9. a) Proof the following assertion about the sign of the eigenvalue parameter \( \lambda \):

There exist a nontrivial solution \( v = v(x) \) of the boundary value problem

\[
v'' + \lambda v = 0 \quad \text{in} \quad (0, \pi) \subset \mathbb{R} ; \quad v(0) = 0 , \ v(\pi) = 0 ,
\]

only if \( \lambda > 0 \).

b) Let \( f \in C^{2k}([0, \pi]), \ k \in \mathbb{N} \) be a function with \( f^{(l)}(0) = f^{(l)}(\pi) = 0 \) for \( l = 0, \ldots, 2k-2 \).

Show that its Fourier coefficients

\[
A_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx) \, dx
\]

satisfy the estimate \( |A_n| \leq \frac{C}{n^{2k}} \)

with a constant \( C \in \mathbb{R}, \ (n \in \mathbb{N}) \).

(Hint: Integration by parts)

10. Use the Fourier method (first ansatz by separation and then ansatz by series) to solve the problem of heat conduction in a finite, quite thin rod with constant temperature at both ends, i.e. solve the initial-boundary value problem

\[
\begin{align*}
u_t &= a^2 u_{xx} \quad \text{in} \quad (0, \pi) \times (0, \infty) ; \\
u(x,0) &= f(x) \quad \text{for} \quad x \in [0, \pi] ; \quad u(0,t) = u(\pi,t) = 0 \quad \text{for} \quad t > 0 .
\end{align*}
\]

11. a) Determine all solutions \( u \) of the (spatially) threedimensional wave equation such that they are only depending on a given direction \( e \) in space (plane waves), i.e. all \( u = u(x,t) = \varphi(e \cdot x, t) \) with \( u_{tt} = a^2 \Delta u \) and \( e \in \mathbb{R}^3, \|e\| = 1 \).

b) Determine all solutions \( u \) of the (spatially) threedimensional wave equation such that they are radially symmetric in space (spherical waves), i.e. all \( u = u(x,t) = \psi(r, t) \) with \( u_{tt} = a^2 \Delta u \) and \( r = \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2} > 0 \).

(Hint: The functions \( \varphi(s,t) \) and \( r\psi(r,t) \) satisfy a (spatially) onedimensional wave equation in the variables \( s,t \) and \( r,t \), respectively.)

12. Solve the initial value problem

\[
\begin{align*}
u_{tt} &= 4\Delta u \quad \text{in} \quad \mathbb{R}^4 ; \\
u(x,0) &= \sin x_2 , \ u_t(x,0) = x_1 - 2x_2 + 2x_3 \quad \text{for} \quad x \in \mathbb{R}^3 .
\end{align*}
\]