Homework (5. Series)

Deadline: 09.05.2019

17. We define \( H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) \) for \( x \in \mathbb{R}, n \in \mathbb{N}_0 \). These functions are called the Hermite polynomials and play an important role in the analysis of the harmonic oscillator in Quantum Mechanics. Show:

a) \( H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \) for \( n \in \mathbb{N} \).

b) \( H_n(x) \) is a polynomial of degree \( n \).

c) \( H'_n(x) = 2nH_{n-1}(x) \) for \( n \in \mathbb{N} \).

d) \( H_n \) is a solution \( y = y(x) \) of Hermite’s differential equation
\[
y'' - 2xy' + 2ny = 0.
\]

(Hint: Use Leibniz’s product rule \( \frac{d^n}{dx^n} (fg) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} f \frac{d^{n-k}}{dx^{n-k}} g \).)

18. Let \( \mathcal{P} \) be the vector space of all real polynomials in \( x \).

a) Check that \( \langle f, g \rangle := \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2} \, dx \) für \( f, g \in \mathcal{P} \) defines a scalar product.

(weighted scalar product on \( (-\infty, \infty) \) with the weight function \( e^{-x^2} \))

b) Apply the Gram–Schmidt process of orthonormalisation to the polynomials \( 1, x, x^2 \) where this weighted scalar product is to be used. Compare the results with the Hermite polynomials \( H_0, H_1, H_2 \) from Problem 17.

19. Let \( l^p := \{ x = (x_1, x_2, x_3, \ldots) \mid x_n \in \mathbb{C} \text{ for } n \in \mathbb{N}, \sum_{n=1}^{\infty} |x_n|^p < \infty \} \).

a) For which \( p > 0 \) does the inclusion \( l^p \subseteq l^2 \) holds true? (Proof or counter example)

b) Determine the closure of \( l^p \) with respect to the norm of the Hilbert space \( l^2 \) for the detected values of \( p \) from part a).

c) Show that \( U := \{ x \in l^2 \mid x_{2k-1} = 0 \text{ for all } k \in \mathbb{N} \} \) is a closed linear subspace of \( l^p \).

D) Compute \( U^\perp \) and the orthogonal projection of \( y = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots) \) onto \( U \).

20. Deduce the following properties of the orthogonal complement of a subset \( M \) of the Hilbert space \( H \) from its definition:

a) \( M \subseteq M^\perp := (M^\perp)^\perp \),

b) \( M_1 \subseteq M_2 \Rightarrow M_1^\perp \supseteq M_2^\perp \),

c) \( M^\perp = \bar{M}^\perp \).