The purpose of this note is to give a proof of the main result of [FLM] (i.e. of Theorem 2, which directly implies Theorem 1) in the case of arrangements of rank two. The result reduces to a non-trivial but elementary calculation. In particular, the commutative algebra results of [FL1] are not necessary in this case.

Since it does not make any difference, we will not restrict to root arrangements but consider general line arrangements in a two-dimensional real vector space. It is not clear whether the validity of Theorems 1 and 2 of [FLM] does really depend on the assumption that the underlying arrangement is a root arrangement. Using the methods of [FLM], it is in fact not difficult to see that Theorems 1 and 2 are also true for every hyperplane arrangement of rank three and every simplicial arrangement of rank four. In [FL2] we formulate a conjectural generalization of Theorem 1 to arbitrary simplicial fans.

Let $V$ be a real vector space of dimension two and $V^*$ its dual space. Let $\beta : \bigwedge^2 V \to \mathbb{R}$ be a fixed isomorphism. The choice of $\beta$ defines an oriented volume element on $V$. Let $\mathcal{A}$ be a finite line arrangement in $V^*$ given by the lines

$$\langle \lambda, \alpha_i^\vee \rangle = 0, \quad i = 1, \dots, N,$$

with pairwise non-collinear vectors $\alpha_i^\vee \in V \setminus \{0\}$. Let $\mathcal{P}$ be the set of connected components of the complement of these lines. The elements of $\mathcal{P}$ are called chambers. We assume in addition that the vectors $\alpha_i^\vee$ are oriented in such a way that there exists $\lambda_0 \in V^*$ with $\langle \lambda_0, \alpha_i^\vee \rangle > 0$ for all $i$. Fix a vector $\lambda_0$ with this property and denote the associated chamber by $P_0$. We order the vectors $\alpha_i^\vee$ in the counterclockwise direction, i.e. we require that $v_{ij} = \beta(\alpha_i^\vee \wedge \alpha_j^\vee) > 0$ for $1 \le i < j \le N$.

For each $P \in \mathcal{P}$ let $\Sigma_P^\vee$ be the subset of those functionals in the set $\{\pm\alpha_1^\vee, \dots, \pm\alpha_N^\vee\}$ which are positive on $P$ and $\Delta_P^\vee \subseteq \Sigma_P^\vee$ the two-element subset of functionals defining the walls of $P$. Set $\Sigma_{P_0;P}^\vee = \Sigma_{P_0}^\vee \cap \Sigma_{\bar{P}}^\vee$ for all $P \in \mathcal{P}$. We can order the chambers in counterclockwise direction as $P_0$, $P_1$, …, $P_{N-1}$, $\bar{P}_0$, $\bar{P}_1$, …, $\bar{P}_{N-1}$, where the bar over a symbol denotes the opposite chamber. Then $\Delta_{P_0}^\vee = \{\alpha_1^\vee, \alpha_N^\vee\}$ and $\Delta_{P_i}^\vee = \{-\alpha_i^\vee, \alpha_{i+1}^\vee\}$ for $1 \le i \le N-1$. Furthermore, $\Sigma_{P_0;P_i}^\vee = \{\alpha_1^\vee, \dots, \alpha_i^\vee\}$ and $\Sigma_{P_0;\bar{P}_i}^\vee = \{\alpha_{i+1}^\vee, \dots, \alpha_N^\vee\}$.

There are precisely two galleries from $P_0$ to $\bar{P}_0$ in the line arrangement $\mathcal{A}$, namely $\mathcal{G}_1 : P_0, P_1, \dots, P_{N-1}, \bar{P}_0$ and $\mathcal{G}_2 : P_0, \bar{P}_{N-1}, \dots, \bar{P}_1, \bar{P}_0$ (cf. [FLM, Lemma 2]). Every $P \in \mathcal{P} \setminus \{P_0, \bar{P}_0\}$ is contained in precisely one of the galleries $\mathcal{G}_i$. For $P \in \mathcal{P}$ and $k = 1$ or $2$ let $\mathcal{G}_{P,k}$ be the unique gallery containing $P$ if $P$ is different from $P_0$ and $\bar{P}_0$, and set $\mathcal{G}_{P_0,k} = \mathcal{G}_{3-k}$, $\mathcal{G}_{\bar{P}_0,k} = \mathcal{G}_k$.

Let $\mathfrak{s}^\vee$ be the polynomial ring in the independent variables $\varpi_1, \dots, \varpi_N$, which are in bijection with the elements of $\Sigma^\vee_{P_0}$, and $\mathcal{S}^\vee \simeq (\mathfrak{s}^\vee)^2$ the free module of maps $\mathfrak{X} \to \mathfrak{s}^\vee$, where $\mathfrak{X} = \{\mathcal{G}_1, \mathcal{G}_2\}$ is the set of all galleries from $P_0$ to $\bar{P}_0$. Let $\mathrm{Rel}^\perp$ be the subset of the vector space $\mathfrak{s}^\vee_1$ consisting of all elements of the form

$$\sum_{i=1}^N \langle \eta, \alpha_i^\vee \rangle \varpi_i, \quad \eta \in V^*.$$

The relation space $\mathcal{R} \subseteq \mathcal{S}^\vee$ is then the set of all elements of the form $r(\mathbf{1}_{\mathcal{G}_1} - \mathbf{1}_{\mathcal{G}_2})$ with $r \in \mathrm{Sym}(\mathrm{Rel}^\perp) \subseteq \mathfrak{s}^\vee$.

In the space $\mathcal{S}^\vee_2$ we consider the element

$$\mathfrak{d} = \frac{1}{2} \sum_{1 \le i < j \le N} v_{ij} \varpi_i \varpi_j (\mathbf{1}_{\mathcal{G}_1} + \mathbf{1}_{\mathcal{G}_2}),$$

and for any $\eta \in V^*$ such that $\langle \eta, \alpha_i^\vee \rangle \neq 0$ for all $i$ and any $k \in \{1, 2\}$ the element

$$\mathfrak{c}_{\eta;k} = \frac{1}{2} \sum_{P \in \mathcal{P}} v(\Delta_P^\vee) \frac{\left( \sum_{i:\, \alpha_i^\vee \in \Sigma^\vee_{P_0;P}} \langle \eta, \alpha_i^\vee \rangle \varpi_i \right)^2}{\prod_{\alpha^\vee \in \Delta_P^\vee} \langle \eta, \alpha^\vee \rangle} \mathbf{1}_{\mathcal{G}_{P,k}}.$$

Here, for a two-element subset $\Delta^\vee = \{v_1, v_2\} \subseteq V$ we write $v(\Delta^\vee) = |\beta(v_1 \wedge v_2)|$. Note that in the case of root arrangements considered in [FLM], the factors $v(\Delta_P^\vee)$ can be omitted if we assign the coroot lattice covolume one. For this reason they do not appear in [FLM]. One observes that the element $\mathfrak{d}$ is the common value of the $\mathfrak{d}_{\underline{\xi}}$ considered in [FLM] (regardless of $\underline{\xi}$), while the elements $\mathfrak{c}_{\eta;k}$ are the possible values of the expressions $\mathfrak{c}_{\eta;(\mu_P)_P}$ there for varying parameters $(\mu_P)_P$.

The assertion of [FLM, Theorem 2] is that for any $\eta$ and $k$ the difference $\mathfrak{c}_{\eta;k} - \mathfrak{d}$ is an element of $\mathcal{R}_2$. We will prove this by an explicit calculation. We first use the explicit information on the set $\mathcal{P}$ summarized above to rewrite the formula for $\mathfrak{c}_{\eta;k}$ in the form

$$\begin{aligned}\mathfrak{c}_{\eta;k} &= -\frac{1}{2}\sum_{i=1}^{N-1}\frac{v_{i,i+1}}{\langle\eta,\alpha_i^\vee\rangle\langle\eta,\alpha_{i+1}^\vee\rangle} \\ &\left(\left(\sum_{j=1}^{i}\langle\eta,\alpha_j^\vee\rangle\varpi_j\right)^2\mathbf{1}_{\mathcal{G}_1}+\left(\sum_{j=i+1}^{N}\langle\eta,\alpha_j^\vee\rangle\varpi_j\right)^2\mathbf{1}_{\mathcal{G}_2}\right) \\ &+\frac{1}{2}\frac{v_{1N}}{\langle\eta,\alpha_1^\vee\rangle\langle\eta,\alpha_N^\vee\rangle}\left(\sum_{j=1}^{N}\langle\eta,\alpha_j^\vee\rangle\varpi_j\right)^2\mathbf{1}_{\mathcal{G}_k}.\end{aligned}$$

One observes immediately that $\mathfrak{c}_{\eta;1}-\mathfrak{c}_{\eta;2}\in\mathcal{R}_2$, which is consistent with the main assertion. To proceed further, we need the following simple identity.

**Lemma 1.** *For $1\le i<j\le N$ we have*

$$\sum_{k=i}^{j-1}\frac{v_{k,k+1}}{\langle\eta,\alpha_k^\vee\rangle\langle\eta,\alpha_{k+1}^\vee\rangle}=\frac{v_{ij}}{\langle\eta,\alpha_i^\vee\rangle\langle\eta,\alpha_j^\vee\rangle}.$$

*Proof.* Use induction on $j$ for fixed $i$, the case $j=i+1$ being trivial. The fact that $v_{ij}=\beta(\alpha_i^\vee\wedge\alpha_j^\vee)$ for $i<j$ implies that

$$v_{ij}\alpha_k^\vee+v_{jk}\alpha_i^\vee=v_{ik}\alpha_j^\vee,\quad i\le j\le k. \tag{1}$$

As a special case we have $v_{i,j-1}\alpha_j^\vee+v_{j-1,j}\alpha_i^\vee=v_{ij}\alpha_{j-1}^\vee$. From this we get

$$\frac{v_{i,j-1}}{\langle\eta,\alpha_i^\vee\rangle\langle\eta,\alpha_{j-1}^\vee\rangle}+\frac{v_{j-1,j}}{\langle\eta,\alpha_{j-1}^\vee\rangle\langle\eta,\alpha_j^\vee\rangle}=\frac{v_{ij}}{\langle\eta,\alpha_i^\vee\rangle\langle\eta,\alpha_j^\vee\rangle},$$

which is precisely what is needed for the induction step.

Write $\mathfrak{c}_{\eta;2}=c_{\eta1}\mathbf{1}_{\mathcal{G}_1}+c_{\eta2}\mathbf{1}_{\mathcal{G}_2}$ with $c_{\eta1}$, $c_{\eta2}\in\mathfrak{s}_2^\vee$. We can now collect the monomials in the $\varpi_i$ in $c_{\eta1}$ and $c_{\eta2}$. Using the Lemma, the coefficient of $\varpi_i^2$ in $c_{\eta1}$ is

$$-\frac{\langle\eta,\alpha_i^\vee\rangle^2}{2}\sum_{k=i}^{N-1}\frac{v_{k,k+1}}{\langle\eta,\alpha_k^\vee\rangle\langle\eta,\alpha_{k+1}^\vee\rangle}=-\frac{v_{iN}\langle\eta,\alpha_i^\vee\rangle}{2\langle\eta,\alpha_N^\vee\rangle}.$$

The coefficient of $\varpi_i\varpi_j$, $1\le i<j\le N$, in $c_{\eta1}$ is

$$-\langle\eta,\alpha_i^\vee\rangle\langle\eta,\alpha_j^\vee\rangle\sum_{k=j}^{N-1}\frac{v_{k,k+1}}{\langle\eta,\alpha_k^\vee\rangle\langle\eta,\alpha_{k+1}^\vee\rangle}=-\frac{v_{jN}\langle\eta,\alpha_i^\vee\rangle}{\langle\eta,\alpha_N^\vee\rangle}.$$

To sum up,

$$c_{\eta 1} = -\frac{1}{2}\sum_{i=1}^{N} v_{iN}\frac{\langle \eta, \alpha_i^\vee\rangle}{\langle \eta, \alpha_N^\vee\rangle}\varpi_i^2 - \sum_{1\le i<j\le N} v_{jN}\frac{\langle \eta, \alpha_i^\vee\rangle}{\langle \eta, \alpha_N^\vee\rangle}\varpi_i\varpi_j. \tag{2}$$

Consider now the following special case of (1):

$$v_{ij}\alpha_N^\vee + v_{jN}\alpha_i^\vee = v_{iN}\alpha_j^\vee, \quad 1 \le i < j \le N,$$

which implies immediately

$$v_{jN}\alpha_i^\vee + v_{iN}\alpha_j^\vee = 2v_{jN}\alpha_i^\vee + v_{ij}\alpha_N^\vee.$$

Combining this identity with the formula (2) it is then easy to verify that

$$c_{\eta 1} = \frac{1}{2}\sum_{1\le i<j\le N} v_{ij}\varpi_i\varpi_j - \frac{1}{2\langle \eta, \alpha_N^\vee\rangle}\left(\sum_{i=1}^{N}\langle \eta, \alpha_i^\vee\rangle\varpi_i\right)\left(\sum_{i=1}^{N} v_{iN}\varpi_i\right).$$

In the same way, we obtain

$$\begin{aligned} c_{\eta 2} &= \frac{1}{2}\sum_{i=1}^{N} v_{iN}\frac{\langle \eta, \alpha_i^\vee\rangle}{\langle \eta, \alpha_N^\vee\rangle}\varpi_i^2 + \sum_{1\le i<j\le N} v_{iN}\frac{\langle \eta, \alpha_j^\vee\rangle}{\langle \eta, \alpha_N^\vee\rangle}\varpi_i\varpi_j \\ &= \frac{1}{2}\sum_{1\le i<j\le N} v_{ij}\varpi_i\varpi_j + \frac{1}{2\langle \eta, \alpha_N^\vee\rangle}\left(\sum_{i=1}^{N}\langle \eta, \alpha_i^\vee\rangle\varpi_i\right)\left(\sum_{i=1}^{N} v_{iN}\varpi_i\right). \end{aligned}$$

We can now observe that

$$\mathfrak{c}_{\eta;2} - \mathfrak{d} = r_{\eta;2}(\mathbf{1}_{\mathcal{G}_1} - \mathbf{1}_{\mathcal{G}_2})$$

with

$$r_{\eta;2} = -\frac{1}{2\langle \eta, \alpha_N^\vee\rangle}\left(\sum_{i=1}^{N}\langle \eta, \alpha_i^\vee\rangle\varpi_i\right)\left(\sum_{i=1}^{N} v_{iN}\varpi_i\right).$$

Since we have $v_{iN} = \beta(\alpha_i^\vee \wedge \alpha_N^\vee)$, there exists $\xi \in V^*$ with $v_{iN} = \langle \xi, \alpha_i^\vee\rangle$ for $1 \le i \le N$. Therefore $r_{\eta;2} \in \mathrm{Sym}^2(\mathrm{Rel}^\perp)$, as required.

If we consider $\mathfrak{c}_{\eta;1}$ instead, we get the result

$$\mathfrak{c}_{\eta;1} - \mathfrak{d} = r_{\eta;1}(\mathbf{1}_{\mathcal{G}_1} - \mathbf{1}_{\mathcal{G}_2})$$

with

$$r_{\eta;1} = \frac{1}{2\langle \eta, \alpha_1^\vee \rangle} \left( \sum_{i=1}^{N} \langle \eta, \alpha_i^\vee \rangle \varpi_i \right) \left( \sum_{i=1}^{N} v_{1i} \varpi_i \right) \in \mathrm{Sym}^2(\mathrm{Rel}^\perp).$$

In Section 4 of [FLM] it is explained how the polynomial identity of Theorem 2 (which we proved directly in the rank two case) implies the formula of Theorem 1 for intertwining families.

# References


[FL1] T. Finis, E. Lapid: Relation spaces of hyperplane arrangements and modules defined by graphs of fiber zonotopes, preprint

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