A CONJECTURAL NON-COMMUTATIVE GENERALIZATION OF A VOLUME FORMULA OF MCMULLEN-SCHNEIDER

TOBIAS FINIS AND EREZ LAPID

1. Preliminaries

Let V be a real vector space of dimension d and V^* its dual space. By a *cone* in V^* we will always mean a closed polyhedral cone σ with apex 0 such that $\sigma \cap -\sigma = \{0\}$. Let Σ be a *fan* in V^*, i.e., a collection of cones such that

(1) if $\sigma \in \Sigma$ then any face of σ belongs to Σ,
(2) if $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face in both.

We will assume that Σ is *complete*, that is $\cup\Sigma = V^*$. The elements of Σ are called *faces*. We denote by $\Sigma(i)$ the set of i-dimensional faces of Σ. In particular, $\Sigma(d)$, $\Sigma(d-1)$ and $\Sigma(1)$ are the sets of *chambers*, *walls* and *rays* of Σ respectively. Two chambers are *adjacent* if they intersect in a wall. Any wall is contained in exactly two chambers (which are adjacent). We will write $\sigma \overset{\tau}{\longleftrightarrow} \tilde{\sigma}$ if σ and $\tilde{\sigma}$ are adjacent with common wall $\tau = \sigma \cap \tilde{\sigma}$. If we want to distinguish σ we will write $\sigma \xrightarrow{\tau} \tilde{\sigma}$ and speak of a *directed wall* ω emerging from σ. We denote by $\tilde{\omega}$ the opposite directed wall $\tilde{\sigma} \xrightarrow{\tau} \sigma$. **Henceforth, we will assume that Σ is simplicial**, that is, each cone in Σ is simplicial. Equivalently, any chamber σ has precisely d directed walls emerging from it.

For any cone σ we denote by $\mathcal{V}(\sigma)$ its linear span and by $\sigma^\perp$ its annihilator in V. A d-tuple $(\tau_1, \dots, \tau_d)$ of walls is called *transversal* if $\sum_{i=1}^d \tau_i^\perp = V$, i.e. if $\cap_{i=1}^d \mathcal{V}(\tau_i) = 0$.

A basic example of a simplicial fan is the normal fan Σ_P of a simple convex polytope P in V, whose affine hull is V. It is given by $\Sigma_P = \{\tau(F) : F \in \mathcal{F}(P)\}$ where $\mathcal{F}(P)$ denotes the lattice of faces of P and

$$\tau(F) = \{\lambda \in V^* : \lambda|_P \text{ attains its maximum on } F\}.$$

Note that $\tau : \mathcal{F}(P) \to \Sigma_P$ is an inclusion reversing bijection and $\dim F + \dim \tau(F) = d$. (We recall that not every fan is the normal fan of a polytope.)

Given $\sigma \in \Sigma$ we say that $v \in V$ is positive with respect to σ if $\langle \lambda, v \rangle > 0$ for any $\lambda \in \operatorname{relint} \sigma$. Given a directed wall $\omega : \sigma \xrightarrow{\tau} \tilde{\sigma}$, a *directed normal* for ω, or an ω-directed normal, is an element of $\tau^\perp$ which is positive with respect to σ. Such a vector is uniquely determined up to multiplication by a positive scalar.

Date: October 31, 2008.
Authors partially supported by a grant from the German-Israeli Foundation.

2. Piecewise polynomial functions

Let $S = \mathrm{Sym}(V)$ be the ring of polynomial functions on V^*. For any face $\sigma \in \Sigma$ we denote by $\mathfrak{I}_\sigma$ the ideal of S generated by the subspace $\sigma^\perp$ of V.

A *piecewise polynomial* with respect to Σ is a function on V^* whose restriction to any chamber (hence, to any face) is a polynomial. We denote by $\mathcal{A} = \mathcal{A}_\Sigma$ the graded algebra of piecewise polynomials with respect to Σ. It is known that $\mathcal{A}$ is a free S-module, and is generated as an algebra by its degree 1 elements (the piecewise linear functionals). Moreover, the dimension of $\mathcal{A}_1$ is the number of rays in Σ.

We can view an element of $\mathcal{A}$ as a collection X_σ of elements of S, one for each chamber σ, such that $X_{\sigma_1} - X_{\sigma_2} \in \mathfrak{I}_{\sigma_1 \cap \sigma_2}$ for any two chambers σ_1, σ_2. (It is enough to check this condition for σ_1, σ_2 adjacent.) For any directed wall $\omega : \sigma \xrightarrow{\tau} \tilde{\sigma}$ we write $X_\omega = X_\sigma - X_{\tilde{\sigma}} \in \mathfrak{I}_\tau$.

More generally if M is an S-module we define $M_\Sigma := M \otimes_S \mathcal{A}_\Sigma$ to be the $\mathcal{A}_\Sigma$-module of Σ-piecewise elements of M. If M is flat over S then an element of M_Σ can be described as a collection $m_\sigma \in M$, $\sigma \in \Sigma(d)$ such that $m_{\sigma_1} - m_{\sigma_2} \in \mathfrak{I}_{\sigma_1 \cap \sigma_2} M$ for any chambers σ_1, σ_2.

Suppose that P is a polytope in V whose normal fan is Σ. Then the vertices of P are indexed by the chambers of Σ and give rise to a piecewise linear form L_P on V^* with respect to Σ. We say that L_P is the piecewise linear form defined by P. These piecewise linear forms are characterized by the property that X_ω is positive with respect to σ for any directed wall $\omega : \sigma \xrightarrow{\tau} \tilde{\sigma}$.

Fix $0 \neq \beta = \beta_\Sigma \in (\wedge^d V)^*$.

Let σ be a chamber and let $\omega_i : \sigma \xrightarrow{\tau_i} \sigma_i$, $i = 1, \dots, d$ be the directed walls emerging from σ. Set

$$\theta_\sigma = \frac{v_1 \dots v_d}{|\beta(v_1 \wedge \dots \wedge v_d)|} \in S$$

where v_i is a directed normal of ω_i. As the notation suggests, θ_σ depends only on σ and not on the choice of the v_i's or the order of the ω_i's.

It is well-known (e.g. [Bri97]) that we have an S-linear map $\delta_\Sigma : \mathcal{A}_\Sigma \to S$ defined by

$$(X_\sigma)_{\sigma \in \Sigma(d)} \mapsto \sum_{\sigma \in \Sigma(d)} \frac{X_\sigma}{\theta_\sigma}.$$

Extending scalars, we get for any S-module M an S-linear map

$$\delta = \delta_{\Sigma; M} : M_\Sigma \to M.$$

3. The setup

Let $\mathbb{C}[[V]]$ denote the algebra of formal power series in V, i.e. the completion of S at the origin, which is also the dual space of the vector space $\mathbb{C}[V]$ of polynomials on V. We denote by $[\cdot, \cdot] : \mathbb{C}[V] \times \mathbb{C}[[V]] \to \mathbb{C}$ the pairing.

For any vector space U we set $U[[V]] = \mathbb{C}[[V]] \otimes U$. We write $[\cdot, \cdot] : \mathbb{C}[V] \times U[[V]] \to U$ for the bilinear pairing. Given a bilinear map $\circ : U_1 \times U_2 \to U_3$ we will continue to denote by $\circ$ the $\mathbb{C}[[V]]$-bilinear map $\circ : U_1[[V]] \times U_2[[V] \to U_3[[V]]$ obtained by extending the scalars.

Henceforth, whenever V' is a subspace of V we will identify $\mathbb{C}[[V']]$ with a subalgebra of $\mathbb{C}[[V]]$.

Definition 3.1. An intertwining family with respect to Σ consists of the following data

(1) For each chamber σ, a finite-dimensional vector space W_σ,
(2) For any pair of chambers $\sigma_1, \sigma_2 \in \Sigma(d)$, an element $A_{\sigma_2|\sigma_1} \in \mathrm{Hom}(W_{\sigma_1}, W_{\sigma_2})[[(\sigma_1 \cap \sigma_2)^\perp]]$,

with the following properties

(1) $A_{\sigma|\sigma} = \mathrm{id}_{W_\sigma} \otimes 1$ for all chambers σ,
(2) For any triple of chambers $\sigma_1, \sigma_2, \sigma_3$ we have

$$A_{\sigma_3|\sigma_1} = A_{\sigma_3|\sigma_2} \circ A_{\sigma_2|\sigma_1}$$

(an equality in $\mathrm{Hom}(W_{\sigma_1}, W_{\sigma_3})[[V]]$) where $\circ$ denotes composition.

Note that the data is determined by $A_{\sigma_2|\sigma_1}$ where σ_1, σ_2 are adjacent. Given a directed wall $\omega : \sigma \xrightarrow{\tau} \tilde{\sigma}$ we write $A_\omega = A_{\tilde{\sigma}|\sigma}$.

We fix a chamber σ_0 and consider for any chamber σ the element $m_\sigma := A_{\sigma_0|\sigma}(0) A_{\sigma|\sigma_0}$ of $M = \mathrm{End}(W_{\sigma_0})[[V]]$. It is easy to see that this defines a Σ-piecewise element of M. Let $\mathcal{D}_{\sigma_0} A := \delta_{\Sigma;M}((m_\sigma)_{\sigma \in \Sigma(d)}) \in M$. Observe that $\mathcal{D}_{\tilde{\sigma}_0} A = A_{\tilde{\sigma}_0|\sigma_0}(0) \circ \mathcal{D}_{\sigma_0} A \circ A_{\sigma_0|\tilde{\sigma}_0}$ for any other chamber $\tilde{\sigma}_0$ in Σ.

Let $\omega : \sigma \xrightarrow{\tau} \tilde{\sigma}$ be a directed wall. For any $f \in \mathbb{C}[\tau^\perp]$ we set

$$f^{\sigma_0}_\omega(A) = A_{\sigma_0|\sigma}(0) \circ A_\omega(0)^{-1} \circ [f, A_\omega] \circ A_{\sigma|\sigma_0}(0) \in \mathrm{End}(W_{\sigma_0}).$$

Note that if $v^* \in (\tau^\perp)^* \subseteq \mathbb{C}[\tau^\perp]$ then

$$(v^*)^{\sigma_0}_{\tilde{\omega}}(A) = -(v^*)^{\sigma_0}_\omega(A).$$

Also,

$$f^{\tilde{\sigma}_0}_\omega(A) = A_{\tilde{\sigma}_0|\sigma_0} \circ f^{\sigma_0}_\omega(A) \circ A_{\sigma_0|\tilde{\sigma}_0}$$

for any chamber $\tilde{\sigma}_0$.

Let $\tau_1, \dots, \tau_d$ be walls. For each $i = 1, \dots, d$ choose a directed wall $\omega_i : \sigma_i \xrightarrow{\tau_i} \tilde{\sigma}_i$ and a directed normal v_i for ω_i. Let $v_i^* \in (\tau_i^\perp)^* \subseteq \mathbb{C}[\tau_i^\perp]$ be such that $\langle v_i^*, v_i \rangle = 1$. We set

$$\partial^{\sigma_0}_{\tau_1,\dots,\tau_d} A = |\beta(v_1 \wedge \cdots \wedge v_d)|\, (v_1^*)^{\sigma_0}_{\omega_1}(A) \circ \cdots \circ (v_d^*)^{\sigma_0}_{\omega_d}(A).$$

Note that this expression depends only on $(\tau_1, \dots, \tau_d)$ and not on the choice of the ω_i's or the v_i's. Also note that

$$\partial^{\tilde{\sigma}_0}_{\tau_1,\dots,\tau_d} A = A_{\tilde{\sigma}_0|\sigma_0}(0) \circ \partial^{\sigma_0}_{\tau_1,\dots,\tau_d} A \circ A_{\sigma_0|\tilde{\sigma}_0}(0)$$

for any chamber $\tilde{\sigma}_0$.

4. The conjectural formula

Let A be an intertwining family with respect to Σ.

Conjecture 4.1. *For any choice of* $\vec{\lambda} = (\lambda_1, \dots, \lambda_d) \in (V^*)^d$ *in general position with respect to* Σ *we have*

$$\mathcal{D}^{\sigma_0} A(0) = \frac{(-1)^d}{d!} \sum_{(\tau_1,\dots,\tau_d) \in \mathcal{X}_{\vec{\lambda}}} \partial^{\sigma_0}_{\tau_1,\dots,\tau_d} A$$

where $\mathcal{X}_{\vec{\lambda}}$ *is the set of* d*-tuples* $(\tau_1, \dots, \tau_d)$ *of transversal walls such that the translates* $\lambda_i + \tau_i$, $i = 1, \dots, d$ *intersect (necessarily in a point).*

The conjecture is trivially true for $d = 1$. It can be also proved for $d = 2$ by direct computation. A special case of the conjecture for Coxeter fans (corresponding to root hyperplane arrangements) was established in [FLM]. It played a role in the analysis of the spectral side of Arthur's trace formula.

5. Remarks

Suppose that P is a polytope in V and Σ is its normal fan in V^*. Let L_σ, $\sigma \in \Sigma(d)$ denote the vertices of P. We can form the intertwining family with $W_\sigma = \mathbb{C}$ for all chambers and $A_{\sigma_2|\sigma_1} = e^{L_{\sigma_2} - L_{\sigma_1}}$. Then $\mathcal{D}^{\sigma_0} A$ is the Fourier transform of the translate of P by σ_0 ([Bri97]). Thus, conjecture 4.1 reduces in this case to the McMullen-Schneider's formula ([MS83]) expressing $\mathrm{vol}(P)$ as $1/d!$ times the sum of the volume of the parallelotope formed by the vectors $\vec{e_1}, \dots, \vec{e_d}$ as $(e_1, \dots, e_d)$ range over the d-tuples of edges of P for which there exists $\mu \in V^*$ such that $\max(\mu + \lambda_i)|_P$ is attained on e_i, $i = 1, \dots, d$. (Here, $\vec{e_i} \in V$ denotes the vector corresponding to e_i.)

Next, we comment about the dependence on $\vec{\lambda}$. Given a fan Σ in U and a a linear surjective map $p : U \to U'$ the *quotient fan* on U' is defined by the common refinement of $p(\sigma)$, $\sigma \in \Sigma$ (cf. [KSZ91], [BS94]). In the case where $U = V^*$ and Σ is the normal fan of a polytope P in V, the quotient fan is the normal fan of the *fiber polytope* of P, in the sense of Billera-Sturmfels, with respect to the projection $V \to V/(\mathrm{Ker}\, p)^\perp = (\mathrm{Ker}\, p)^*$ ([KSZ91, Proposition 2.3]).

In particular, consider V^* embedded diagonally in $(V^*)^d$ and let $p : (V^*)^d \to (V^*)^d / V^*$ be the canonical projection. Let $\Sigma^d = \overbrace{\Sigma \times \cdots \times \Sigma}^{d \text{ times}}$ (a fan in $(V^*)^d$), and let $\overline{\Sigma}$ be the quotient fan in $(V^*)^d / V^*$. The precise condition on $\vec{\lambda}$ to be in general position with respect to Σ is that it lies outside the walls of $\overline{\Sigma}$, i.e. it lies in the interior of a chamber of $\overline{\Sigma}$. Moreover, the set $\mathcal{X}_{\vec{\lambda}}$ depends only on the chamber to which $\vec{\lambda}$ belongs.

References

[Bri97] Michel Brion, *The structure of the polytope algebra*, Tohoku Math. J. (2) **49** (1997), no. 1, 1–32. MR MR1431267 (98a:52019)

[BS94] Louis J. Billera and Bernd Sturmfels, *Iterated fiber polytopes*, Mathematika **41** (1994), no. 2, 348–363. MR MR1316614 (96a:52013)

[FLM] Tobias Finis, Erez Lapid, and Werner Müller, *On the spectral side of Arthur's trace formula II*, preprint.

[KSZ91] M. M. Kapranov, B. Sturmfels, and A. V. Zelevinsky, *Quotients of toric varieties*, Math. Ann. **290** (1991), no. 4, 643–655. MR MR1119943 (92g:14050)

[MS83] Peter McMullen and Rolf Schneider, *Valuations on convex bodies*, Convexity and its applications, Birkhäuser, Basel, 1983, pp. 170–247. MR MR731112 (85e:52001)