Homework (2. Series)

Deadline: 18.04.2019

5. a) Let \( Q_1 := \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\} \) be the first quadrant of the \( x \)-\( y \)-plane. Use the function \( f(x, y) = xy \) to construct infinitely many solutions of the Dirichlet problem:

\[ \Delta u = 0 \quad \text{in} \quad Q_1, \quad u = 0 \quad \text{on} \quad \partial Q_1. \]

Is this contradictory to the uniqueness theorem for the Dirichlet problem?

b) Use an ansatz for \( u \) as a polynomial in \( x \) and \( y \) to solve the following Dirichlet problem for the unit circle:

\[ \Delta u = 0 \quad \text{in} \quad U_1(0), \quad u = x^2 \quad \text{on} \quad \partial U_1(0). \]

6. a) Show that, for fixed \( y \in \mathbb{R}^N \), the kernel \( K(x, y, t) := (4a^2\pi t)^{-\frac{N}{2}} \exp \left( -\frac{\|x - y\|^2}{4a^2t} \right) \) satisfies the \( N \)-dimensional heat equation \( u_t = a^2\Delta u \) in \( \mathbb{R}^N \times (0, \infty) \).

b) Let \( \|f\| := \sup_{x \in \mathbb{R}} |f(x)| \). Proof the stability inequality

\[ \|u_1(\cdot, t) - u_2(\cdot, t)\| \leq \|u_1(\cdot, 0) - u_2(\cdot, 0)\| \quad \text{for all} \quad t > 0, \]

provided that \( u_1, u_2 \) are two bounded solutions of the one-dimensional heat equation and are from the space \( C^2(\mathbb{R} \times (0, \infty)) \cap C(\mathbb{R} \times [0, \infty)) \).

7. a) Let \( u_j(x, t), j = 1, \ldots, N, \) be solutions of the one-dimensional heat equation. Show that the function

\[ u(x_1, \ldots, x_N, t) := \prod_{j=1}^{N} u_j(x_j, t) \]

satisfies the \( N \)-dimensional heat equation for \( (x_1, \ldots, x_N, t) \in \mathbb{R}^N \times (0, \infty) \). Does this assertion hold true also for the wave equation?

b) Solve the initial value problem

\[ 3u_{tt} - 2u_{xx} = 0 \quad \text{in} \quad \mathbb{R}^2; \quad u(x, 0) = \cos x, \quad u_t(x, 0) = -2 \quad \text{for} \quad x \in \mathbb{R}. \]

8. Solve the initial value problem

\[ u_{tt} - a^2u_{xx} = x \quad \text{in} \quad \mathbb{R}^2; \quad u(x, 0) = x^2, \quad u_t(x, 0) = 0 \quad \text{für} \quad x \in \mathbb{R}. \]

(Hint: Find firstly a time independent solution of the partial differential equation and then transform the above problem in an initial value problem with homogeneous differential equation.)