A note on polynomially growing C_0-semigroups

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Abstract

We characterize polynomial growth of a C_0-semigroup in terms of the first power of the resolvent of its generator. We do this for a class of semigroups which includes C_0-semigroups on Hilbert spaces and analytic semigroups on Banach spaces. Furthermore, we characterize polynomial growth for discrete semigroups.

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1 Introduction

A classical problem in semigroup theory is to characterize (in a useful way and in terms of its generator) boundedness of a strongly continuous semigroup. The characterization given by the Hille-Yosida theorem involves all powers of the generator's resolvent and is difficult to use in concrete situations.

Recently, (see Gomilko [5], Shi and Feng [13], Malejki [10], Eisner and Zwart [14, 2, 3]) bounded and polynomially bounded semigroups and groups have been characterized using only the first and the second power of the resolvent of the generator. In this note we use the method from [14, 3] to describe polynomial growth in terms of the first power of the resolvent. We do this for a class of C_0-semigroups which includes semigroups on Hilbert spaces and analytic semigroups on Banach spaces, see Theorem 2.1. An analogous result for discrete semigroups is presented as well. Finally, we also give the corresponding characterization of polynomially bounded C_0-groups. Note that polynomially bounded groups satisfy certain important properties such as the weak spectral mapping theorem, see Nagel (ed.) [9, Theorem A-III.7.4].

To be more precise we recall the following definitions.

A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is called *polynomially bounded* if $\|T(t)\| \leq C(1+t^d)$ for some constants C, $d \geq 0$ and all $t \geq 0$. Please note that d need not to be a positive integer, and hence t^d need not to be a (proper) polynomial.

It is well-known that every matrix semigroup is polynomially bounded if and only if the spectrum of the generator belongs to the closed left half-plane. For the same reason every quasi-compact semigroup on a Banach space is polynomially bounded under the same assumption on the spectrum of its generator.

We denote by $\omega_0(T)$ the growth bound of a C_0-semigroup $(T(t))_{t\geq 0}$, by $R(\lambda, A)$ the resolvent of A at λ, and by

$$s_0(A) := \inf\{a \in \mathbb{R} : R(\lambda, A) \text{ is bounded on } \{\lambda : Re\lambda > a\}\}$$

the *pseudo-spectral bound* of A (also called *abscissa of uniform boundedness of the resolvent* of A, see Arendt, Batty, Hieber, and Neubrander [1]).

In this paper we mainly consider semigroups whose generators have the following resolvent property. We define that an operator A has a *p-integrable resolvent* if for some/all $a, b > s_0(A)$ the following conditions hold

$$\int_{-\infty}^{\infty} \|R(a+is, A)x\|^p ds < \infty \quad \text{for all } x \in X, \tag{1}$$

$$\int_{-\infty}^{\infty} \|R(b+is, A')y\|^q ds < \infty \quad \text{for all } y \in X', \tag{2}$$

where $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Plancherel's theorem applied to the functions $t \mapsto e^{-at}T(t)x$ and $t \mapsto e^{-at}T^*(t)y$ for sufficiently large $a > 0$ implies that every generator of a C_0-semigroup on a Hilbert space has 2-integrable resolvent. Moreover, for generators on a Banach space with Fourier type $p > 1$ condition (1) is satisfied automatically. Finally, every generator of an analytic semigroup (in particular, every bounded operator) on an arbitrary Banach space has p-integrable resolvent for every $p > 1$. Intuitively, for a generator A the property of having p-integrable resolvent for some p means good properties of A or/and good properties of the space X.

2 Characterization of polynomial growth

The main result is the following characterization of polynomial growth of semigroups.

Theorem 2.1. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ having p-integrable resolvent for some $p > 1$. Assume that $\mathbb{C}_0^+ = \{\lambda : \ Re\lambda > 0\}$ is contained in the resolvent set of A and there exist $a_0 > 0$ and $M > 0$ such that the following conditions hold:*

(a) $\|R(\lambda, A)\| \leq \frac{M}{(Re\lambda)^d}$ for all λ with $0 < Re\lambda < a_0$ and for some $d \geq 0$;

(b) $\|R(\lambda, A)\| \leq M$ for all λ with $Re\lambda \geq a_0$.

Then $\|T(t)\| \leq K(1 + t^{2d-1})$ holds for some constant $K > 0$ and all $t \geq 0$.

Conversely, if $(T(t))_{t\geq 0}$ is a C_0-semigroup on a Banach space with

$$\|T(t)\| \leq K(1 + t^{\gamma})$$

for some constants $\gamma \geq 0$, $K > 0$ and all $t \geq 0$, then for every $a_0 > 0$ there exists a constant $M > 0$ such that the resolvent of the generator satisfies conditions (a) and (b) above for $d = \gamma + 1$.

Proof. The second part of the theorem follows easily from the representation $R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)xdt$. The idea of the proof of the first part is based on the inverse Laplace transform representation of the semigroup and the technique from [14, 3].

We first note that by conditions (a) and (b) we obtain $s_0(A) \leq 0$.

Next, by condition (1) and the uniform boundedness principle there exists a constant $M_0 > 0$ such that

$$\|R(a + i\cdot, A)x\|_{L^p(\mathbb{R},X)} \leq M_0\|x\| \tag{3}$$

holds for all $x \in X$. Similarly, one obtains by (2) the dual result, i.e.,

$$\|R(b + i\cdot, A')y\|_{L^q(\mathbb{R},X')} \leq \tilde{M}_0\|y\| \tag{4}$$

for all $y \in X'$.

Take now $0 < r < a_0$. By the resolvent equality we have

$$R(r+i\omega,A)x = [I + (a-r)R(r+i\omega,A)]\, R(a+i\omega,A)x$$

and hence

$$\begin{aligned}\|R(r+i\omega,A)x\| &\le [1+|a-r|\|(R(r+i\omega,A)\|]\,\|R(a+i\omega,A)x\| \\ &\le \left[1+|a-r|\frac{M}{r^d}\right]\|R(a+i\omega,A)x\|,\end{aligned}$$

where we have used (a). Combining this with estimate (3), we find that

$$\begin{aligned}\|R(r+i\cdot,A)x\|_{L^p(\mathbb{R},X)} &\le \left[1+|a-r|\frac{M}{r^d}\right] M_0\|x\| \\ &\le M_1\left[1+\frac{1}{r^d}\right]\|x\|. \qquad (5)\end{aligned}$$

Similarly, we find that

$$\|R(r+i\cdot,A')y\|_{L^q(\mathbb{R},X')} \le \tilde{M}_1\left[1+\frac{1}{r^d}\right]\|y\|. \qquad (6)$$

By the estimates (5), (6) and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}|\langle R(r+i\omega,A)^2x,y\rangle|d\omega \\ &= \int_{-\infty}^{\infty}|\langle R(r+i\omega,A)x,R(r+i\omega,A')y\rangle|d\omega \\ &\le \|R(r+i\cdot,A)x\|_{L^p(\mathbb{R},X)}\|R(r+i\cdot,A')y\|_{L^q(\mathbb{R},X')} \\ &\le M_1\tilde{M}_1\|x\|\|y\|\left[1+\frac{1}{r^d}\right]^2. \qquad (7)\end{aligned}$$

Convergence of the integral on the left hand side of (7) implies that the inverse formula for the semigroup

$$T(t)x = \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{(r+is)t}R(r+is,A)^2x\,ds$$

holds for all $x \in X$ (see, e.g., Kaashoek and Verduyn Lunel [7], Kaiser and Weis [8] or [2]). Notice that the condition $r > s_0(A)$ is essential. Combining this formula with (7) we obtain

$$\begin{aligned}|\langle T(t)x,y\rangle| &\le \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{rt}|\langle R(r+i\omega,A)^2x,y\rangle|d\omega \\ &\le \frac{1}{2\pi t}e^{rt}M_1\tilde{M}_1\|x\|\|y\|\left[1+\frac{1}{r^d}\right]^2. \qquad (8)\end{aligned}$$

Since this holds for all $0 < r < a_0$, we may choose $r = \frac{1}{t}$ for t large enough, which gives

$$|\langle T(t)x, y\rangle| \leq \frac{1}{2\pi t} e M_1 \tilde{M_1} \|x\| \|y\| \left[1 + t^d\right]^2. \tag{9}$$

So for large t the norm of the semigroup is bounded by Ct^{2d-1} for some constant C. Uniform boundedness of C_0-semigroups on compact time intervals finishes the proof. ■

As mentioned above every generator on a Hilbert space has 2-integrable resolvent, hence we have the following immediate corollary.

Corollary 2.2. *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space H. If A satisfies conditions (a) and (b) of Theorem 2.1 for some $d \geq 0$ and $a_0 > 0$, then there exists $K > 0$ such that $\|T(t)\| \leq K[1 + t^{2d-1}]$ for all $t \geq 0$.*

Remark 2.3. Using the power series expansion for the resolvent, it is not hard to show that if $0 \leq d < 1$, then conditions (a) and (b) already imply $s_0(A) < 0$. On the other hand, for generators with p-integrable resolvent the equality $\omega_0(T) = s_0(A)$ holds (see [7]). Combining these facts, we obtain that in this case the semigroup is even uniformly exponentially stable. On the other hand, the exponential stability follows from the Theorem 2.1 only for $d < \frac{1}{2}$. So for $\frac{1}{2} \leq d < 1$ Theorem 2.1 does not give the best information about the growth of the semigroup. Nevertheless, for $d = 1$ the growth stated in Theorem 2.1 is best possible, i.e., the exponent $2d - 1$ cannot be decreased in general (see [3]). For $d > 1$ it is not clear whether Theorem 2.1 gives the best possible constant γ.

Note that the parameter $d = \gamma + 1$ in the converse implication of Theorem 2.1 is optimal for $\gamma \in \mathbb{N}$. Indeed, for $X := \mathbb{C}^n$ and

$$A := \begin{pmatrix} 0\ 1\ 0\ \ldots\ 0 \\ 0\ 0\ 1\ \ldots\ 0 \\ \ldots \\ 0\ 0\ 0\ \ldots\ 0 \end{pmatrix}$$

conditions (a) and (b) in Theorem 2.1 are fulfilled for $d = n$ and the semigroup generated by A grows exactly as t^{n-1}.

By Corollary 2.2 we see that the class of generators of polynomially bounded semigroups on a Hilbert space coincides with the class of generators of C_0-semigroups with resolvent conditions (a) and (b). For semigroups on Banach spaces this is not true since there exist semigroups such that $w_0(T) > s_0(A)$ holds (see [4, Examples IV.3.2 and IV.3.3]).

We conclude this section with the discrete version of Theorem 2.1. Note that in this case both the formulation and the proof are simpler than their continuous analogues. For related results concerning the behavior of the resolvent and growth of the powers of an operator see e.g. Nagy and Zemánek [11], Nevanlinna [12], Gomilko and Zemánek [6].

Theorem 2.4. *Let T be a bounded operator on a Banach space X with $r(T) \le 1$. If*

$$\limsup_{|z|\to 1+} (|z|-1)^d \|R(z,T)\| < \infty \quad \textit{for some } d \ge 0, \tag{10}$$

then

$$\|T^n\| \le Cn^d \quad \textit{for some } C > 0 \textit{ and all } n \in \mathbb{N}. \tag{11}$$

Moreover, if (11) holds for $d = k$, then (10) holds with $d = k+1$.

Proof. Assume that condition (10) holds and take $n \in \mathbb{N}$ and $r > 1$. By the Dunford functional calculus and (10) we have

$$\|T^n\| \le \frac{r^{n+1}}{2\pi} \int_0^{2\pi} \|R(re^{i\varphi})\| d\varphi \le \frac{Mr^{n+1}}{(r-1)^d}$$

for $M := \limsup_{|z|\to 1+} (|z|-1)^d \|R(z,T)\|$. Taking $r := 1 + \frac{1}{n}$ we obtain $\|T^n\| \le 2Men^d$ and the first part of the theorem is proved.

For the second part we assume that condition (11) holds for $d = k$. Take $n \in \mathbb{N}$, $r > 1$, $\varphi \in [0, 2\pi)$, and $q := \frac{1}{r} < 1$. Then

$$\begin{aligned} \|R(re^{i\varphi}, T)\| &\le \sum_{n=0}^{\infty} \frac{\|T^n\|}{r^{n+1}} \le Cq \sum_{n=0}^{\infty} n^k q^n \le C \sum_{n=0}^{k-1} n^k + C \sum_{n=k}^{\infty} n^k q^n \\ &\le C \sum_{n=0}^{k-1} n^k + C\tilde{C} q^k \frac{d^k}{dq^k} \sum_{n=0}^{\infty} q^n \le C \sum_{n=0}^{k-1} n^k + \frac{C\tilde{C}k!}{(1-q)^{k+1}}, \end{aligned}$$

where $\tilde{C}$ is such that $n^k \le \tilde{C} \cdot n(n-1)\dots(n-k+1)$ for all $n \ge k$. For $k = 0$ we suppose the first sum on the right hand side to be equal to zero. Substituting q by $\frac{1}{r}$ we obtain condition (10) for $d = k+1$. ∎

3 Case of groups

As a corollary of Theorem 2.1 we have the following characterization of polynomially bounded groups in terms of the resolvent of the generator.

Theorem 3.1. *Let A be the generator of a C_0-group $(T(t))_{t\geq 0}$. Assume that A has p-integrable resolvent for some $p > 1$. Then the group $(T(t))_{t\in\mathbb{R}}$ is polynomially bounded if and only if the following conditions on the operator A are satisfied:*

(a) $\sigma(A) \subset i\mathbb{R}$*;*

(b) There exist $a_0 > 0$ *and* $d \geq 0$ *such that* $\|R(\lambda, A)\| \leq \frac{M}{|Re\lambda|^d}$ *for some constant* M *and all* λ *with* $0 < |Re\lambda| < a_0$*;*

(c) $R(\lambda, A)$ *is uniformly bounded on* $\{\lambda : \ |Re\lambda| \geq a_0\}$.

Proof. It is enough to show that also the operator $-A$ has p-integrable resolvent for A satisfying (a)–(c). Take any $a > 0$. Then by (b) or (c), respectively, $R(\lambda, A)$ is bounded on the vertical line $-a+i\mathbb{R}$. By the resolvent equation we obtain

$$\|R(-a + is, A)x\| \leq [1 + 2a\|R(-a + is, A)\|]\|R(a + is, A)x\|, \qquad (12)$$

and therefore the function $s \mapsto \|R(-a+is, A)x\|$ also belongs to $L^p(\mathbb{R})$. The rest follows immediately from Theorem 2.1. ■

Again, this yields a characterization of generators of polynomially bounded groups on Hilbert spaces. Note that the relation between the growth of the group and the growth of the resolvent appearing in (b) of Theorem 3.1 is the same as in Theorem 2.1.

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