Pricing and Hedging of Credit Derivatives via the Innovations Approach to Nonlinear Filtering

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Abstract In this paper we propose a new, information-based approach for modelling the dynamic evolution of a portfolio of credit risky securities. In our setup market prices of traded credit derivatives are given by the solution of a nonlinear filtering problem. The innovations approach to nonlinear filtering is used to solve this problem and to derive the dynamics of market prices. Moreover, the practical application of the model is discussed: we analyze model calibration, the pricing of exotic credit derivatives and the computation of risk-minimizing hedging strategies. The paper closes with a small numerical case study.

Keywords Credit derivatives, incomplete information, nonlinear filtering, hedging

1 Introduction

Credit derivatives - derivative securities whose payoff is linked to default events in a given portfolio - are an important tool in managing credit risk. However, the recent turmoil in credit markets highlights the need for a sound methodology for the pricing and the risk management of these securities. Credit portfolio products pose a particular challenge in this regard: the main difficulty is to capture the dependence structure of the defaults and the dynamic evolution of the credit spreads in a realistic way.

In this paper we propose a new, information-based approach to this problem. We consider a reduced-form model driven by an unobservable background

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factor process X. For tractability reasons X is modelled as finite-state Markov chain. We consider a market for defaultable securities related to m firms and assume that the default times are conditionally independent doubly stochastic random times with default intensity of firm i given by $\lambda_{t,i} = \lambda_i(X_t)$. This setup is akin to the model of di Graziano & Rogers (2006). If, for a moment, X is considered observable, pricing of credit derivatives turns out to be highly tractable and the Markovian structure of the model implies that prices are given as functions of the past defaults and the current state of X.

The factor process X is however not observable in our setup. Rather, we assume that prices of liquidly traded securities are given as conditional expectation with respect to a filtration $\mathbb{F}^{\mathbb{M}} = (\mathcal{F}_t^{\mathbb{M}})_{t\geq 0}$ which we call *market information.* $\mathbb{F}^{\mathbb{M}}$ is generated by a process Z giving observations of X in additive noise and by the default history of the firms under consideration. It turns out that in order to compute the prices of the traded securities one needs to determine the conditional distribution of X_t given $\mathcal{F}_t^{\mathbb{M}}$. Since X is a finite-state Markov chain this distribution is represented by a vector of probabilities denoted $\boldsymbol{\pi}_t$. Computing the dynamics of the process $\boldsymbol{\pi} = (\boldsymbol{\pi}_t)_{t\geq 0}$ is a nonlinear filtering problem which is solved in Section 3 using martingale representation results and the innovations approach to nonlinear filtering. By the same token we derive the dynamics of the market price of traded credit derivatives.

In Section 4 these results are then applied to the the pricing and the hedging of non-traded credit derivatives. It is shown that the price of most credit derivatives common in practice - defined as conditional expectation of the associated payoff given $\mathcal{F}_t^{\mathbb{M}}$ - depends on the realization of $\boldsymbol{\pi}_t$ and on past default information. Here a major issue arises for the application of the model: we view the process Z as abstract source of information which is not directly linked to economic quantities. Hence the process $\boldsymbol{\pi}$ is not directly accessible for typical investors. As we aim at pricing formulas and hedging strategies which can be evaluated in terms of publicly available information, a crucial point is to determine estimates of $\boldsymbol{\pi}_t$ from the prices of traded securities, and we propose two calibration strategies in this regard. Thereafter we derive risk-minimizing hedging strategies. Finally, in Section 5, we illustrate the applicability of the model to practical problems with a number of small empirical and numerical case studies.

The proposed modelling approach has a number of advantages. First, actual computations are done mostly in the context of the hypothetical model where X is fully observable. Since the latter has a simple Markovian structure, computations become relatively straightforward. Second, the fact that prices of traded securities are given by the conditional expectation given the market filtration $\mathbb{F}^{\mathbb{M}}$ leads to rich credit-spread dynamics: the proposed approach accommodates *spread risk* (random fluctuations of credit spreads between defaults) and *default contagion* (the observation that at the default of a company the credit spreads of related companies often react drastically). A prime example for contagion effects is the reaction of credit spreads after the default of Lehman brothers. Both features are important in the derivation of robust dynamic hedging strategies. Third, the model has a natural factor structure with factor process $\boldsymbol{\pi}$. Finally, the approach gives great flexibility in terms of calibration methodologies as is discussed in detail in Section 4.

Reduced form credit risk models with incomplete information have been considered previously by Schönbucher (2004), Collin-Dufresne, Goldstein & Helwege (2003), Duffie, Eckner, Horel & Saita (2006) and Frey & Runggaldier (2008). Schönbucher and Collin-Dufresne et. al.were the first to point out that the successive updating of $\boldsymbol{\pi}_t$ in reaction to incoming default observation has the potential to generate contagion effects. The paper by Duffie et al. (2006) has a strong empirical focus. Frey & Runggaldier (2008) finally concentrate on the mathematical analysis of filtering problems in reduced-form credit risk models. None of these contributions addresses the dynamics of credit-derivative prices under incomplete information or issues related to hedging. The innovations approach to nonlinear filtering has been used previously by Landen (2001) in the context of default-free term-structure models. Moreover, nonlinear filtering problems arise in a natural way in structural credit risk models with incomplete information about the current value of assets or liabilities such as Kusuoka (1999), Duffie & Lando (2001), Jarrow & Protter (2004), Coculescu, Geman & Jeanblanc (2008) or Frey & Schmidt (2007).

2 The Model

Our model is constructed on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions; all processes considered are by assumption $\mathbb{F}$-adapted. $\mathbb{Q}$ is the risk-neutral martingale measure used for pricing. For simplicity we work directly with discounted quantities so that the default-free money market account satisfies $B_t \equiv 1$.

Defaults and losses. We consider a fixed credit portfolio consisting of a set of m firms. The default time of firm i is denoted by τ_i and the current *default state* of the portfolio is $Y_t = (Y_{t,1}, \dots, Y_{t,m})$ with $Y_{t,i} = \mathbb{1}_{\{\tau_i \leq t\}}$. Note that $Y_t \in \{0,1\}^m$. We assume that $Y_0 = 0$. The percentage *loss given default* of firm i is denoted by the random variable $\ell_i \in (0,1]$. We assume that $\ell_1, \dots, \ell_m$ are independent random variables, independent of all other quantities introduced in the sequel. The *loss state* of the portfolio is given by the process $L = (L_{t,1}, \dots, L_{t,m})_{t\geq 0}$ where $L_{t,i} = \ell_i Y_{t,i}$.

Marked-point-process representation. Denote by $0 = T_0 < T_1 < \cdots < T_m < \infty$ the *ordered default times* and by ξ_n the identity of the firm defaulting at T_n. Then the sequence

$$(T_n, (\xi_n, \ell_{\xi_n})) =: (T_n, E_n), \quad 1 \leq n \leq m$$

gives a representation of L as marked point process with mark space $E := \{1, \dots, m\} \times (0,1]$. Let $\mu^L(ds, de)$ be the random measure associated to L with support $[0,\infty) \times E$. Note that any random function $R : \Omega \times [0,\infty) \times E \to \mathbb{R}$ can be written in the form

$$R(s,e) = R(s,(\xi,\ell)) = \sum_{i=1}^{m} \mathbb{1}_{\{\xi=i\}} R_i(s,\ell)$$

with $R_i(s,\ell) := R(s,(i,\ell))$. Then integrals with respect to $\mu^L(ds,de)$ can be written in the form

$$\int_0^t \int_E R(s,e)\,\mu^L(ds,de) = \sum_{T_n \le t} R_{\xi_n}(T_n,\ell_{\xi_n}) = \sum_{\tau_i \le t} R_i(\tau_i,\ell_i).$$

2.1 The underlying Markov model

The default intensities of the firms under consideration are driven by the so-called factor or state process X. The process X is modelled as a finite-state Markov chain; in the sequel its state space S^X is identified with the set $\{1,\ldots,K\}$. The following assumption states that the default times are conditionally independent, doubly-stochastic random times with default intensity $\lambda_{t,i} := \lambda_i(X_t)$. Set $\mathcal{F}_\infty^X = \sigma(X_s : s \ge 0)$.

A1 There are functions $\lambda_i : S^X \to (0,\infty)$, such that for all $t_1,\ldots,t_m \ge 0$

$$\mathbb{Q}\big(\tau_1 > t_1,\ldots,\tau_m > t_m \mid \mathcal{F}_\infty^X\big) = \prod_{i=1}^{m} \exp\Big(-\int_0^t \lambda_i(X_s)ds\Big).$$

It is well-known that under **A1** there are no joint defaults, i.e. $\tau_i \ne \tau_j$, for $i \ne j$ almost surely. Moreover, for all $1 \le i \le m$

$$Y_{t,i} - \int_0^{t\wedge\tau_i} \lambda_i(X_{s-})ds \tag{2.1}$$

is an $\mathbb{F}$-martingale; see for instance Chapter 9 in McNeil, Frey & Embrechts (2005). Furthermore, the process (X,L) is jointly Markov. Finally, denote by F_{ℓ_i} the distribution function of ℓ_i. Then the $\mathbb{F}$-compensator ν^L of the random measure μ^L is given by

$$\nu^L(dt,de) = \nu^L(dt,d\xi,d\ell) = \sum_{i=1}^{m} \delta_{\{i\}}(d\xi)\,F_{\ell_i}(d\ell)\,(1-Y_{t-,i})\lambda_i(X_{t-})dt\,, \tag{2.2}$$

where $\delta_{\{i\}}$ stands for the Dirac-measure in i.

Example 2.1 To illustrate the flexibility in modelling under **A1**, we give two examples. First, consider a homogeneous model where the default intensities of all firms are identical. Let X be the state of the economy with 1 representing the best state and K the worst state. A good state of economy corresponds to a low default intensity so that the default intensities $\lambda_{t,i} = \lambda(X_t)$ are modelled via some increasing function $\lambda : \{1, \dots, K\} \to (0, \infty)$.

As a second example consider a portfolio with global and industry specific factors. Assume the portfolio entails companies from $\tilde{m} \le m$ different industry sectors. Let $S^X = \{0,1\}^{\tilde{m}} \times \{1, \dots, \kappa\}$ and write X^i, $i = 1, \dots, \tilde{m}+1$ for the ith component of X. The components $X^1, \dots, X^{\tilde{m}+1}$ are assumed to be independent Markov chains. For $i = 1, \dots, \tilde{m}$, X^i refers to the state of industry sector i, which is good ($X^i = 0$) or bad ($X^i = 1$), while $X^{\tilde{m}+1}$ represents a global factor. In line with this interpretation we model the default intensities via $\lambda_i(x) = f_i(x^i) + g_i(x^{m+1})$ for increasing functions f_i and g_i from $\{0,1\}$ respectively $\{1, \dots, \kappa\}$ to $(0, \infty)$.

2.2 Market information

In our setting the factor process X is not directly observable. We assume that prices of traded credit derivatives are determined as conditional expectation with respect to some filtration $\mathbb{F}^{\mathbb{M}}$ which we call *market information*. The following assumptions states that the market information is generated by the loss history $\mathbb{F}^L$ and observations of functions of X in additive Gaussian noise.

A2 $\mathbb{F}^{\mathbb{M}} = \mathbb{F}^L \vee \mathbb{F}^Z$, where the l-dimensional process Z is given by

$$Z_t = \int_0^t \mathbf{a}(X_s)\, ds + B_t. \tag{2.3}$$

Here, B is an l-dimensional standard $\mathbb{F}$-Brownian motion independent of X and L, and $\mathbf{a}(\cdot)$ is a function from S^X to $\mathbb{R}^l$.

Example 2.2 We consider the two cases from Example 2.1. First, in a homogeneous-portfolio situation with $\lambda_{t,i} = \lambda(X_t)$ one could assume that $a(\cdot) = c \ln \lambda(\cdot)$. The constant $c \ge 0$ models the information-content of Z: for $c = 0$, Z carries no information, whereas for c large the state X_t can be observed with high precision. Second, in the one-factor case one could take analogously $l = \tilde{m}+1$, $a_l(x) = c_l \ln f_l(x^l)$, $1 \le l \le \tilde{m}$, and $a_{\tilde{m}+1}(x) = c_{\tilde{m}+1} \ln g(x^{m+1})$. This models a situation where the market has noisy observations regarding the current state of each industry factor and of the global economy factor.

3 Dynamics of traded credit derivatives and filtering

In this section we study in detail traded credit derivatives. First, we give a general description of this type of derivatives and discuss the relation between pricing and filtering. In Section 3.2 we then to study the dynamics of market prices, using the innovations approach to nonlinear filtering.

3.1 Traded securities

We consider a market of N liquidly traded credit derivatives, with - for notational simplicity - common maturity T. Most credit derivatives have intermediate cash flows such as payments at default dates. Hence it is convenient to describe the payoff of the nth derivative by the cumulative *dividend stream* D_n. We assume that D_n takes the form

$$D_{t,n} = \int_0^t d_{1,n}(s, L_s) db(s) + \int_0^t \int_E d_{2,n}(s, L_{s-}, e) \mu^L(ds, de) \tag{3.1}$$

with bounded functions d_1, d_2 and an increasing deterministic function $b : [0,T] \to \mathbb{R}$. Dividend streams of the form (3.1) can be used to model many important credit derivatives:

Example 3.1 In the following examples the function b encodes pre-scheduled payments: considering pre-scheduled payments dates $t_1 < \cdots < t_{\tilde{n}} = T$ we set $b(s) = |\{i \colon t_i \le s\}|$.

Zero-bond. A defaultable bond without coupon payments and with zero recovery on firm i pays 1 at T if $\tau_i > T$ and zero otherwise. Hence we have $d_1(t, L_t) = \mathbb{1}_{\{t=T, L_{T,i}=0\}}$ and $d_2 = 0$.

Credit default swap (CDS). A protection seller position in a CDS on firm i offers regular payments of size S at the pre-scheduled dates $t_1 < \cdots < t_{\tilde{n}}$ until default. In exchange for this, the holder pays the loss ℓ_i at τ_i, provided $\tau_i < T$. (Accrued premium payments are ignored for simplicity.) This can be modelled by taking $d_1(t, L_t) = S\mathbb{1}_{\{L_{t,i}=0\}}$ and $d_2(t, L_{t-}, \xi, \ell) = -\mathbb{1}_{\{\xi=i\}}\ell$; note that

$$\int_0^t \int_E d_{2,n}(s, L_{s-}, e) \mu^L(ds, de) = -\ell_i \mathbb{1}_{\{L_{t,i}>0\}} = -L_{t,i}.$$

Collateralized debt obligation (CDO). A single tranche CDO on the underlying portfolio is specified by an lower and upper detachment point $0 \le x_1 < x_2 \le m$ and a fixed rate S[1]. Define the function

$$H(x) := (x_2 - x)^+ - (x_1 - x)^+$$

and denote the *cumulative portfolio loss* by $\bar{L}_t = \sum_{i=1}^m L_{t,i}$. $H(\bar{L}_t)$ can be viewed as the remaining notional of the tranche at time t. An investor in the tranche receives the cumulative income $\int_0^t SH(\bar{L}_s)db(s)$, so that $d_1(t, L_{t-}) = SH(\bar{L}_{t-})$. On the other side, the investor pays at the successive default times

[1] In practice, lower and upper detachment points are stated in percentage points, say $0 \le l < u \le 1$. Then $x_1 = l \cdot m$ and $x_2 = u \cdot m$.

T_n, $1 \le n \le m$, the amount $-\Delta H(\bar{L}_{T_n}) = -\big(H(\bar{L}_{T_n}) - H(\bar{L}_{T_n-})\big)$. This can be modelled by setting

$$d_2(t, L_{t-}, \xi, \ell) = H\left(\ell + \bar{L}_{t-}\right) - H\left(\bar{L}_{t-}\right).$$

Pricing of traded credit derivatives. Recall that we work with discounted quantities and that $\mathbb{Q}$ represents the underlying pricing measure. Moreoever, the information available to market participants is the market information $\mathbb{F}^M$. As a consequence we assume that the *current market value* of the traded credit derivatives is given by

$$\widehat{p}_{t,n} := \mathbb{E}\big(D_{T,n} - D_{t,n} | \mathcal{F}_t^{\mathbb{M}}\big), \quad 1 \le n \le N. \tag{3.2}$$

The *gains process* $\widehat{g}_n$ of the n-th credit derivative sums the current market value and the dividend payments received so far and is thus given by

$$\widehat{g}_{t,n} := \widehat{p}_{t,n} + D_{t,n} = \mathbb{E}\big(D_{T,n} \mid \mathcal{F}_t^{\mathbb{M}}\big). \tag{3.3}$$

Next we show that the computation of market values leads to a nonlinear filtering problem. We call the conditional expectation $\mathbb{E}\big(D_{T,n} - D_{t,n} | \mathcal{F}_t\big)$ *hypothetical value* of D_n in the underlying Markov model. While this quantity will be an important tool in our analysis it does not correspond to market prices as we consider X as unobservable. Observe that by (3.1) $D_{T,n} - D_{t,n}$ is a function of the future path $(L_s)_{s \in (t,T]}$. Hence, the $\mathbb{F}$-Markov property of the pair (X, L) implies that

$$\mathbb{E}\big(D_{T,n} - D_{t,n} | \mathcal{F}_t\big) = p_n(t, X_t, L_t) \tag{3.4}$$

for functions $p_n : [0,T] \times S^X \times [0,1]^m \to \mathbb{R}$, $n = 1, \ldots, N$. By iterated conditional expectations we obtain

$$\widehat{p}_{t,n} = \mathbb{E}\Big(\mathbb{E}\big(D_{T,n} - D_{t,n} | \mathcal{F}_t\big) | \mathcal{F}_t^{\mathbb{M}}\Big) = \mathbb{E}\big(p_n(t, X_t, L_t) | \mathcal{F}_t^{\mathbb{M}}\big).$$

Hence, in order to compute the market values $\widehat{p}_{t,n}$ we need to determine the conditional distribution of X_t given $\mathcal{F}_t^{\mathbb{M}}$. This a nonlinear filtering problem which we solve in Section 3.3 below.

3.2 Asset Price Dynamics under the Market Filtration

In the sequel we use the innovations approach to nonlinear filtering in order to derive a representation of the martingales $\widehat{g}_n$ as a stochastic integral with respect to certain $\mathbb{F}^{\mathbb{M}}$-adapted martingales. Throughout the rest of the paper we denote by $\widehat{U}_t := \mathbb{E}(U_t | \mathcal{F}_t^{\mathbb{M}})$ the *optional projection* of a generic process U w.r.t. the market filtration $\mathbb{F}^{\mathbb{M}}$.

We begin by introducing the martingales needed for the representation result. First, define for $i = 1, \ldots, l$

$$m_{t,i}^Z := Z_{t,i} - \int_0^t \widehat{a_i(X_s)}\, ds\,; \tag{3.5}$$

it is well-known that m^Z is an $\mathbb{F}^{\mathbb{M}}$-Brownian motion and thus the martingale part in the $\mathbb{F}^{\mathbb{M}}$-semimartingale decomposition of Z. Second, denote by

$$\widehat{\nu}^{\,L}(dt,de) = \sum_{i=1}^{m} \delta_{\{i\}}(d\xi)\, F_{\ell_i}(d\ell)\, (1 - Y_{t-,i})\widehat{\lambda_i(X_t)}dt \tag{3.6}$$

the compensator of μ^L w.r.t. $\mathbb{F}^{\mathbb{M}}$ and define the compensated random measure

$$m^L(dt,de) := \mu^L(dt,de) - \widehat{\nu}^{\,L}(dt,de)\,. \tag{3.7}$$

It is well known that for every $\mathbb{F}^{\mathbb{M}}$-predictable random function f such that $\mathbb{E}\big(\int_0^T |f(s,e)|\,\widehat{\nu}^{\,L}(ds,de)\big) < \infty$ the integral $\int_0^t f(s,e)\, m^L(ds,de)$ is a martingale with respect to $\mathbb{F}^{\mathbb{M}}$.

The following martingale representation result is a key tool in our analysis; its proof is relegated to the appendix.

Lemma 3.2 *For every $\mathbb{F}^{\mathbb{M}}$-martingale $(U_t)_{0\le t\le T}$ there exists a $\mathbb{F}^{\mathbb{M}}$-predictable function γ and an $\mathbb{R}^l$-valued $\mathbb{F}^{\mathbb{M}}$-adapted process $\boldsymbol{\alpha}$ such that U has the representation*

$$U_t = U_0 + \int_0^t \gamma(s,e)\, m^L(ds,de) + \int_0^t \boldsymbol{\alpha}_s^\top dm_s^Z, \quad 0 \le t \le T. \tag{3.8}$$

The next theorem is the basis for the mathematical analysis of the model under the market filtration. In order to ease the notation, we denote for a generic function $f : S^X \to \mathbb{R}$ by $\widehat{f}$ the optional projection of the process $(f(X_s))_{0\le s\le T}$ with respect to $\mathbb{F}^{\mathbb{M}}$.

Theorem 3.3 *Consider a real-valued $\mathbb{F}$-semimartingale*

$$J_t = J_0 + \int_0^t A_s ds + M_t^J, \quad t \le T$$

such that $[M^J, B] = 0$. Assume that

(i) $\mathbb{E}(|J_0|) < \infty$, $\mathbb{E}(\int_0^T |A_s|ds) < \infty$ *and* $\mathbb{E}(\int_0^T |J_s|\lambda_i(X_s)ds) < \infty$, $i = 1, \ldots, m$.
(ii) $\mathbb{E}([M^J]_T) < \infty$.
(iii) For all $1 \le i \le m$ there is some $\mathbb{F}^{\mathbb{M}}$-predictable $R_i : \Omega \times [0,T] \times (0,1] \to \mathbb{R}$ such that

$$[J, Y_i]_t = \int_0^t \int_E \mathbb{1}_{\{\xi=i\}} R_i(s,\ell)\, \mu^L(ds,d\xi,d\ell). \tag{3.9}$$

Moreover, $\mathbb{E}(\int_0^T \int_0^1 |R_i(s,\ell)| F_{\ell_i}(dl)(1 - Y_{s-,i})\lambda_i(X_s)ds) < \infty$.
(iv) $\int_0^t J_{s-}dB_{s,j}$ *and* $\int_0^t Z_{s,j}dM_s^J$, $1 \le j \le l$ *are true martingales.*

Then the optional projection $\widehat{J}$ has the representation

$$\widehat{J}_t = \widehat{J}_0 + \int_0^t \widehat{A}_s ds + \int_0^t \int_E \gamma(s,e)\, m^L(ds,de) + \int_0^t \boldsymbol{\alpha}_s^\top dm_s^Z, \quad t \le T. \tag{3.10}$$

Here, $\gamma(s,e) = \gamma\big(s,(\xi,\ell)\big) = \sum_{i=1}^m \mathbb{1}_{\{\xi=i\}}\gamma_i(s,\ell)$, and $\boldsymbol{\alpha}, \gamma_i$ are given by

$$\boldsymbol{\alpha}_s = (\widehat{J\mathbf{a}})_{s-} - \widehat{J}_{s-}(\widehat{\mathbf{a}})_{s-}, \tag{3.11}$$

$$\gamma_i(s,\ell) = \frac{1}{(\widehat{\lambda}_i)_{s-}}\Big[(\widehat{J\lambda_i})_{s-} - \widehat{J}_{s-}(\widehat{\lambda}_i)_{s-} + (\widehat{R_i(\cdot,\ell)\lambda_i})_{s-}\Big]. \tag{3.12}$$

Proof The proof uses the following two well-known facts.

1. For every true $\mathbb{F}$-martingale N, the projection $\widehat{N}$ is an $\mathbb{F}^{\mathbb{M}}$-martingale.
2. For any progressively measurable process ϕ with $\mathbb{E}\big(\int_0^T |\phi_s|ds\big) < \infty$ the process $\widehat{\int_0^t \phi_s ds} - \int_0^t \widehat{\phi}_s\, ds$, $s \le T$, is an $\mathbb{F}^{\mathbb{M}}$ martingale.

The first fact is simply a consequence of iterated expectations, while the second follows from the Fubini theorem, see for instance Davis & Marcus (1981).

As M^J is a true martingale by (ii), Fact 1 and 2 immediately yield that $\widehat{J}_t - \widehat{J}_0 - \int_0^t \widehat{A}_s ds$ is an $\mathbb{F}^{\mathbb{M}}$-martingale. Lemma 3.2 thus gives the existence of the representation (3.10).

It remains to identify γ and $\boldsymbol{\alpha}$. The idea is to use the elementary identity

$$\widehat{J\phi} = \widehat{J}\,\phi$$

for any $\mathbb{F}^{\mathbb{M}}$-adapted ϕ. Each side of this equation gives rise to a different semimartingale decomposition of $\widehat{J\phi}$; comparing those for suitably chosen ϕ one obtains γ and $\boldsymbol{\alpha}$.

In order to identify γ, fix i and let

$$\phi_t^i = \int_0^t \int_E \varphi(s,\ell)\mathbb{1}_{\{\xi=i\}}\, \mu^L(ds,d\xi,d\ell)$$

for a bounded and $\mathbb{F}^{\mathbb{M}}$-predictable φ. Note that ϕ^i is $\mathbb{F}^{\mathbb{M}}$-adapted. We first determine the $\mathbb{F}$-semimartingale decomposition of $J\phi^i$. Itô's formula gives

$$dJ_t\phi_t^i = \phi_{t-}^i dJ_t + J_{t-}d\phi_t^i + d[J,\phi^i]_t. \tag{3.13}$$

Now with (3.9)

$$[J,\phi^i]_t = \sum_{s\le t} \Delta J_s \Delta\phi_s^i = \int_0^t \int_E R_i(s,\ell)\varphi(s,\ell)\mathbb{1}_{\{\xi=i\}}\mu^L(ds,d\xi,d\ell).$$

Hence, using (2.2), the predictable compensator of $[J,\phi^i]$ is

$$\langle J,\phi^i\rangle_t = \int_0^t\int_0^1 R_i(s,\ell)\varphi(s,\ell)F_{\ell_i}(d\ell)(1-Y_{s-,i})\lambda_i(X_s)ds. \tag{3.14}$$

Moreover, $[J,\phi^i]-\langle J,\phi^i\rangle$ is a true martingale by (iii), as φ is bounded. Using (3.13) and (3.14) the finite variation part in the $\mathbb{F}$-semimartingale decomposition of $J\phi^i=\tilde{A}+\tilde{M}$ computes to

$$\begin{aligned}\tilde{A}_t = \int_0^t \Bigg(\phi^i_s A_s + J_{s-}(1-Y_{s-,i})\lambda_i(X_s)\int_0^1 \varphi(s,\ell)F_{\ell_i}(d\ell)\\ + \int_0^1 R_i(s,\ell)\varphi(s,\ell)(1-Y_{s-,i})\lambda_i(X_s)F_{\ell_i}(d\ell)\Bigg)ds.\end{aligned}$$

Moreover, $\tilde{M}$ is a true $\mathbb{F}$-martingale by (i) - (iii). Using Fact 1 and 2 the finite variation part in the $\mathbb{F}^{\mathbb{M}}$-semimartingale decomposition of $\widehat{J\phi^i}$ turns out to be

$$\begin{aligned}\int_0^t \Bigg(\phi^i_s\widehat{A}_s + (1-Y_{s-,i})(\widehat{J\lambda_i})_s\int_0^1 \varphi(s,\ell)F_{\ell_i}(d\ell)\\ + \int_0^1 \varphi(s,\ell)(1-Y_{s-,i})(\widehat{R_i(\cdot,\ell)\lambda_i})_s\, F_{\ell_i}(d\ell)\Bigg)ds.\end{aligned} \tag{3.15}$$

On the other hand, we get from Lemma 3.2 that

$$\widehat{J}_t = \int_0^t \widehat{A}_s ds + \int_0^t\int_E \gamma(s,e)m^L(ds,de) + \int_0^t \boldsymbol{\alpha}_s^\top dm^Z_s.$$

Hence, Itô's formula gives

$$\begin{aligned}\widehat{J}_t\phi^i_t = M_t + \int_0^t \Bigg(\phi^i_s\widehat{A}_s + \widehat{J}_s\int_0^1 \varphi(s,\ell)F_{\ell_i}(d\ell)(1-Y_{s-,i})(\widehat{\lambda}_i)_s\\ + \int_0^1 \gamma_i(s,\ell)\varphi(s,\ell)F_{\ell_i}(d\ell)(\widehat{\lambda}_i)_s(1-Y_{s-,i})\Bigg)ds\end{aligned} \tag{3.16}$$

where M is a local $\mathbb{F}^{\mathbb{M}}$-martingale. Recall that $\widehat{J\phi}=\widehat{J}\,\phi$. By the uniqueness of the semimartingale decomposition, (3.15) must equal the finite variation part in (3.16) which leads to

$$0=\int_0^t\int_0^1\varphi(s,\ell)(1-Y_{s-,i})\Big((\widehat{J\lambda_i})_s-\widehat{J}_s(\widehat{\lambda}_i)_s$$
$$+(\widehat{R_i(\cdot,\ell)\lambda_i})_s-\gamma_i(s,\ell)(\widehat{\lambda}_i)_s\Big)F_{\ell_i}(d\ell)ds$$

for all $0\leq t\leq T$. Since φ was arbitrary, we get (3.12).

In order to establish (3.11) we use a similar argument with $\phi=Z_i$. For this, note that the arising local martingales in the semimartingale decomposition of JZ_i are true martingales by (iv). □

The following theorem describes the dynamics of the gains processes of the traded credit derivatives and gives their instantaneous quadratic covariation.

Theorem 3.4 *Under* **A1** *and* **A2** *the gains processes $\widehat{g_1},\dots,\widehat{g_N}$ of the traded securities have the martingale representation*

$$\widehat{g}_{t,n}=\widehat{g}_{0,n}+\sum_{i=1}^m\int_0^t\int_0^1\mathbb{1}_{\{\xi=i\}}\gamma_i^{\widehat{g}_n}(s,\ell)\,m^L(ds,d\xi,d\ell)+\int_0^t(\boldsymbol{\alpha}_s^{\widehat{g}_n})^\top dm_s^Z. \quad (3.17)$$

Here the integrands are given by

$$\boldsymbol{\alpha}_t^{\widehat{g}_n}=\widehat{p_{t,n}\cdot\mathbf{a}}_t-\widehat{p}_{t,n}\,\widehat{\mathbf{a}}_t\,, \quad (3.18)$$

$$\gamma_i^{\widehat{g}_n}(t,\ell)=\frac{1}{(\widehat{\lambda}_i)_{s-}}\Big[(\widehat{p_n\lambda_i})_{s-}-(\widehat{p_n})_{s-}(\widehat{\lambda}_i)_{s-}+(\widehat{R_{i,n}(\cdot,\ell)\lambda_i})_{s-}\Big]\ \text{with} \quad (3.19)$$

$$R_{i,n}(s,\ell)=p_n\left(s,X_s,L_s+\ell\boldsymbol{e}_i\right)-p_n(s,X_s,L_s)+d_{2,n}\left(s,X_s,L_s+\ell\boldsymbol{e}_i\right) \quad (3.20)$$

and $\boldsymbol{e}_i$ the ith unit vector in $\mathbb{R}^m$. The predictable quadratic variation of the gains processes $\widehat{g_1},\dots,\widehat{g_N}$ with respect to $\mathbb{F}^{\mathbb{M}}$ satisfies $d\langle\widehat{g}_i,\widehat{g}_j\rangle_t^{\mathbb{M}}=v_t^{ij}\,dt$ with

$$v_t^{ij}:=\sum_{k=1}^m\int_0^1\gamma_k^{\widehat{g}_i}(t,\ell)\,\gamma_k^{\widehat{g}_j}(t,\ell)\,F_{\ell_k}(d\ell)\,\widehat{\lambda}_{t-,k}(1-Y_{t-,k})+\sum_{k=1}^l\alpha_{t-,k}^{\widehat{g}_i}\alpha_{t-,k}^{\widehat{g}_j}\,. \quad (3.21)$$

Proof We apply Theorem 3.3 to the the $\mathbb{F}$-martingale $J_t=\mathbb{E}(D_{T,n}|\mathcal{F}_t)$. To this we verify the conditions therein: first, $[J,B]=0$ as B is independent of X and L. As $d_{1,n}$ and $d_{2,n}$ from (3.1) are bounded, so is J. By **A1** λ_i is bounded and hence (i) holds. Second, $M^J=J$ is bounded and hence a square-integrable true martingale which gives (ii). Next, note that $J_t = p_n(t, X_t, L_t) + D_{t,n}$. Hence

$$
\begin{aligned}
[J, Y_i]_t &= (\Delta J_{\tau_i} \Delta Y_{\tau_i,i}) \mathbb{1}_{\{\tau_i \le t\}} \\
&= \mathbb{1}_{\{\tau_i \le t\}} \Big(p_n(\tau_i, X_{\tau_i}, L_{\tau_i}) - p_n(\tau_i, X_{\tau_i -}, L_{\tau_i -}) + \Delta D_{\tau_i,n} \Big) \\
&= \int_0^t \int_E \mathbb{1}_{\{\xi = i\}} R_{i,n}(s,\ell) \mu^L(ds, d\xi, d\ell)
\end{aligned}
$$

with $R_{i,n}$ as in (3.20). As R is bounded, (iii) follows. Next, as J is bounded, $\int J dB_j$ is a true martingale. Moreover,

$$
\int_0^t Z_{s,j} dJ_s = \int_0^t \int_0^s a_j(X_u) du \, dJ_s + \int_0^t B_{s,j} dJ_s .
$$

As $\mathbf{a}(\cdot)$ is bounded, the first term has integrable quadratic variation and is thus a true martingale. Since B and J are independent, we get

$$
\mathbb{E}\Big(\int_0^t (B_{s,j})^2 d[J]_s \Big) = \mathbb{E}\Big(\int_0^t \mathbb{E}(B_{s,j}^2) d[J]_s \Big) \le T \mathbb{E}([J]_T) < \infty .
$$

This together yields (iv) and hence (3.17) where in (3.18) and (3.19) $\widehat{p}_{t,n}$ is replaced by J. Recall that $\widehat{g}_{t,n} = \widehat{p}_{t,n} + D_{t,n}$ where $D_{t,n}$ is $\mathbb{F}_t^{\mathbb{M}}$-measurable. This allows as to replace J by $p_{t,n}$ and yields the first part of the theorem.

The second part (the statement regarding the predictable quadratic variations) follows immediately from (3.17) and (3.6). □

Remark 3.5 The assumption that X is a finite state Markov chain was only used to insure integrability conditions in Theorem 3.3 and in Theorem 3.4 so that these results are easily extended to a more general setting. The filtering results in Section 3.3 below on the other hand do exploit the specific structure of X.

3.3 Filtering and factor representation of market prices

Since X is a finite state Markov chain, the conditional distribution of X_t given $\mathcal{F}_t^{\mathbb{M}}$ is given by the vector $\boldsymbol{\pi}_t = (\pi_t^1, \ldots, \pi_t^K)^\top$ with $\pi_t^k := \mathbb{Q}(X_t = k | \mathcal{F}_t^{\mathbb{M}})$. The following proposition shows that the process $\boldsymbol{\pi}$ is the solution of a K-dimensional SDE system driven by m^Z and the $\mathbb{F}^{\mathbb{M}}$-martingale M given by

$$
M_{t,j} := Y_{t,j} - \int_0^t (1 - Y_{s-,j}) \, (\widehat{\lambda_j})_s ds = \int_0^t \int_E \mathbb{1}_{\{\xi = j\}} m^L(ds, d\xi, d\ell), \quad j = 1, \ldots, m.
$$

Proposition 3.6 *Denote the generator matrix of X by $(q(i,k))_{1\le i,k,\le K}$. Then, for $k=1,\dots,K$,*

$$d\pi_t^k = \sum_{i\in S^X} q(i,k)\pi_t^i dt + (\boldsymbol{\gamma}^k(\boldsymbol{\pi}_{t-}))^\top dM_t + (\boldsymbol{\alpha}^k(\boldsymbol{\pi}_t))^\top dm_t^Z \,, \tag{3.22}$$

with coefficients given by

$$\gamma_j^k(\boldsymbol{\pi}_t) = \pi_t^k\Big(\frac{\lambda_j(k)}{\sum_{i\in S^X}\lambda_j(i)\pi_t^i} - 1\Big), \quad 1\le j\le m, \tag{3.23}$$

$$\boldsymbol{\alpha}^k(\boldsymbol{\pi}_t) = \pi_t^k\Big(\mathbf{a}(k) - \sum_{i\in S^X}\pi_t^i\mathbf{a}(i)\Big)\,. \tag{3.24}$$

Proof Denote the generator of X by $\mathscr{L}$ and set $f_k(x) = \mathbb{1}_{\{x=k\}}$. Then the $\mathbb{F}$-semimartingale decomposition of $(f_k(X_t))_{t\ge 0}$ is

$$f_k(X_t) = f_k(X_0) + \int_0^t \mathscr{L} f_k(X_s)\,ds + \Big(f_k(X_t) - f_k(X_0) - \int_0^t \mathscr{L} f_k(X_s)\,ds\Big)\,.$$

Note that $\pi^k = \widehat{f_k(X_t)}$ and that $\mathscr{L} f_k(X_t) = q(X_t,k)$. We apply Theorem 3.3 with $J = f_k(X_t) = \mathbb{1}_{\{X_t=k\}}$. First, $[f_k(\cdot),B] = [M^J,B]\equiv 0$, as B is continuous and $f_k(\cdot)$ is of finite variation. Moreover, Note that $[f_k(\cdot),Y_i]=0$ for all i as X and Y have a.s. no common jumps, so that the random function $R_i = 0$ in Condition (iv) of Theorem 3.3 vanishes for all i. Boundedness of J implies Conditions (i)-(iv) from that theorem by a similar argument as in the proof of Theorem 3.4. Hence

$$d\pi_t^k = \widehat{q(X_t,k)}dt + \int_E \sum_{i=1}^m \gamma_i(t,\ell)\mathbb{1}_{\{\xi=i\}}\, m^L(dt,d\xi,d\ell) + \boldsymbol{\alpha}_t^\top dm_t^Z$$

with γ_i given by

$$\gamma_i(t,\ell) = \frac{1}{(\widehat{\lambda}_i)_{t-}}\Big((\widehat{\lambda_i(k)J})_{t-} - (\widehat{\lambda}_i)_{t-}\widehat{J}_{t-}\Big) = \frac{1}{(\widehat{\lambda}_i)_{t-}}\Big(\lambda_i(k)\pi_{t-}^k - (\widehat{\lambda}_i)_{t-}\,\pi_{t-}^k\Big)\,.$$

Note that $(\widehat{\lambda}_i)_{t-} = \sum_{k\in S^X}\lambda_i(k)\pi_{t-}^k$. As $\gamma_i(t,\ell)$ does not depend on ℓ,

$$\int_0^t\int_0^1 \gamma_i(s,\ell)\mathbb{1}_{\{\xi=i\}} m^L(ds,d\xi,d\ell) = \int_0^t \gamma_i^k(\boldsymbol{\pi}_{s-})dM_{s,i}\,,$$

and (3.23) follows. For (3.24), note finally that

$$\boldsymbol{\alpha}_t^k = \widehat{f_k(X_t)\mathbf{a}(X_t)} - \widehat{f_k(X_t)}\widehat{\mathbf{a}(X_t)} = \pi_t^k\,\mathbf{a}(k) - \pi_t^k\sum_{i\in S^X}\pi_t^i\mathbf{a}(i)\,.$$

Remark 3.7 Related results have previously appeared in the filtering literature. For the case of diffusion observations, (3.22) is the Wonham filter (Wonham 1965). For the case of marked-point-process observations we refer to Brémaud (1981) and further references therein.

Contagion. The previous results permit us to give an explicit expression for the *contagion effects* (the observation that at default of a company the credit spreads of related companies react significantly) induced in our model. For $i \neq j$ we get from (3.23) that

$$\begin{aligned} \widehat{\lambda}_{\tau_j,i} - \widehat{\lambda}_{\tau_j-,i} &= \sum_{k=1}^{K} \lambda_i(k) \cdot \pi^k_{\tau_j-} \left(\frac{\lambda_j(k)}{\sum_{l=1}^{K} \lambda_j(l) \pi^l_{\tau_j-}} - 1 \right) \\ &= \frac{\operatorname{cov}^{\boldsymbol{\pi}_{\tau_j-}} \big(\lambda_i, \lambda_j\big)}{\mathbb{E}^{\boldsymbol{\pi}_{\tau_j-}} (\lambda_j)} \,. \end{aligned} \tag{3.25}$$

Moreover, $\boldsymbol{\pi}_{\tau_j-}$ gives the conditional distribution of X immediately prior to the default event. According to (3.25), default contagion increases (i) with increasing correlation of the random variables $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ under $\boldsymbol{\pi}_{\tau_j-}$, and (ii) with increasing variance of $\lambda_i(\cdot)$ or $\lambda_j(\cdot)$, i.e. with increasing dispersion of the measure $\boldsymbol{\pi}_{\tau_j-}$. Both effects are very intuitive.

The process $(L, \boldsymbol{\pi})$ is a natural state variable process for the model: first, $(L, \boldsymbol{\pi})$ is a Markov process (see Lemma 3.8 below); second, all quantities of interest at time t can be represented in terms of L_t and $\boldsymbol{\pi}_t$. In particular, the market values from (3.2) can be expressed as follows

$$\widehat{p}_{t,n} = \sum_{k \in S^X} p_n(t, k, L_t)\, \pi^k_t ,$$

and a similar representation can be obtained for the integrands $\boldsymbol{\alpha}^{\widehat{g}_n}_t$ and $\gamma^{\widehat{g}_n}_i(t, \ell)$ from Theorem 3.4. Motivated by these two observations we call $(L, \boldsymbol{\pi})$ the *market state* process.

Lemma 3.8 *The market state process* $(L, \boldsymbol{\pi})$ *is an* $\mathbb{F}^{\mathbb{M}}$*-Markov process of jump-diffusion type with generator* $\mathscr{L}$ *given by formula* (A.1) *in the appendix.*

To proof this claim we use Itô's formula to identify the generator of $(L, \boldsymbol{\pi})$ and show uniqueness of the related martingale problem; see Appendix A.2 for details.

4 Practical issues: pricing, calibration and hedging

In this section we discuss the pricing, the calibration and the hedging of credit derivatives. Consider a non-traded credit derivative. In accordance with (3.2), we define the price at time t of the credit derivative as conditional expectation of the associated payoff given $\mathcal{F}^{\mathbb{M}}_t$. For the credit derivatives common in practice this conditional expectation is given by a function of the current market state $(L_t, \boldsymbol{\pi}_t)$, as we show in Section 4.1. Here a major issue arises for the application of the model: we view the process Z generating the market filtration $\mathbb{F}^{\mathbb{M}}$ as abstract source of information so that the process $\boldsymbol{\pi}$ is not directly observable for investors. On the other hand, pricing formulas and hedging strategies need to be evaluated using only publicly available information. A key point in this section is therefore model calibration: in Section 4.2 we explain how to determine and estimate of $\boldsymbol{\pi}_t$ from prices of traded securities observed at time t.

4.1 Pricing

Basically all credit derivatives common in practice fall in one of the following two classes:

Options on the loss state. This class comprises derivatives with payoff given by an $\mathbb{F}^L$-adapted dividend stream D of the form (3.1); examples are typical basket derivatives or bespoke CDOs. As in (3.4), the hypothetical value of an option on the loss state in the underlying Markov model, $\mathbb{E}\big(D_T - D_t|\mathcal{F}_t\big)$, is equal to $p(t, X_t, L_t)$ for some function $p : [0, T] \times S^X \times [0, 1]^m \to \mathbb{R}$. Hence the price of the option at time t is given by

$$\widehat{p}_t := \mathbb{E}(D_T - D_t|\mathcal{F}_t^{\mathbb{M}}) = \sum_{k \in S^X} p(t, k, L_t)\pi_t^k \,. \tag{4.1}$$

Note that for an option on the loss state the price $\widehat{p}_t$ depends only on the current market state $(L_t, \boldsymbol{\pi}_t)$ and on the function $p(\cdot)$ that gives the hypothetical value of the option in the underlying Markov model; the precise form of the the function $\mathbf{a}(\cdot)$ from **A2** and thus of the dynamics of $\boldsymbol{\pi}$ is irrelevant. The dynamics of $\boldsymbol{\pi}$ do however matter in the computation of hedging strategies for these claims, see Subsection 4.3 below.

Options on traded assets. This class contains derivatives whose payoff depends on the future market value of traded securities: the payoff is of the form $\tilde{h}(L_{\tilde{T}}, \widehat{p}_{\tilde{T},1}, \ldots, \widehat{p}_{\tilde{T},N})$ to be paid at maturity $\tilde{T} \le T$. Examples include options on corporate bonds or calls and puts on CDS indices.

Denote by $\mathcal{M} = \{\boldsymbol{\pi} \ge 0 \colon \sum_{k \in S^X} \pi_k = 1\}$ the unit simplex in $\mathbb{R}^K$. Inspecting the pricing formula (4.1) it turns out that the payoff of the option can be written in the form $h\,(L_{\tilde{T}}, \boldsymbol{\pi}_{\tilde{T}})$, where h is implicitly defined by $\tilde{h}(L_{\tilde{T}}, \widehat{\mathbf{p}}_{\tilde{T}}) = h\,(L_{\tilde{T}}, \boldsymbol{\pi}_{\tilde{T}})$. Since the market state $(L, \boldsymbol{\pi})$ is a $\mathbb{F}^{\mathbb{M}}$-Markov process, the price of the option at time t satisfies

$$\mathbb{E}\left(h(L_{\tilde{T}}, \boldsymbol{\pi}_{\tilde{T}})|\mathcal{F}_t^{\mathbb{M}}\right) =: h(t, L_t, \boldsymbol{\pi}_t), \tag{4.2}$$

for some $h \colon [0, \tilde{T}] \times [0, 1]^m \times \mathcal{M} \to \mathbb{R}$. By standard results the function h is a solution of the backward equation

$$\partial_t h(\cdot) + \mathscr{L} h(\cdot) = 0.$$

However, the market state is usually a high-dimensional process so that the practical computation of $h(\cdot)$ has to be based on Monte Carlo methods. Note that in contrast to the case of an option on the loss state case, here the function $h(\cdot)$ typically depends on the entire generator $\mathscr{L}$ of $(L, \boldsymbol{\pi})$ and hence on the form of $\mathbf{a}(\cdot)$; numerical examples are given in Section 5.

4.2 Calibration

Model calibration involves two separate tasks: on the one hand, at fixed current time t one needs to determine $\boldsymbol{\pi}_t$, the current value of the process $\boldsymbol{\pi}$. On the other hand, the model parameters, i.e. the generator matrix of X, $(q(i,k))_{1\le i,k\le K}$, and (parameters of) the functions $\mathbf{a}(\cdot)$ and $\lambda_i(\cdot), i=1,\dots,m$ need to be estimated. The latter task is context specific. In the following we therefore concentrate on the determination of $\boldsymbol{\pi}_t$. The key point is the observation that the set of all probability vectors consistent with the price information at a given point in time t can be described in terms of a set of linear inequalities. Details depend on the way the traded credit derivatives are quoted in practice, and we discuss zero coupon bonds and CDSs (see Example 3.1) as representative examples.

Zero-bond. Consider a zero coupon bond on firm i. Its hypothetical value prior to default in the underlying market model is given by

$$\mathbb{E}\Big(\exp(-\int_t^T \lambda_i(X_s)ds)|X_t=k\Big) =: p_i(t,k)$$

The precise form of this function is irrelevant here. Suppose that at t we observe bid and ask quotes $\underline{p}\le\overline{p}$ for the bond. In order to be consistent with this information, a solution $\boldsymbol{\pi}^*\in\mathcal{M}$ of the calibration problem at t needs to satisfy the linear inequalities

$$\underline{p}\le\sum_{k\in S^X} p_i(t,k)\pi_k^*\le\overline{p}\,.$$

Credit default swap. A CDS on firm i is quoted by its spread S_t. The spread is chosen in such a way that the market value of the contract is zero. In our setup this translates as follows. Let

$$\begin{aligned} V_i^{\mathrm{def}}(t,k) &:= \mathbb{E}\big(L_{T,i}|X_t=k, L_{t,i}=0\big),\\ V_i^{\mathrm{prem}}(t,k) &:= \sum_{t_j\in(t,T]}\mathbb{Q}\big(L_{t_j,i}=0|X_t=k,L_{t,i}=0\big). \end{aligned}\tag{4.3}$$

The quoted CDS spread solves for $t<\tau_i$

$$\sum_{k\in S^X}\pi_t^k\left(S_t\,V_i^{\mathrm{prem}}(t,k)-V_i^{\mathrm{def}}(t,k)\right)=0.$$

Suppose now that at time t we observe bid and ask spreads $\underline{S}\le\overline{S}$ for the CDS contract. Then $\boldsymbol{\pi}^*$ must satisfy the following two inequalities:

$$\begin{aligned}&\sum_{k\in S^X}\pi_k^*\big(\underline{S}V_i^{\mathrm{prem}}(t,k)-V_i^{\mathrm{def}}(t,k)\big)\le 0\,,\\ &\sum_{k\in S^X}\pi_k^*\big(\overline{S}V_i^{\mathrm{prem}}(t,k)-V_i^{\mathrm{def}}(t,k)\big)\ge 0\,.\end{aligned}$$

Standard linear programming techniques can be used to detect if the system of linear inequalities corresponding to the available market quotes is nonempty and to determine a solution $\boldsymbol{\pi}^* \in \mathcal{M}$. In case that there is more than one probability vector $\pi \in \mathcal{M}$ consistent with the given price information at time t, a unique solution $\boldsymbol{\pi}^*$ of the calibration problem can be determined by a suitable *regularization procedure.* More precisely, given a reference measure $\boldsymbol{\nu}$ on S^X and a distance d, $\boldsymbol{\pi}^*$ is given by

$$\boldsymbol{\pi}^* = \operatorname{argmin}\big\{d(\boldsymbol{\pi}, \boldsymbol{\nu}) : \boldsymbol{\pi} \text{ is consistent with the price information in } t\big\}.$$

A possible choice is to minimize relative entropy to the uniform distribution; in that case $d(\boldsymbol{\pi}, \boldsymbol{\nu}) \propto \sum_{k \in S^X} \pi_k \ln \pi_k$.

Calibration via filtering. The probability distribution $\boldsymbol{\pi}_t$ can alternatively be estimated by nonlinear filtering. For this one assumes that investors observe the prices of the traded securities in additive Gaussian noise. By (4.1), the market prices are functions of $\boldsymbol{\pi}_t$. In order to determine the conditional distribution of $\boldsymbol{\pi}_t$ given the information available to investors at time t one therefore has to solve a second nonlinear filtering problem with signal process $(\boldsymbol{\pi}_t)_{t\geq 0}$ and observations given by the loss state L and the noisy price information. For details on this approach we refer to Frey & Runggaldier (2008).

4.3 Hedging

Hedging is a key issue in the management of portfolios of credit derivatives. The market standard practice is to use sensitivity-based hedging strategies computed by ad hoc rules within the static base-correlation framework; see for instance Neugebauer (2006). Clearly, it is desirable to work with hedging strategies which are based on a methodologically sound approach instead. In this section we therefore use our results from Section 3 to derive model-based dynamic hedging strategies. We expect the market to be incomplete, as the prices of the traded credit derivatives follow a jump-diffusion process. In order to deal with this problem we use the concept of risk minimization as introduced by Föllmer & Sondermann (1986). The hedging of credit derivatives via risk minimization is also studied in Frey & Backhaus (2007) and Cont & Kan (2008), albeit in a different setup; other relevant contributions include the papers Laurent, Cousin & Fermanian (2007) or Bielecki, Jeanblanc & Rutkowski (2007).

We begin by recalling the notion of a risk-minimizing hedging strategy. Denote by $\widehat{\mathbf{g}} = (\widehat{g}_1, \ldots, \widehat{g}_n)^\top$ the vector of gains processes of the traded securities and by $\mathbf{v}_t = (v_t^{ij})_{1\leq i,j\leq N}$ their instantaneous quadratic variation as given in Theorem 3.4, and let $L^2(\widehat{\mathbf{g}}, \mathbb{F}^{\mathbb{M}})$ be the space of $\mathbb{F}^{\mathbb{M}}$-predictable processes $\boldsymbol{\theta}$ such that $\mathbb{E}(\int_0^T \boldsymbol{\theta}_s^\top \mathbf{v}_s \boldsymbol{\theta}_s ds) < \infty$. An *admissible strategy* is given by a pair $\varphi = (\boldsymbol{\theta}, \eta)$ where $\boldsymbol{\theta} \in L^2(\widehat{\mathbf{g}}, \mathbb{F}^{\mathbb{M}})$ and η is $\mathbb{F}^{\mathbb{M}}$-adapted. Moreover the value process $V_t = V_t(\varphi) = \boldsymbol{\theta}_t^\top \widehat{\mathbf{p}}_t + \eta_t$ is RCLL and $\mathbb{E}(\sup_{0\leq t\leq T} V_t^2) < \infty$. The *cost* process $C = C(\varphi)$ and the *remaining risk* process $R = R(\varphi)$ of the trading strategy φ are finally defined by

$$C_t = V_t - \int_0^t \boldsymbol{\theta}_s^\top d\widehat{\mathbf{g}}_s \quad \text{and} \quad R_t = \mathbb{E}\big((C_T - C_t)^2 | \mathcal{F}_t^{\mathbb{M}}\big), \quad t \le T.$$

Consider now a claim H with square integrable, $(\mathbb{F}^L \vee \mathbb{F}^{\widehat{\mathbf{p}}})$-adapted cumulative dividend stream D such as the credit derivatives considered in Section 4.1. An admissible strategy φ is called a *risk-minimizing hedging strategy* for H if $V_T(\varphi) = D_T$ and if moreover for any $t \in [0, T]$ and any admissible strategy $\tilde{\varphi}$ satisfying $V_T(\tilde{\varphi}) = D_T$ we have $R_t(\varphi) \le R_t(\tilde{\varphi})$.

Risk-minimization is well-suited for our setup as the ensuing hedging strategies are relatively easy to compute and as it suffices to know the risk-neutral dynamics of credit derivative prices. From a methodological point of view it might however be more natural to minimize the remaining risk under the historical probability measure. This would lead to alternative quadratic-hedging approaches; see for instance Schweizer (2001). However, the computation of the corresponding strategies becomes a very challenging problem. Moreover, it is quite hard to determine the dynamics of CDS and CDO spreads under the historical measure as this requires the estimation of historical default intensities.

Proposition 4.1 *Consider a claim H with cumulative dividend stream $D_T \in L^2(\Omega, \mathcal{F}_T^L \vee \mathcal{F}_T^{\widehat{\mathbf{p}}}, \mathbb{Q})$ and gains process $\widehat{g}_t^H = \mathbb{E}(D_T | \mathcal{F}_t^{\mathbb{M}})$. A risk-minimizing hedging strategy $\varphi = (\boldsymbol{\theta}, \eta)$ for H is given by*

$$\boldsymbol{\theta}_t = \mathbf{v}_t^{\text{inv}} \frac{d\langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle_t^{\mathbb{M}}}{dt} \quad \text{and } \eta_t = \widehat{g}_t^H - \theta_t^\top \widehat{\mathbf{p}}_t, \quad t \le T \tag{4.4}$$

where $\mathbf{v}_t^{\text{inv}}$ denotes the pseudo inverse of the instantaneous quadratic variation $\mathbf{v}_t$ and where $d\langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle_t^{\mathbb{M}}/dt$ is the predictable Lebesgue-density of $\langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle_t^{\mathbb{M}}$.

Proof It is well-known that risk-minimizing hedging strategies relate to the *Galtchouk-Kunita-Watanabe* decomposition of the martingale g^H with respect to the gains processes of traded securities:

$$\widehat{g}_t^H = \widehat{g}_0^H + \sum_{n=1}^N \int_0^t \xi_{s,n}^H \, d\widehat{g}_{s,n} + H_t^\perp, \quad t \le T \tag{4.5}$$

with $\xi_i^H \in L^2(\widehat{\mathbf{g}}, \mathbb{F}^{\mathbb{M}})$ and $\langle H^\perp, \widehat{\mathbf{g}} \rangle^{\mathbb{M}} \equiv 0$: one has that $\boldsymbol{\theta} = \xi^H$, $V_t(\varphi) = \widehat{g}_t^H$ and $C = H^\perp$. From $\langle H^\perp, \widehat{\mathbf{g}} \rangle^{\mathbb{M}} \equiv 0$ we get the following equation for ξ^H:

$$\frac{\langle \widehat{g}^H, \widehat{g}_j \rangle_t^{\mathbb{M}}}{dt} = \sum_{n=1}^N \xi_{t,j}^H v_t^{n,j}, \quad t \le T; \tag{4.6}$$

by definition of $\mathbf{v}_t^{\text{inv}}$ a solution of (4.6) is given by $\mathbf{v}_t^{\text{inv}} \frac{d\langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle_t^{\mathbb{M}}}{dt}$. □

The crucial step in applying Proposition 4.1 is to compute the quadratic variation $\langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle^{\mathbb{M}}$, and we now explain how this can be achieved for the claims considered in Subsection 4.1. First, if H represents an option on the loss state, by an argument analogous to the proof of Theorem 3.4 one obtains that $\widehat{g}_t^H$ has a representation of the form (3.17) with integrands $\boldsymbol{\alpha}^H$ and $\boldsymbol{\gamma}^H$ given by the analogous expressions to (3.18) and (3.19). Then, $\langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle^{\mathbb{M}}$ is given by

$$d\langle \widehat{g}^H, \widehat{g}_i \rangle_t^{\mathbb{M}} = \Big(\sum_{j=1}^{m} \int_0^1 \Big(\gamma_j^H(t,l) \gamma_j^{\widehat{g}_i}(t,l) \Big) F_{l_j}(dl) \widehat{\lambda}_{t-,j} (1 - Y_{t-,j}) \tag{4.7}$$

$$+ \sum_{j=1}^{l} \alpha_{t-,j}^H \alpha_{t-,j}^{\widehat{g}_i} \Big) dt, \quad 1 \leq i \leq N. \tag{4.8}$$

The main step in computing $\boldsymbol{\alpha}^H$ and $\boldsymbol{\gamma}^H$ is to compute the function $p(t,k,L)$ that gives the hypothetical value of the derivative in the underlying Markov model; a numerical example is presented in Section 5.

Second, if the option H offers the payoff $\tilde{h}(\widehat{\mathbf{p}}_S, L_S)$ for $S \leq T$, we have $\widehat{g}_t^H = h(t, L_t, \boldsymbol{\pi}_t)$, compare (4.2). Applying Itô's formula to $h(t, L_t, \boldsymbol{\pi}_t)$ gives a martingale representation of g^H, see the proof of Lemma 3.8 in the appendix and the related comment A.1. From this $[\widehat{g}^H, \widehat{g}_n]$ its compensator $\langle \widehat{g}^H, \widehat{g}_n \rangle^{\mathbb{M}}$ can be computed via standard arguments. Note finally that in both cases $\boldsymbol{\theta}_t$ depends only on the current market state $(L_t, \boldsymbol{\pi}_t)$.

5 Numerical case studies

In this section we present results from a number of small numerical and empirical case studies that serve to further illustrate the application of the model to practical problems.

The setup. We assume throughout that the factor process is in fact a random variable, $X_t \equiv X$ for all t. In this case the filtering problem in Section 3 reduces to Bayesian analysis and the process $\boldsymbol{\pi} = (\boldsymbol{\pi}_t)_{t \geq 0}$ of conditional probabilities solves the SDE (3.22) with $q(i,k) = 0$ for all $i, j \in S^X$. A model of this type is known as *frailty model*; see also Schönbucher (2004) and Collin-Dufresne et al. (2003). The frailty model has a number of attractive features: first, defaults are independent given $X = k$ so that computing the hypothetical value of traded securities $p_n(t,k,L)$ introduced in (3.4) is particularly easy; moreover, there is no need to calibrate or estimate the generator matrix of X. In the frailty model the conditional survival function of $(\tau_1, \ldots, \tau_m)$ takes the simple form

$$\mathbb{Q}\big(\tau_1 > t_1, \ldots, \tau_m > t_m \mid \mathcal{F}_t^{\mathbb{M}}\big) = \sum_{k \in S^X} \pi_t^k \prod_{i=1}^{m} \boldsymbol{e}^{-\lambda_i(k)(t_i - t)}, \quad t_i \geq t,$$

assuming $t < T_1$ for simplicity. A static model of this type (no dynamics for the process $\boldsymbol{\pi}$) has been studied by Hull & White (2006) under the label implied copula model. Recall from the previous section that the price of options on the loss state such as CDS-indices and CDO tranches is independent of the model for the dynamics of $\boldsymbol{\pi}$. Hence for these products pricing and calibration in the frailty model and in the Hull & White (2006)-model coincides. However, our framework permits us to study model-based hedging strategies as well as the pricing of options on the market state. Both issues cannot be addressed in the static implied copula model.

Now we turn to a number of specific applications of the frailty model.

1. Credit-spread dynamics. The fact that in our model prices of traded securities are given by the conditional expectation given the market filtration leads to rich credit-spread dynamics with spread risk (random fluctuations of credit spreads between defaults) and default contagion. This is illustrated in Figure 5.1 where we plot a simulated of credit-spread trajectory. The random fluctuation of the credit spreads between defaults as well contagions effects at default times can be spotted clearly.

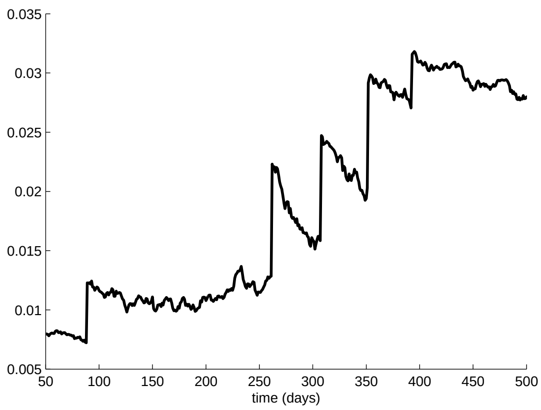

Fig. 5.1 A simulated path of credit spreads under zero recovery.

2. Calibration to observed CDO spreads. Next we present an example for a calibration of the model to observed tranche and index spreads of the itraxx. We consider a homogeneous model with $|S^X| = 9$; the values of the one-year default intensity are given in Table 5.1 below. The model was calibrated to tranche and index spread data from 2004, 2006, 2008 and 2009; the market data can be found in Table B.1 in the appendix. The data from 2004 and 2006 are typical for tranche and index spreads before the credit crisis; the data from 2008 and 2009 on the other hand represent the state of the market during the crisis. In order to determine a solution $\boldsymbol{\pi}^*$ of the calibration problem we use the methodology described in Section 4.2, with very satisfactory results. The resulting values for $\boldsymbol{\pi}$ are given in Table 5.1, the spreads generated by the model can be found in Table B.1. We clearly see that with the emergence of the credit crisis the calibration procedure puts more mass on states where the default intensity is high; in particular, the extreme state where $\lambda = 70\%$ gets a probability of around 3%. This reflects the increased awareness of future defaults and the increasing risk aversion in the market after the advent of the crisis.

λ (in %)	0.01	0.3	0.6	1.2	2.5	4.0	8.0	20	70
$\boldsymbol{\pi}^*$, data from 2004	12.6	22.9	42.0	17.6	2.5	1.45	0.54	0.13	0.03
$\boldsymbol{\pi}^*$, data from 2006	22.2	29.9	39.0	7.6	1.2	0.16	0.03	0.03	0.05
$\boldsymbol{\pi}^*$, data from 2008	1.1	7.9	57.6	10.8	11.7	4.9	1.26	1.79	2.60
$\boldsymbol{\pi}^*$, data from 2009	0.0	13.6	6.35	42.2	22.3	12.5	0.0	0.00	3.06

Table 5.1 Results of the calibration to itraxx spread data for different data sets from several years; the components of $\boldsymbol{\pi}^*$ are given in percentage points. The spread data used in the calibration are given in Table B.1 in the appendix.

3. Hedging of CDO tranches. Next we consider the hedging of synthetic CDO tranches using the underlying CDS index as hedging instrument. As explained in Section 4.3, the form of the diffusion part in the dynamics of the process $\boldsymbol{\pi}$, which is essentially determined by the function$a(\cdot)$ from **A2**, does have an impact on the form of the hedge ratios generated within the model. Here we want to study this impact numerically. We use the same values for the one-year default intensity as in the previous paragraph. We take $l = 1$ in **A2** and assume that $a(x) = c\ln(\lambda(x))$. For c small, the process $\boldsymbol{\pi}$ and hence also the $\mathbb{F}^{\mathbb{M}}$ default intensity $\widehat{\lambda}$ evolves in an almost deterministic fashion between defaults; for c large on the other hand $\boldsymbol{\pi}$ and $\widehat{\lambda}$ exhibit stronger fluctuations. The hedge ration θ_t giving the number of CDS index contracts to be held in the portfolio was computed from Proposition 4.1 using relation (4.7); numerical results are given in Table 5.2. We see that θ is affected by c and that higher values of c mostly lead to larger hedge ratios. In practice an analysis of the time series properties of CDS and CDO spreads is called for to determine an appropriate value of c; this is however beyond the scope of the present paper.

4. Pricing of options on a CDS index. Options on a CDS index are a typical example for an option on traded asset. Denote $\tilde{T}$ the maturity of the contract. Upon exercise the owner of the option holds a protection-buyer position on the underlying CDS index with a pre-specified spread S (the exercise spread of the option); moreover, he obtains the cumulative loss $\bar{L}_{\tilde{T}}$. Denote by $V^{\text{def}}(t, X_t, L_t)$ and $V^{\text{prem}}(t, X_t, L_t)$ the hypothetical value of the default and the premium payment leg of the CDS index in the underlying Markov model. In our setup

Tranche	[0-3]	[3-6]	[6-9]	[9-12]	[12-22]
θ, $c = 0.5$	0.0674	0.0376	0.0359	0.0342	0.1073
θ, $c = 6$	0.0948	0.0516	0.0436	0.0370	0.1055

Table 5.2 Risk-minimizing hedge ratio θ for hedging a CDO tranche with the underlying CDS index. The numbers were computed using the probability vector $\boldsymbol{\pi}^*$ obtained via calibration to the itraxx data from 2008.

c	0.5	1	2
moneyness $S/S_0 = 0.9$	2.33	2.50	2.66
moneyness $S/S_0 = 1$	1.93	2.12	2.35
moneyness $S/S_0 = 1.1$	1.61	1.77	2.05

Table 5.3 Prices of a call on the CDS index with $\lambda = (0.015, 0.06)$. We state the prices for $c \in \{0.5, 1, 2\}$ and moneyness $S/S_0 \in \{0.9, 1, 1.1\}$.

the value of the option at maturity is then given by the following function of the market state at $\tilde{T}$:

$$h(L_{\tilde{T}}, \boldsymbol{\pi}_{\tilde{T}}) = \Big(\bar{L}_{\tilde{T}} + \sum_{k \in S^X} \pi_{\tilde{T}}^K \big(V^{\mathrm{def}}(\tilde{T}, k, L_{\tilde{T}}) - SV^{\mathrm{prem}}(\tilde{T}, k, L_{\tilde{T}})\big)\Big)^+,$$

using the notation from (4.3). In practice this contract is usually priced by an application of the Black formula; see Brigo & Morini (2007) for further details.

The framework developed in the present paper leads to an alternative approach for pricing pricing this product. In this example we use a very simple version of the frailty model where the default intensity can take only two values; $\boldsymbol{\pi}^*$ is then uniquely determined by calibrating the model to the index spread at time $t = 0$. As before we assume that $a(x) = c \ln(\lambda(x))$ for varying values of c. Prices were computed via Monte-Carlo simulation; numerical results are presented in Table 5.3. Note that with increasing c the volatility of the index spread increases which leads to an increase in the prices of the option.

A Proofs

A.1 Proof of Lemma 3.2

The proof goes in three steps. First, we introduce a new measure $\mathbb{Q}^*$, so that under $\mathbb{Q}^*$ the $\mathbb{F}^{\mathbb{M}}$-compensator of μ^L is independent of X, and Z is a $\mathbb{Q}^*$-Brownian motion. Next, we use available martingale representation results under $\mathbb{Q}^*$ and finally we change back to the original measure $\mathbb{Q}$.

In the following, we simply write $\mathbf{a}_s := \mathbf{a}(X_s)$. Define the density martingale

$$\eta_t := \prod_{T_n \le t} (\widehat{\lambda}_{T_n-,\xi_n})^{-1} \exp\Bigg(\int_0^t \sum_{i=1}^m (1 - Y_{s,i})(\widehat{\lambda}_{s-,i} - 1) ds - \int_0^t \widehat{\mathbf{a}}_s^\top dm_s^Z - \frac{1}{2} \int_0^t \|\widehat{\mathbf{a}}_s\|^2 \, ds \Bigg), \ t \le T,$$

and note that the dynamics of η is

$$d\eta_t = \eta_{t-} \left(\sum_{i=1}^m ((\widehat{\lambda}_{t-,i})^{-1} - 1)\big(dY_{t,i} - \widehat{\lambda}_{t,i}(1 - Y_{t,i})dt\big) - \widehat{\mathbf{a}}_t^\top dm_t^Z \right).$$

By **A1**, $\lambda_j > 0$. As S^X is finite, the functions $\lambda, \widehat{\lambda}, \widehat{\lambda}^{-1}$, and $\widehat{\mathbf{a}}$ are bounded, hence η is a true martingale; see for instance Protter (2004), Exercise V.14. Define a measure $\mathbb{Q}^*$ by $d\mathbb{Q}^*/d\mathbb{Q}_{|\mathcal{F}_T^{\mathbb{M}}} = \eta_T$. Then, by the Girsanov theorem, Z is a $\mathbb{Q}^*$-Brownian motion and the $\mathbb{F}^{\mathbb{M}}$-compensator of μ^L under $\mathbb{Q}^*$ is

$$\widehat{\nu}^*(dt, de) := \sum_{i=1}^m \delta_{\{i\}}(d\xi)\, F_{\ell_i}(d\ell)\, (1 - Y_{t-,i})dt.$$

Consider now the $(\mathbb{Q}, \mathbb{F}^{\mathbb{M}})$-martingale U and define the $\mathbb{Q}^*$-integrable random variable $N_T := U_T \eta_T^{-1}$ and the associated martingale $N_t = \mathbb{E}^{\mathbb{Q}^*}(N_T \mid \mathcal{F}_t^{\mathbb{M}})$. Note that by the Bayes formula,

$$N_t = \frac{1}{\eta_t} \mathbb{E}^{\mathbb{Q}}(N_T \eta_T \mid \mathcal{F}_t^{\mathbb{M}}) = \frac{1}{\eta_t} \mathbb{E}^{\mathbb{Q}}(U_T \mid \mathcal{F}_t^{\mathbb{M}}) = \frac{U_t}{\eta_t}.$$

Due to the independence of Z and $m^* := \mu^L - \widehat{\nu}^*$ we have a martingale representation of the $\mathbb{Q}^*$-martingale $(N_t)_{0 \le t \le T}$ using standard representation results such as Jacod & Shiryaev (2003), Theorem III.4.34:

$$N_t = \mathbb{E}(U_T) + \int_0^t \tilde{\boldsymbol{\alpha}}_s^\top dZ_s + \int_0^t \int_E \tilde{\gamma}(s, e) m^*(ds, de).$$

The final step is to compute the differential of $U_t = \eta_t N_t$. We have

$$\begin{aligned}
dU_t &= d(\eta_t N_t) = \eta_{t-} dN_t + N_{t-} d\eta_t + d[\eta, N]_t \\
&= \eta_{t-} \tilde{\boldsymbol{\alpha}}_t^\top \big(dm_t^Z + \widehat{\mathbf{a}}_t dt\big) + \int_E \eta_{t-} \tilde{\gamma}(t, e) m^*(dt, de) \\
&+ \sum_{i=1}^m N_{t-} \eta_{t-} ((\widehat{\lambda}_{t-,i})^{-1} - 1)\big(dY_{t,i} - \widehat{\lambda}_{t,i}(1 - Y_{t,i})dt\big) - \eta_{t-} N_{t-} \widehat{\mathbf{a}}_t^\top dm_t^Z - \eta_{t-} \widehat{\mathbf{a}}_t^\top \tilde{\boldsymbol{\alpha}}_t dt \\
&+ \int_E \tilde{\gamma}(t, \xi, \ell) \eta_{t-} (\widehat{\lambda}_{t-,\xi}^{-1} - 1)\, \mu^L(dt, d\xi, d\ell).
\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}
dU_t &= \eta_{t-} \left(\tilde{\boldsymbol{\alpha}}_t^\top - N_{t-} \widehat{\mathbf{a}}_t^\top \right) dm_t^Z \\
&+ \int_E \eta_{t-} \left(N_{t-}((\widehat{\lambda}_{t-,\xi})^{-1} - 1) + (\widehat{\lambda}_{t-,\xi})^{-1} \tilde{\gamma}(t, \xi, \ell) \right) m^L(dt, d\xi, d\ell),
\end{aligned}$$

which is the desired martingale representation for U. □

A.2 Proof of Lemma 3.8

First, we identify the generator of the process $(L,\boldsymbol{\pi})$. Consider $f:[0,T]\times[0,1]^m\times\mathcal{M}\to\mathbb{R}$, sufficiently regular. The Itô formula gives

$$\begin{aligned}
f(t,L_t,\boldsymbol{\pi}_t) = f(0,L_0,\boldsymbol{\pi}_0) &+ \int_0^t \partial_t f(\cdot)ds + \sum_{k\in S^X}\int_0^t \partial_{\pi^k} f(\cdot)d\pi^k_s \\
&+ \frac{1}{2}\sum_{k,l\in S^X}\int_0^t \partial_{\pi^k}\partial_{\pi^l} f(\cdot)d[\pi^k,\pi^l]^c_s \\
&+ \sum_{T_n\le t}\Big(f(T_n,L_{T_n},\boldsymbol{\pi}_{T_n}) - f(T_n,L_{T_n-},\boldsymbol{\pi}_{T_n-}) - \sum_{k\in S^X}\partial_{\pi^k} f(\cdot)\Delta\pi^k_{T_n}\Big),
\end{aligned}$$

where $f(\cdot)$ stands for $f(s,L_{s-},\boldsymbol{\pi}_{s-})$. With (3.22) we obtain

$$\begin{aligned}
\sum_{k\in S^X}\int_0^t \partial_{\pi^k} f(\cdot)d\pi^k_s = &\sum_{k\in S^X}\int_0^t \partial_{\pi^k} f(\cdot)\boldsymbol{\alpha}^k(\boldsymbol{\pi}_s)^\top dm^Z_s \\
&+ \sum_{k\in S^X}\int_0^t \partial_{\pi^k} f(\cdot)\boldsymbol{\gamma}^k(\boldsymbol{\pi}_{s-})^\top dM_s \\
&+ \sum_{k\in S^X}\int_0^t \partial_{\pi^k} f(\cdot)\Big(\sum_{i\in S^X} q(i,k)\pi^i_s\Big)ds.
\end{aligned}$$

Letting $c_{lk}(\boldsymbol{\pi}) := \boldsymbol{\alpha}^k(\boldsymbol{\pi})^\top\boldsymbol{\alpha}^l(\boldsymbol{\pi})$, we have that $d[\pi^k,\pi^l]^c_s = c_{lk}(\boldsymbol{\pi}_s)ds$. Finally, with $\mathbf{e}_i$ being the i-th unit vector in $\mathbb{R}^m$,

$$\begin{aligned}
&\sum_{T_n\le t}\Big(f(T_n,L_{T_n},\boldsymbol{\pi}_{T_n}) - f(T_n,L_{T_n-},\boldsymbol{\pi}_{T_n-}) - \sum_{k\in S^X}\partial_{\pi^k} f(\cdot)\Delta\pi^k_{T_n}\Big) \\
&= \int_0^t \sum_{i=1}^m \mathbb{1}_{\{\xi=i\}}\Big(f\Big(s, L_{s-}+\ell\mathbf{e}_i, \pi^1_{s-}\frac{\lambda_i(1)}{(\widehat{\lambda}_i)_{s-}},\dots,\pi^K_{s-}\frac{\lambda_i(K)}{(\widehat{\lambda}_i)_{s-}}\Big) - f(s,L_{s-},\boldsymbol{\pi}_{s-}) \\
&\qquad\qquad + \sum_{k\in S^X}\partial_{\pi^k} f(\cdot)\pi^k_{s-}\big(\frac{\lambda_i(k)}{(\widehat{\lambda}_i)_{s-}} - 1\big)\Big)\mu^L(ds,d\xi,d\ell).
\end{aligned}$$

Let $\widehat{\lambda}_i(\boldsymbol{\pi}) := \sum_{l\in S^X}\lambda_i(k)\pi^l$. The above shows that $f(t,L_t,\boldsymbol{\pi}_t) - \int_0^t \mathscr{L}f(s,L_s,\boldsymbol{\pi}_s)ds$ is a (local) martingale where

$$\begin{aligned}
\mathscr{L}f(t,L,\boldsymbol{\pi}) = &\sum_{k\in S^X}\partial_{\pi^k} f(t,L,\boldsymbol{\pi})\Big(\sum_{i\in S^X} q(i,k)\pi^i\Big) \\
&+ \frac{1}{2}\sum_{k,l\in S^X}\partial_{\pi^k}\partial_{\pi^l} f(t,L,\boldsymbol{\pi})c_{lk}(\boldsymbol{\pi}) \\
&+ \sum_{i=1}^m \mathbb{1}_{\{\ell_i=0\}}\sum_{k\in S^X}\partial_{\pi^k} f(t,L,\boldsymbol{\pi})\pi^k\big(\lambda_i(k) - \widehat{\lambda}_i(\boldsymbol{\pi})\big) \qquad\qquad (A.1) \\
&+ \sum_{i=1}^m \mathbb{1}_{\{\ell_i=0\}}\widehat{\lambda}_i(\boldsymbol{\pi})\int_0^1 \Big(f\Big(t, L+\ell\mathbf{e}_i, \pi^1\frac{\lambda_i(1)}{\widehat{\lambda}_i(\boldsymbol{\pi})},\dots,\pi^K\frac{\lambda_i(K)}{\widehat{\lambda}_i(\boldsymbol{\pi})}\Big) - f(t,L,\boldsymbol{\pi})\Big)F_{\ell_i}(d\ell).
\end{aligned}$$

By standard results, $(L, \boldsymbol{\pi})$ is a Markov process if the martingale problem associated to the operator $\mathscr{L}$ has a unique solution. By Theorem III.2.26 from Jacod & Shiryaev (2003) this is equivalent to uniqueness in law of an associated SDE. The latter follows from Theorem 2.2 in Kliemann, Koch & Marchetti (1990). □

Remark A.1 The above proof shows additionally, that the martingale part of $f(t, L_t, \boldsymbol{\pi}_t)$ can be written as

$$\sum_{k \in S^X} \int_0^t \partial_{\pi^k} f(\cdot) \boldsymbol{\alpha}^k(\boldsymbol{\pi}_s)^\top dm_s^Z + \sum_{k \in S^X} \int_0^t \partial_{\pi^k} f(\cdot) \boldsymbol{\gamma}^k(\boldsymbol{\pi}_{s-})^\top dM_s$$
$$+ \int_0^t \sum_{i=1}^m \mathbb{1}_{\{\xi = i\}} \Big(f\Big(s, L_{s-} + \ell \mathbf{e}_i, \pi^1_{s-} \frac{\lambda_i(1)}{(\widehat{\lambda}_i)_{s-}}, \dots, \pi^K_{s-} \frac{\lambda_i(K)}{(\widehat{\lambda}_i)_{s-}}\Big) - f(s, L_{s-}, \boldsymbol{\pi}_{s-})$$
$$+ \sum_{k \in S^X} \partial_{\pi^k} f(\cdot) \pi^k_{s-} \Big(\frac{\lambda_i(k)}{(\widehat{\lambda}_i)_{s-}} - 1 \Big) \Big) m^L(ds, d\xi, d\ell).$$

B Data

	Tranche	[0-3]	[3-6]	[6-9]	[9-12]	[12-22]	Index
2004	market	0.2760	0.0168	0.0070	0.0043	0.0020	0.0042
	model	0.2739	0.0169	0.0069	0.0043	0.0020	0.0042
2006	market	0.1450	0.0062	0.0018	0.0007	0.0003	0.0026
	model	0.1441	0.0062	0.0018	0.0007	0.0003	0.0026
2008	market	0.4650	0.0567	0.0370	0.0235	0.0145	0.0150
	model	0.4615	0.0571	0.0367	0.0237	0.0144	0.0149
2009	market	0.6700	0.1148	0.0595	0.0340	0.0105	0.0175
	model	0.6795	0.1165	0.0604	0.0335	0.0115	0.0173

Table B.1 The table gives the market date of quoted itraxx spreads in the years 2004, 2006, 2008 and 2009. Moreover we give the model spreads corresponding to the calibrated values $\boldsymbol{\pi}^*$ from Table 5.1. We clearly see that the model spreads are very close to the market spreads for all four data sets.

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