sim \text{finite} \quad d>2 . \end{aligned} \qquad (11)$$

For $d=3$ we obtain finite free energy density and pressure; the energy density, however, diverges for $\beta \to \beta_{\mathrm{H}}$,

$$\rho^{(2)} \sim \frac{1}{\sqrt{\beta-\beta_{\mathrm{H}}}} , \quad d=3 , \qquad (12)$$

as can be seen from the relation (9). This is the main of this paper. It should be contrasted with the one-loop result where for $d=3$ the energy density is finite as $\beta \to \beta_{\mathrm{H}}$. For higher genus the degree of divergence will not increase further as long as the two-loop energy is finite, i.e. $d>2$.

We now turn to a discussion of some of the cosmological consequences of the presence of a dilaton and the divergence of the energy density at the Hagedorn temperature. We look for homogeneous Robertson–Walker solutions $g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2)$ coupled to a homogeneous dilaton $\phi(t)$ in the presence of the incoherent matter contribution characterized by

$$f = \sum_g (g-1) \exp(2g\phi) f_g , \qquad (13)$$

$$\rho = \sum_g \exp(2g\phi)\, \rho^{(g)} ,$$

$$p = \sum_g \exp(2g\phi)\, p^{(g)} . \qquad \text{(13 cont'd)}$$

Introducing the notation $K=\dot{a}/a$, $\sigma=\dot{\phi}$ we obtain from eq. (6) the evolution equations

$$\begin{aligned} \dot{K} &= -3K^2 + 2\sigma K + \tfrac{1}{2}(p-f) , \\ \dot{\sigma} &= \sigma^2 - 3/2K^2 + \tfrac{1}{4}(3p-2f) \end{aligned} \qquad (14)$$

together with the constraint

$$3K^2 - 6\sigma K + \sigma^2 = \tfrac{1}{2}\rho . \qquad (15)$$

Since the free energy density f vanishes to one-loop order in string perturbation theory, and is non-singular near the Hagedorn temperature, it will be neglected in the following. The equation of state could be parametrized by

$$p \simeq \alpha\rho \qquad (16)$$

with $\alpha \to \frac{1}{3}$ at low temperatures. Near the Hagedorn temperature one finds $\alpha \to \bar{\alpha} < \frac{1}{3}$, where to one-loop order $\bar{\alpha}$ is some small non-vanishing number which depends on the compactification scheme. The two-loop singularity of ρ implies

$$\alpha \to 0 \quad \text{for } T \to T_{\mathrm{H}} , \qquad (17)$$

instead of the non-vanishing one-loop limit $\bar{\alpha}$. Near the singularity it is actually more convenient to use

$$p = \tilde{p}(T) \exp(2\phi) + \mathrm{O}(\exp(4\phi)) , \qquad (18)$$

where $\tilde{p}(T)$ goes to a finite limit $\bar{p}$ for $T \to T_{\mathrm{H}}$. $\bar{p}$ again depends on the compactification scheme.

The first observation concerning the solution space of eqs. (14), (15) is that it contains in a natural way the standard radiation-dominated universe, with the dilaton field approaching a constant. For small σ and $\alpha = \frac{1}{3}$ we obtain

$$\dot{K} \simeq -2K^2 + \mathrm{O}(\sigma^2) , \quad \dot{\sigma} \simeq -3\sigma K + \mathrm{O}(\sigma^2) , \qquad (19)$$

leading to

$$a(t) \simeq t^{1/2} , \quad \phi(t) \simeq \text{const.} + \mathrm{O}(1/\sqrt{t}) , \qquad (20)$$

consistent with the assumption of σ being small, $\mathrm{O}(1/t^{3/2})$. The other interesting problem is the behaviour near the Hagedorn temperature, i.e. close to an initial or final singularity. It is not difficult to solve eq. (14) in this limit [using eq. (18)]. Considering e.g. a collapsing universe we find to leading order

$$K \simeq -\frac{1}{\sqrt{3}}\frac{1}{t_f - t}, \quad \sigma \simeq -\frac{1}{1+\sqrt{3}}\frac{1}{t_f - t}, \tag{21}$$

i.e.

$$a(t) \simeq (t_f - t)^{1/\sqrt{3}}, \quad \exp(2\phi) \simeq (t_f - t)^{\sqrt{3}-1}. \tag{22}$$

Thus the universe is driven to zero string coupling at the singularity. The energy diverges at the singularity like

$$\rho \simeq c\frac{1}{t_f - t}, \tag{23}$$

where c is an arbitrary constant of integration. T approaches T_H at the singularity like

$$T_H - T \simeq (t_f - t)^{4\sqrt{3}-2}. \tag{24}$$

Of course, there remains the possibility of having a phase transition occurring *below* T_H [5], i.e. in the present scenario before reaching the singularity.

There is one problem that occurs in any case before the Hagedorn temperature is reached, and that is the breakdown of sigma model perturbation theory, simply because the curvature diverges at the singularity. This makes it even more urgent to study string propagation directly on curved manifolds.

To summarize, we have found the multi-loop contribution to the $d=3$ heterotic string energy density to diverge at the Hagedorn temperature, in contrast to the one-loop contribution which is finite at T_H. This changes cosmology at high temperature significantly. For a collapsing universe the Hagedorn temperature is reached only at the singularity. Curvature fluctuations are now expected to be dramatic near the Hagedorn temperature, in addition to thermodynamic fluctuations which are better described in a microcanonical formulation [9-11,22].

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