Homework (7. Series)

Deadline: 23.05.2019

25. Let \( X, Y \) be normed vector spaces and \( \mathcal{L}(X,Y) \) be the vector space of all linear continuous mappings (operators) \( A : X \to Y \). Show:

a) A linear operator \( A \) is continuous if and only if \( A \) is bounded, i.e., if there is a nonnegative constant \( c \) such that

\[
\|A(u)\| := \|A(u)\|_Y \leq c\|u\|_X =: c\|u\| \quad \text{for all} \quad u \in X.
\]

(Hint: Prove the necessity of the boundness of \( A \) by contradiction)

b) If \( A \in \mathcal{L}(X,Y) \), there holds:

\[
\sup_{u \neq 0} \frac{\|A(u)\|}{\|u\|} = \sup_{\|u\|=1} \|A(u)\| = \sup_{\|u\| \leq 1} \|A(u)\| < \infty.
\]

c) If \( A \in \mathcal{L}(X,Y) \), a norm is defined by

\[
\|A\| := \sup_{u \neq 0} \frac{\|A(u)\|}{\|u\|}.
\]

26. Use Problem 25 to analyze whether or not the following operators \( A : X \to Y \) are elements from \( \mathcal{L}(X,Y) \) with the given vector spaces \( X, Y \). If so determine \( \|A\|_\cdot \).

a) \( A(x) = \left( \sum_{n=1}^{\infty} \frac{x_n}{n}, 0, 0, \ldots \right) \), \( X = Y = l^2 \).

b) \( A(f) = \frac{df}{dx} \), \( X = C^1([0,1]), Y = C([0,1]) \), both spaces equipped with the maximum norm on \([0,1] \).

c) \( A(f) = \int_0^\pi e^{ix} \left( \int_0^x f(t) \, dt \right) \, dx \), where \( X \) is the closure of the complex vector space \( C([0,\pi],\mathbb{C}) \) with respect of the integral norm, \( Y = \mathbb{C} \).

27. Analyze whether or not the following mappings \( L \) are linear continuous functionals on the Hilbert space \( l^2 \). If so determine the associated vector due to the Riesz–Fréchet representation theorem and \( \|L\| \).

a) \( L(x) = -4 x_6 + 2.6 x_{123} - \pi x_{290} + x_{5555} \),  

b) \( L(x) = \sum_{n=1}^{\infty} (-1)^n x_n \),  
c) \( L(x) = \sum_{n=1}^{\infty} \frac{x_n}{n} \).

28. a) Let \( X, Y \) be normed vector spaces and \( \mathcal{L}(X,Y) \) be the space defined in Problem 25. Prove that \( A \in \mathcal{L}(X,Y), B \in \mathcal{L}(Y,Z) \) imply

\[
BA := B \circ A \in \mathcal{L}(X,Z) \quad \text{with} \quad \|BA\| \leq \|A\| \|B\|.
\]

Find an example for \( \|BA\| < \|A\| \|B\| \).

b) Let \( H \) be a complex Hilbert space with the scalar product \( \langle \cdot, \cdot \rangle \). For fixed elements \( u, v \in H \) we set \( (u \otimes v)(w) := \langle w, u \rangle v \). Test the operator \( u \otimes v \) for linearity and boundness. If so compute its norm.