Weak stability for orbits of C_0-semigroups on Banach spaces

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Dedicated to the memory of Günter Lumer

Abstract. A result of Huang and van Neerven [11] establishes weak individual stability for orbits of C_0-semigroups under boundedness assumptions on the local resolvent of the generator. We present an elementary proof for this using only the inverse Fourier-transform representation of the orbits of the semigroup in terms of the local resolvent.

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1. Introduction

This paper is originally motivated by the structure theory of relatively weakly compact semigroups on Banach spaces as presented, for example, in Engel, Nagel [6, Ch. V]. Suppose that a C_0-semigroup $(T(t))_{t\geq 0}$, with generator $(A, D(A))$, is *relatively weakly compact*, that is each of the orbits $\{T(t)x : t \geq 0\}$ is a relatively weakly compact subset of the Banach space X. Then the Jacobs–Glicksberg–de Leeuw decomposition yields the existence of a projection $Q \in \mathcal{L}(X)$ commuting with the semigroup $(T(t))_{t\geq 0}$ such that

$$\begin{aligned} \ker Q &= \big\{x \in X : \ 0 \in \overline{\{T(t)x : \ t \geq 0\}}^{\sigma}\big\}, \\ \operatorname{rg} Q &= \overline{\operatorname{lin}}\big\{x \in D(A) : \ \exists\, \alpha \in \mathbb{R} \text{ with } Ax = i\alpha x\big\}. \end{aligned}$$

In particular, if $(T(t))_{t\geq 0}$ is a bounded semigroup on a reflexive Banach space X, then the semigroup is of course relatively weakly compact, and we always have the existence of such a projection. If now the generator does not have point spectrum on the imaginary axis, then we have $\ker Q = X$. So 0 belongs to the weak closure of *each* orbit. There are however examples showing that generally we can not expect weak stability, i.e., that all orbits converge to 0 in the weak topology (see [6, Example V.2.11 ii)]). In fact, the "no eigenvalues on the imaginary axis" assumption is roughly speaking equivalent to *almost weak stability* (i.e., convergence to zero along a large set of time values) but, in general, not to weak stability, see [7], [8], and also [9].

Concerning stability questions for bounded semigroups the size of the spectrum on the imaginary line and the growth of the resolvent $R(\lambda, A)$ in a neighbourhood of it play an important role. The celebrated theorem of Arendt, Batty [1] and Lyubich, Vũ [14] gives a sufficient condition on the boundary spectrum for *strong stability.* They show that, in case of reflexive X, countable spectrum $\sigma(A)$ on the imaginary axis and no eigenvalues on $i\mathbb{R}$ imply strong stability of bounded C_0-semigroups. Later Batty [2] gave similar results for *weak individual stability* of the orbit $T(t)x_0$ under the above spectral assumptions and the boundedness of the orbit (see also Batty, Vũ [5]).

In connection with individual stability or growth of orbits, the boundedness of the *local resolvent* has gained wide recognition. In the sequel, we will say, with a slight abuse of terminology, that a bounded local resolvent $R(\lambda)x_0$ exists on $\mathbb{C}_+$ if the function $\rho(A) \ni \lambda \mapsto R(\lambda, A)x_0$ admits a bounded, holomorphic extension $R(\lambda)x_0$ to the whole right halfplane $\mathbb{C}_+ := \{\mu : \Re\mu > 0\}$.

Huang and van Neerven [11] proved that if the Banach space X has Fourier type $p \in (1, 2]$, then the existence of a bounded local resolvent $R(\lambda)x_0$ on $\mathbb{C}_+$ already implies the strong convergence $T(t)R(\mu, A)^\alpha x_0 \to 0$ as $t \to +\infty$, for all $\mu > \omega_0(A)$ and $\alpha > 1$ (see also [10]).

Interestingly enough, *weak convergence* of the orbit may be also concluded from the existence of bounded local resolvent. In [4] a functional calculus method was developed for investigating asymptotic behaviour of C_0-semigroups with bounded local resolvents. A corollary of this approach is an alternative proof of the next theorem (see [11] Theorem 0.3]).

Theorem (Huang, van Neerven [11], **Theorem 0.3).** *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator $(A, D(A))$, $x_0 \in X$, and suppose that the local resolvent $R(\lambda)x_0$ exists on the open right halfplane and that it is bounded, i.e., there exists some $M > 0$ such that*

$$\|R(\lambda)x_0\| \leq M \quad \textit{for all } \lambda \in \mathbb{C}_+.$$

Then it holds

$$T(t)R(\mu, A)^\alpha x_0 \to 0 \quad \textit{weakly as } t \to +\infty \textit{ for all } \alpha > 1 \textit{ and } \mu > \omega_0(A).$$

In [17] van Neerven obtains even the exponent $\alpha = 1$ under an additional positivity assumption.

Theorem (van Neerven [17]**).** *Suppose that X is an ordered Banach space with weakly closed normal cone C. If for some $x_0 \in X$*

i) $T(t)x_0 \in C$ for all sufficiently large t, and

ii) $R(\cdot, A)x_0$ has a bounded holomorphic extension to $\mathbb{C}_+$, *then for all* $\mu \in \rho(A)$ *and* $y \in X'$

$$\langle T(t)R(\mu, A)x_0, y\rangle \to 0 \qquad \text{as } t \to +\infty.$$

It is also known that the above *eventual positivity* assumption *cannot* be omitted (see Batty [3], van Neerven [17]).

Reformulating van Neerven's assertion we can write

$$\langle T(t)x_0, y\rangle \to 0 \qquad \text{as } t \to +\infty \text{ for all } y \in D(A'). \tag{1}$$

This is an individual stability result for the orbit of x_0 under the semigroup. Our aim is to give an *elementary* proof of such convergence in the presence of bounded local resolvent without assumption on the Banach space, but only for $y \in D({A'}^2)$. This is the above mentioned result in the case $\alpha = 2$. That we assume $\alpha = 2$ instead of $\alpha > 1$ is only technical to keep the arguments the simplest possible.

At the end, we formulate the analogous individual stability result for bi-continuous semigroups (see [13] for general theory).

2. The Result

Theorem 1. *Let* $(T(t))_{t\geq 0}$ *be a* C_0*-semigroup with generator* $(A, D(A))$*,* $x_0 \in X$*, and suppose that the local resolvent* $R(\lambda)x_0$ *exists on the open right halfplane and that it is bounded, i.e., there exists some* $M > 0$ *such that*

$$\|R(\lambda)x_0\| \leq M \quad \text{for all } \lambda \in \mathbb{C}_+.$$

Then the convergence

$$\langle T(t)x_0, y\rangle \to 0 \quad \text{as } t \to +\infty \text{ for all } y \in D({A'}^2)$$

holds.

Remark. To make distinction between the resolvent operator and the local resolvent, for the latter we will use the notation $R(\mu)x_0$, while the use of the symbol $R(\lambda, A)$ tacitly assumes that λ belongs to the resolvent set $\rho(A)$, hence $(\lambda - A)^{-1}$ is a bounded linear operator.

To prove the theorem we need the following series of lemmas.

Lemma 1. *For all* $\lambda \in \rho(A)$ *and* $\mu \in \mathbb{C}_+$

$$R(\lambda, A)x_0 - R(\mu)x_0 = (\mu - \lambda)R(\lambda, A)R(\mu)x_0 \tag{2}$$

holds.

Proof. For a fixed $\lambda \in \rho(A)$ both functions on the two sides of (2) are analytic on $\mathbb{C}_+$. For large $\Re\mu$ the resolvent identity holds, so the assertion follows by uniqueness of analytic functions. □

Lemma 2. *For all* $\lambda \in \rho(A)$ *and* $\mu \in \mathbb{C}_+$ *we have*

$$\|R(\lambda, A)R(\mu)x_0\| \leq \frac{M + \|R(\lambda, A)x_0\|}{|\lambda - \mu|}.$$

Proof. Use Lemma 1. $\square$

Lemma 3. *For $y \in D(A'^2)$ and $a > \omega_0(T)$ there exists a constant $c := c(y,a)$ such that*

$$\|R^2(a+is, A')y\| \le \frac{c}{a^2+s^2} \quad \text{for all } s \in \mathbb{R}.$$

Proof. Let us write $\lambda = a + is$. Then we have

$$R(\lambda, A')y = \frac{1}{\lambda}\left(R(\lambda, A')A'y + y\right).$$

Thus

$$\begin{aligned} R(\lambda, A')^2 y &= \frac{1}{\lambda}\left(R(\lambda, A')R(\lambda, A')A'y + R(\lambda, A')y\right) \\ &= \frac{1}{\lambda^2}\left(R(\lambda, A')A'R(\lambda, A')A'y + R(\lambda, A')A'y + R(\lambda, A')A'y + y\right). \end{aligned}$$

The assertion follows by noticing that the terms in parenthesis are bounded. $\square$

Lemma 4. *For $y \in D(A'^2)$, $x \in X$ and $a > \omega_0(T)$ we have*

$$\begin{aligned} \langle T(t)x, y\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{(a+is)t}\langle R(a+is, A)x, y\rangle \,\mathrm{d}s \\ &= \frac{1}{2\pi t}\int_{-\infty}^{+\infty} \mathrm{e}^{(a+is)t}\langle R^2(a+is, A)x, y\rangle \,\mathrm{d}s. \end{aligned} \tag{3}$$

Proof. The integral in (3) is just

$$\frac{1}{2\pi t}\int_{-\infty}^{+\infty} \mathrm{e}^{(a+is)t}\langle x, R^2(a+is, A')y\rangle \,\mathrm{d}s,$$

and it is absolutely convergent by Lemma 3. Integration by parts yields equality of the two integrals. In particular, since the first integral converges, we obtain immediately that it coincides with $\langle T(t)x, y\rangle$ as the inverse Laplace transform of the resolvent (see [12, Lemma 2.4]). $\square$

Lemma 5. *For $y \in D(A'^2)$, $x \in X$, $a > \omega_0(T)$ and $0 < \delta < a$ we have*

$$\begin{aligned} \langle T(t)x_0, y\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{(a+is)t}\langle R(a+is, A)x_0, y\rangle \,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{(\delta+is)t}\langle R(\delta+is)x_0, y\rangle \,\mathrm{d}s. \end{aligned}$$

Proof. Let N be positive, then using the analyticity of $R(\lambda)x_0$ on $\mathbb{C}_+$ and Cauchy's theorem, we obtain for some $\mu \in \rho(A)$

$$\begin{aligned}
&\Big|\frac{1}{2\pi}\int_{-N}^{+N} \mathrm{e}^{(a+is)t}\langle R(a+is,A)x_0, y\rangle \,\mathrm{d}s - \frac{1}{2\pi}\int_{-N}^{+N} \mathrm{e}^{(\delta+is)t}\langle R(\delta+is)x_0, y\rangle \,\mathrm{d}s\Big| \\
&\le (a-\delta)\max_{b\in[\delta,a]}\big|\mathrm{e}^{(b+iN)}\langle R(b+iN)x_0, y\rangle\big| + (a-\delta)\max_{b\in[\delta,a]}\big|\mathrm{e}^{(b-iN)}\langle R(b-iN)x_0, y\rangle\big| \\
&= (a-\delta)\Big(\max_{b\in[\delta,a]}\big|\mathrm{e}^{b}\langle R(\mu,A)R(b+iN)x_0, (\mu-A')y\rangle\big| \\
&\qquad + \max_{b\in[\delta,a]}\big|\mathrm{e}^{b}\langle R(\mu,A)R(b-iN)x_0, (\mu-A')y\rangle\big|\Big),
\end{aligned}$$

but this converges to 0 by Lemma 2 as $N \to +\infty$. □

Proof of Theorem 1. According to (2) Lemma 1

$$\begin{aligned}
R(\delta+is)x_0 &= R(a+is,A)x_0 + (a-\delta)R(a+is,A)R(\delta+is)x_0 \\
&= R(a+is,A)x_0 + (a-\delta)R^2(a+is,A)x_0 \\
&\qquad\qquad + (a-\delta)^2R^2(a+is,A)R(\delta+is)x_0.
\end{aligned}$$

Using Lemma 5 we obtain for $y \in D({A'}^2)$

$$\begin{aligned}
2\pi\mathrm{e}^{-\delta t}\langle T(t)x_0, y\rangle &= \int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R(\delta+is)x_0, y\rangle \,\mathrm{d}s \\
&= \int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R(a+is,A)x_0, y\rangle \,\mathrm{d}s \\
&\quad + (a-\delta)\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R^2(a+is,A)x_0, y\rangle \,\mathrm{d}s \\
&\quad + (a-\delta)^2\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R^2(a+is,A)R(\delta+is)x_0, y\rangle \,\mathrm{d}s.
\end{aligned}$$

The functions $f_\delta(s) := \langle R^2(a+is,A)R(\delta+is)x_0, y\rangle$ form a relatively compact subset of $L^1(\mathbb{R})$. Indeed, we have

$$\begin{aligned}
|f_\delta(s)| &= |\langle R^2(a+is,A)R(\delta+is)x_0, y\rangle| = |\langle R(\delta+is)x_0, R^2(a+is,A')y\rangle| \\
&\le M\|R^2(a+is,A')y\|,
\end{aligned}$$

and the function on the right hand side belongs to $L^1(\mathbb{R})$. This shows the family f_δ to be uniformly integrable (and bounded), thus relatively compact. So by compactness we find a sequence $\delta_n \to 0$ such that $f_{\delta_n} \to f$ in $L^1(\mathbb{R})$ $(n \to \infty)$. Thus substituting δ_n in the above equality and letting $n \to \infty$ we obtain

$$
\begin{aligned}
2\pi\langle T(t)x_0, y\rangle = &\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R(a+is, A)x_0, y\rangle \,\mathrm{d}s \\
&+ a\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle R^2(a+is, A)x_0, y\rangle \,\mathrm{d}s \\
&+ a^2\int_{-\infty}^{+\infty} \mathrm{e}^{ist} f(s)\,\mathrm{d}s =: I_1(t) + I_2(t) + I_3(t).
\end{aligned}
$$

It is easy to deal with the last term I_3. Since f belongs to $L^1(\mathbb{R})$ so by the Riemann–Lebesgue Lemma its Fourier transform vanishes at $+\infty$, i.e., $I_3(t) \to 0$ as $t \to +\infty$. Since $y \in D(A'^2)$, we can rewrite I_1 by Lemma 4 as

$$
I_1(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle x_0, R(a+is, A')y\rangle \,\mathrm{d}s = \frac{1}{t}\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle x_0, R^2(a+is, A')y\rangle \,\mathrm{d}s.
$$

The last integral is absolutely convergent by Lemma 3, hence

$$
|I_1(t)| \le \frac{1}{t}\int_{-\infty}^{+\infty} \|x_0\| \cdot \|R^2(a+is, A')y\| \,\mathrm{d}s \to 0 \qquad \text{as } t \to +\infty.
$$

As for I_2 we first notice that $\langle x_0, R^2(a+i\cdot, A')y\rangle \in L^1(\mathbb{R})$, so by the Riemann–Lebesgue Lemma we have

$$
I_2(t) = a\int_{-\infty}^{+\infty} \mathrm{e}^{ist}\langle x_0, R^2(a+is, A')y\rangle \,\mathrm{d}s \to 0 \qquad \text{as } t \to +\infty.
$$

This conclude the proof. □

Let us draw the following consequences of the above result.

Corollary 1. *Let $(T(t))_{t\ge 0}$ be a C_0-semigroup with generator $(A, D(A))$, and suppose that $\{T(t)x_0 : t \ge 0\}$ is bounded and that the local resolvent $R(\lambda)x_0$ exists and is bounded on $\mathbb{C}_+$. Then*

$$
\langle T(t)x_0, y\rangle \to 0 \quad \textit{as } t \to +\infty \textit{ for all } y \in \overline{D(A')}.
$$

Proof. Since $(A', D(A'))$ is a Hille–Yosida operator, its part A'_0 generates a C_0-semigroup on $\overline{D(A')}$. But $D({A'_0}^2) \subseteq D(A'^2) \subseteq D(A'_0)$, so $D(A'^2)$ is dense in $\overline{D(A')}$. Now let $\varepsilon > 0$. For $y \in \overline{D(A')}$ take $y' \in D(A'^2)$ with $\|y - y'\| \le \varepsilon/2M$, where $\|T(t)x_0\| < M$, $t \ge 0$. For large t we have $|\langle T(t)x_0, y'\rangle| \le \varepsilon/2$ by Theorem 1. So

$$
|\langle T(t)x_0, y\rangle| \le |\langle T(t)x_0, y'\rangle| + |\langle T(t)x_0, y - y'\rangle| \le \varepsilon/2 + M \cdot \|y - y'\| \le \varepsilon,
$$

for large t. □

Corollary 2. *Let $(T(t))_{t\ge 0}$ be a bounded C_0-semigroup with generator $(A, D(A))$, and suppose $\sigma_p(A) \cap i\mathbb{R} = \emptyset$. If $(is - A)^{-1}x_0$ exists and is bounded in $s \in \mathbb{R}$ for some $x_0 \in X$, then*

$$
\langle T(t)x_0, y\rangle \to 0 \quad \textit{as } t \to +\infty \textit{ for all } y \in \overline{D(A')}.
$$

Proof. A version of the resolvent identity states that

$$(is - A)^{-1}x_0 - R(a + is, A)x_0 = (\lambda - is)R(a + is, A)(is - A)^{-1}x_0.$$

Here the right-hand side is bounded for $a > 0$ and $s \in \mathbb{R}$ by the Hille–Yosida theorem and by the assumption, so $R(a + is, A)x_0$ is bounded. The proof is concluded by applying Corollary 1. □

The above proof of Theorem 1 remains valid if the semigroup $(T(t))_{t\geq 0}$ is only strongly continuous for some coarser locally convex topology τ. More precisely, one has to assume that the semigroup is τ-*bi-continuous*, see [13] for the theory. Then the infinitesimal generator $(A, D(A))$ is a Hille–Yosida operator, but $D(A)$ is not necessarily dense with respect to the norm in X. It is dense however for the topology τ, so in the following the adjoint A' of A is understood with respect to τ. In addition, the resolvent identity, and replacing the vector-valued integrals by τ-strong integrals, all the above integral formulas remain valid, which were the essential ingredients of the proof. This proves the following.

Theorem 2. *For a bi-continuous semigroup $(T(t))_{t\geq 0}$ with generator $(A, D(A))$ and $x_0 \in X$ suppose that the local resolvent $R(\lambda)x_0$ exists on the open right halfplane and that it is bounded. Then for all $y \in D(A'^2)$*

$$\langle T(t)x_0, y\rangle \to 0$$

holds for $t \to +\infty$.

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