Continuous-time Kreiss resolvent condition on infinite-dimensional spaces

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Dedicated to M.N. Spijker on the oocasion of his 65th birthday.

Abstract

Given the infinitesimal generator A of a C_0-semigroup on the Banach space X which satisfies the Kreiss resolvent condition, i.e., there exists an $M > 0$ such that $\|(sI - A)^{-1}\| \leq \frac{M}{\mathrm{Re}(s)}$ for all complex s with positive real part. We show that for general Banach spaces this condition does not give any information on the growth of the associated C_0-semigroup. For Hilbert spaces the situation is less dramatic. In particular, we show that the semigroup can grow as much like t. Furthermore, we show that for every $\gamma \in (0, 1)$ there exists an infinitesimal generator satisfying the Kreiss resolvent condition, but whose semigroup grows like t^γ. As a consequence, we find that for $\mathbb{R}^N$ with the standard Euclidian norm, the estimate $\|\exp(At)\| \leq M_1 \min(N, t)$ cannot be replaced by a lower power of N or t.

1 Introduction

Let us begin by introducing some notation; $(T(t))_{t\geq 0}$ will denote a C_0-semigroup on the Banach space X, and A its infinitesimal generator. The celebrated Hille-Yosida Theorem states that $\|T(t)\| \leq M$ for all $t \geq 0$ if and only if $\|\mathrm{Re}(s)^n(sI-A)^{-n}\| \leq M$ for all $s \in \mathbb{C}_0^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ and $n \in \mathbb{N}$. Unfortunately, this theorem can be very hard to check. Hence people have tried to find conditions which are easier to check. One of the conditions which has been proposed is the Kreiss resolvent condition, originally stated in Kreiss [11] for A being a matrix. This condition corresponds precisely to the first condition in the Hille-Yosida Theorem, i.e.,

$$\|(sI-A)^{-1}\| \leq \frac{M}{\mathrm{Re}(s)} \tag{1}$$

for $s \in \mathbb{C}_0^+$. From the Hille-Yosida Theorem it is clear that if the semigroup is bounded, then (1) holds. Furthermore, it is easy to see that if $M = 1$, then (1) is equivalent with the Hille-Yosida conditions. For Banach spaces and $M > 1$ it is known that the Kreiss resolvent condition does not imply the boundedness of the semigroup, see e.g. Engel and Nagel [8, section V.1.b.]. For a finite-dimensional space, i.e., $X = \mathbb{R}^N$ it was shown in Dorsselaer et al [6] that (1) implies that $\|\exp(At)\| \leq eMN$, see also [13]. Furthermore, if the norm on $\mathbb{R}^N$ is the maximum-norm, then there exists an A such that $\sup_{t\geq 0} \|\exp(At)\| \geq \frac{2N-1}{\pi+1} M$, see Kraaijevanger [10]. For discrete-time, finite-dimensional systems, it is known that $\|A^k\| \leq eM_d \min\{(N+1), k\}$, where M_d is the constant in the Kreiss resolvent estimate for the unit disc, see [6]. However, we were not able to find a continuous-time counterpart of this result in the literature. Using a scaled version of the example of [10], one can construct an example satisfying the Kreiss resolvent condition (1), but $\|\exp(At)\| \geq c \min\{N, t\}$ for some constant c independent of t and N. For a nice overview of these and related results and for historic remarks, we refer the reader to [6]. We remark that the discrete time counterpart of the Kreiss resolvent estimate has attracted more attention, than the continuous time version as we study in this article. A cited reference reach showed that there are 29 citation to the original Kreiss paper [11] on the continuous-time version, whereas there are 68 citations to [12] discussing the discrete-time version. For an overview and many more references we refer to [1, 6, 20].

On basis of the discrete-time result and the above mentioned example, one might hope that the Kreiss resolvent condition (1) implies that the semigroup grows at most like t. In Section 2 we show that this indeed holds X being a Hilbert space. Unfortunately, in general a Banach space exponential growth is possible, see Section 3. We show that for any $M > 1$ and $\alpha > 0$ there exists an infinitesimal generator which satisfies (1) with this M, but grows like $\exp(\alpha t)$. In Section 4 we construct for every $\gamma \in [0, 1)$ a Hilbert space and an infinitesimal generator A satisfying the Kreiss resolvent condition on this Hilbert space, but the corresponding semigroup grows like t^γ.

The example leads to a finite dimensional example, showing that if a $N \times N$ matrix A satisfies (1), then the supremum of $\exp(At)$ over $t > 0$ can be of the order N^γ.

2 An upperbound on the growth

In this section we show that for every Hilbert space, the Kreiss resolvent condition implies that the semigroup grows at most like t. The proof uses the following lemma of Eisner [7], which holds on a general Banach space.

Lemma 2.1. *Assume that A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ and let $s_0(A)$ be the pseudo spectral bound, i.e.,*

$$s_0(A) = \inf\{r \in \mathbb{R} \mid (sI - A)^{-1} \textit{is uniformly bounded on } Re(s) > r\}.$$

If there exists a $r > s_0(A)$ such that for all $x \in X$ and $y \in X^*$

$$\int_{-\infty}^{\infty} |\langle ((r + i\omega)I - A)^{-2}x, y\rangle| d\omega < \infty, \tag{2}$$

then for all $x \in X$ and $t > 0$ the following equality holds

$$T(t)x = \frac{1}{2\pi t} \int_{-\infty}^{\infty} e^{(r+i\omega)t}((r + i\omega)I - A)^{-2}x d\omega. \tag{3}$$

Note that since the left-hand side does not depend on r, the right-hand side should give the same answer for all $r > s_0(A)$. A consequence of this lemma is the following result.

Theorem 2.2. *Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$. Assume that $\mathbb{C}_0^+$ is contained in the resolvent set of A and that the following conditions hold:*

1. *A satisfies the Kreiss estimate (1);*

2. *There exists a $\rho > 0$ such that for all $x \in X$, $y \in X^*$ the function $\omega \mapsto |\langle ((\rho + i\omega)I - A)^{-2}x, y\rangle|$ is integrable.*

Then there exists a $K > 0$ such that $\|T(t)\| \leq K(1+t)$ for all $t \geq 0$.

Proof. The proof consists out of several steps. First we show that item 2. holds for all $\rho, \tilde{\rho} > 0$. Secondly, we estimate the L_1-norm of the function appearing in item 2. Finally, we apply Lemma 2.1 to show the assertion.

Step 1 Since the Kreiss estimate holds, we have that $s_0(A) \leq 0$, see Lemma 2.1.

Step 2 Since the function $\omega \mapsto \langle((\rho + i\omega)I - A)^{-2}x, y\rangle$ is an element of $L^1(\mathbb{R}, \mathbb{C})$ for all $x \in X$ and $y \in X^*$, we conclude from the uniform boundedness theorem that there exists a constant $M_0 > 0$ such that

$$\|\langle((\rho + i\cdot)I - A)^{-2}x, y\rangle\|_{L^1(\mathbb{R},X)} \leq M_0\|x\|\|y\| \tag{4}$$

for all $x \in X$, $y \in X^*$.

Step 3 Since $\mathbb{C}_0^+$ is contained in the resolvent set of A, we obtain by the resolvent identity that for $r > s_0(A)$

$$((r + i\omega)I - A)^{-2}x = \left[I + (\rho - r)((r + i\omega)I - A)^{-1}\right]^2 ((\rho + i\omega)I - A)^{-2}x$$

for all $r, \rho > 0$. Hence

$$|\langle((r + i\omega)I - A)^{-2}x, y\rangle| \leq \left[1 + |\rho - r|\frac{M}{r}\right]^2 |\langle((\rho + i\omega)I - A)^{-2}x, y\rangle|,$$

where we have used the Kreiss estimate (1). Combining this with the estimate (4), we find that for $s_0(A) < r \leq 1$

$$\begin{aligned} \|\langle((r + i\cdot)I - A)^{-2}x, y\rangle\|_{L^1} &\leq \left[1 + |\rho - r|\frac{M}{r}\right]^2 M_0\|x\|\|y\| \\ &\leq M_1\left[1 + \frac{1}{r}\right]^2 \|x\|\|y\| \end{aligned} \tag{5}$$

holds for some constant M_1.

Step 4 Combining equation (3) with (5), we see that

$$\begin{aligned} |\langle T(t)x, y\rangle| &\leq \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{rt}|\langle((r + i\omega)I - A)^{-2}x, y\rangle| d\omega \\ &\leq \frac{1}{2\pi t}e^{rt}M_1\|x\|\|y\|\left[1 + \frac{1}{r}\right]^2. \end{aligned} \tag{6}$$

Since this holds for all $s_0(A) < r < 1$, and since $s_0(A) \leq 0$, we may choose r in (6) equal to $1/t$. This gives

$$|\langle T(t)x, y\rangle| \leq \frac{1}{2\pi t} e M_1 \|x\| \|y\| \left[1+t\right]^2. \tag{7}$$

So for large t the norm of the semigroup is bounded by $\frac{eM_1}{\pi}t$. Since any C_0-semigroup is uniformly bounded on a compact time interval, the result follows. ■

Remark 2.3. We have the following remarks concerning this theorem.

1. As is clear from the proof of the above theorem, the relation between the constants M (the constant in the Kreiss resolvent condition) and K (the constant in the growth) involves other constants as well. We have that $K = \frac{eM_0}{\pi} \max\{\rho M, \frac{1+(1-\rho)M}{2}\}$. Hence one does not have that K is a universal constant times M.

2. The second condition in Theorem 2.2 is satisfied if

   (a) There exists a $\rho > 0$ such that for all $x \in X$ the function $\omega \mapsto ((\rho + i\omega)I - A)^{-1}x$ lies in $L^p(\mathbb{R}, X)$,

   (b) There exists a $\tilde{\rho} > 0$ such that for all $y \in X^*$ the function $\omega \mapsto ((\tilde{\rho} + i\omega)I - A^*)^{-1}y$ lies in $L^q(\mathbb{R}, X)$

   for some $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Indeed, analogously to step 2 of Theorem 2.2 it is easy to see that these two conditions hold for all $\rho,\ \tilde{\rho} > 0$, whenever $s_0(A) \leq 0$. The rest follows immediately from Cauchy-Schwarz inequality. In particular, we obtain the following corollary.

Corollary 2.4. *If A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space H, and if A satisfies the Kreiss estimate (1), then there exists a $K > 0$ such that $\|T(t)\| \leq K[1+t]$ for all $t \geq 0$.*

Proof. Since the Kreiss resolvent condition holds, we have that $s_0(A) \leq 0$. By Proposition 2 of [17] this implies that the growth bound of the semigroup is less or equal than zero. Hence we have that the function $t \mapsto e^{-t}T(t)x$ is square integrable for every $x \in H$. By Paley-Wiener theorem this implies that the function $\omega \mapsto ((1 + i\omega)I - A)^{-1}x$ is square integrable. Since a similar argument holds for the dual semigroup, we conclude from Theorem 2.2 and Remark 2.3 for $p = q = 2$ that there exists a $K > 0$ such that $\|T(t)\| \leq K[1+t]$ for all $t \geq 0$. ■

3 Worst growth on a Banach space

In this section we construct an example showing that if the generator A on a Banach space satisfies the Kreiss estimate, then the corresponding semigroup need not to be bounded by a constant times t. The example will show that for all $M > 1$ in (1) any exponential growth is possible.

Example 3.1. The example is based on the counterexample on page 254 in Engel and Nagel [8]. Let α be a positive number. As state space we choose $X_\alpha := C_0([0,\infty)) \cap L_1((0,\infty), e^s ds)$ with norm

$$\|f\|_{X_\alpha} = \alpha \sup_{\eta \geq 0} |f(\eta)| + \int_0^\infty |f(\eta)| e^\eta d\eta.$$

With this norm X_α becomes a Banach space, and this space is similar to X_1. On the space X_α we define the operator

$$A_0 f = \dot{f}$$

with $D(A_0) = \{f \in X \mid f \in C^1([0,\infty)), \dot{f} \in X_\alpha\}$. In Engel and Nagel it is shown that the resolvent set of A_0 on X_1 contains every complex number with real part greater than -1. Since X_α is similar to X_1, the same assertion holds for the resolvent set of A_0 on X_α. It is easy to see that A_0 is the infinitesimal generator of the C_0-semigroup $(T_0(t))_{t\geq 0}$ with $(T_0(t)f)(\eta) = f(t+\eta)$. From this expression, one easily sees that $\|T_0(t)\| = 1$ for all $t \geq 0$. Furthermore, it is not hard to see that the inverse of $sI - A_0$ is given by

$$\left((sI - A_0)^{-1} f\right)(\eta) = \int_\eta^\infty e^{s(\eta - \xi)} f(\xi) d\xi \tag{8}$$

for $\mathrm{Re}(s) > -1$. Using this expression one can show that

$$\|(sI - A_0)^{-1} f\|_{X_\alpha} = \|(\mathrm{Re}(s)I - A_0)^{-1} \left(e^{-i\mathrm{Im}(s)\cdot} f(\cdot)\right)\|_{X_\alpha}.$$

Since $\|e^{-i\mathrm{Im}(s)\cdot} f(\cdot)\|_{X_\alpha} = \|f\|_{X_\alpha}$, we find that

$$\|(sI - A_0)^{-1}\| = \|(\mathrm{Re}(s)I - A_0)^{-1}\| \tag{9}$$

Using equation (8) and the definition of the norm, we find for $r > -1$

$$\begin{aligned}
\|(rI - A_0)^{-1} f\|_{X_\alpha} &= \alpha \sup_{\eta \geq 0} \left| \int_\eta^\infty e^{r(\eta-\xi)} f(\xi) d\xi \right| + \\
&\quad \int_0^\infty \left| \int_\eta^\infty e^{r(\eta-\xi)} f(\xi) d\xi \right| e^\eta d\eta \\
&= \alpha \sup_{\eta \geq 0} \left| \int_\eta^\infty e^{(r+1)(\eta-\xi)} e^{-\eta} e^\xi f(\xi) d\xi \right| + \\
&\quad \int_0^\infty \left| \int_\eta^\infty e^{(r+1)(\eta-\xi)} e^\xi f(\xi) d\xi \right| d\eta \\
&\leq \alpha \sup_{\eta \geq 0} \int_\eta^\infty |e^\xi f(\xi)| d\xi + \\
&\quad \int_0^\infty e^{-(r+1)\xi} d\xi \int_0^\infty |e^\xi f(\xi)| d\xi \\
&\leq \alpha \|f\|_X + \frac{1}{r+1} \|f\|_X.
\end{aligned} \tag{10}$$

Since A_0 is the infinitesimal generator of a contraction semigroup, we know that

$$\|(rI - A_0)^{-1}\| \leq \frac{1}{r}$$

for all $r > 0$. Hence by combining this with (10) and (9) we find

$$\|(sI - A_0)^{-1}\| \leq \begin{cases} \alpha + \frac{1}{\mathrm{Re}(s)+1} & \mathrm{Re}(s) \in (-1, 0] \\ \min\{\frac{1}{\mathrm{Re}(s)}, \alpha + \frac{1}{\mathrm{Re}(s)+1}\} & \mathrm{Re}(s) > 0 \end{cases}. \tag{11}$$

Choose an $\varepsilon > 0$, then on $[1/\varepsilon, \infty)$ we have that $r^{-1} \leq (1+\varepsilon)(r+1)^{-1}$ and for $\alpha = \varepsilon^2/(1+\varepsilon)$ we have that $\alpha + (r+1)^{-1} \leq (1+\varepsilon)(r+1)^{-1}$ for $r \in (-1, 1/\varepsilon]$. Thus for any $\varepsilon > 0$ we can find an α such that

$$\|(sI - A_0)^{-1}\| \leq \frac{1+\varepsilon}{\mathrm{Re}(s)+1} \qquad \mathrm{Re}(s) > -1. \tag{12}$$

Yet we construct the infinitesimal generator with exponential growth. Define for $\gamma > 0$

$$\mathcal{A}_\gamma := \gamma A_0 + \gamma I.$$

Then we have that for s with positive real part that

$$\begin{aligned}
\|(sI - \mathcal{A}_\gamma)^{-1}\| &= \|((s-\gamma)I - \gamma A_0)^{-1}\| \\
&= \frac{1}{\gamma} \left\| \left(\left(\frac{s}{\gamma} - 1 \right) I - A_0 \right)^{-1} \right\| \leq \frac{1+\varepsilon}{\mathrm{Re}(s)},
\end{aligned} \tag{13}$$

where we have used (12). Thus $\mathcal{A}_\gamma$ satisfies the Kreiss estimate. Since A_0 is the infinitesimal generator, we have that $\mathcal{A}_\gamma$ is it too. The corresponding semigroup is given by

$$\mathcal{T}_\gamma(t) = e^{\gamma t} T_0(\gamma t).$$

Since $\|T_0(t)\| = 1$ for all $t \geq 0$, we find

$$\|\mathcal{T}_\gamma(t)\| = e^{\gamma t}. \tag{14}$$

Thus we have constructed an infinitesimal generator satisfying the Kreiss estimate (1), but having exponential growth.

4 Worst growth on a Hilbert space

In this section we construct an infinitesimal generator on a Hilbert space which satisfies the Kreiss resolvent condition (1), but its corresponding semigroup grows like t^γ. We can do this construction for any $\gamma < 1$. It turns out that the generator is a bounded operator. As a consequence of this construction, we find $N \times N$ matrices Q_N satisfying the Kreiss resolvent condition for the same constant, and the supremum of $e^{Q_N t}$ of the order N^γ for any $\gamma < 1$.

The idea of this example is based on the papers by Spijker, Tracogna, and Welfert [18], Borovykh and Spijker [3], and on the one page note by Kalton and Montgomery-Smith [16]. Note that the basis of the idea is already in the paper by McCarthy and Schwartz [15] from 1965.

Let w be a positive measurable function from the interval $(-\pi, \pi)$ to $\mathbb{R}$. By $L_2((-\pi,\pi), w)$ we denote the set of all measurable functions from $(-\pi,\pi)$ to $\mathbb{C}$ for which $\int_{-\pi}^{\pi} |f(x)|^2 w(x) dx < \infty$. This space is a Hilbert space with inner product

$$\langle f, g \rangle_w = \int_{-\pi}^{\pi} f(x) \overline{g(x)} w(x) dx. \tag{15}$$

By $\text{span}_{k \in \mathbb{Z}}\{e^{ikx}\}$, we denote the finite span, i.e., $f \in \text{span}_{k \in \mathbb{Z}}\{e^{ikx}\}$ can be written as $f = \sum_{k \in \mathbb{Z}} \alpha_k e^{ikx}$ with all but finitely many α_k's equal to zero. Using a result by Hunt, Muckenhoupt, and Wheeden [9] it is not hard to show the following.

Lemma 4.1. *For $n \in \mathbb{Z}$ and $f \in \text{span}_{k \in \mathbb{Z}}\{e^{ikx}\}$, we define*

$$(P_n f)(x) = \sum_{k=-\infty}^{n} \alpha_k e^{ikx}. \tag{16}$$

Let w satisfies the condition

$$\sup_{I\subset[-\pi,\pi]} \frac{1}{|I|^2}\int_I w(x)dx \int_I w(x)^{-1}dx < \infty, \tag{17}$$

where I is an interval and $|I|$ is the length of this interval. Then the following holds.

1. *The P_n's are bounded linear projections on $L_2((-\pi,\pi),w)$, and they are uniformly bounded, i.e.,*

$$\|P_n f\|_w \leq c_w\|f\|_w \tag{18}$$

   for all $f\in L_2((-\pi,\pi),w)$, where c_w does not depend on f and n.

2. *For all $f\in L_2((-\pi,\pi),w)$ we have that*

$$\lim_{n\to-\infty} P_n f = 0, \qquad \lim_{n\to\infty} P_n f = f, \tag{19}$$

The above results imply that the set $\{\ldots, e^{-i2x}, e^{-ix}, 1, e^{ix}, e^{i2x},\ldots\}$ is a conditional basis on $L_2((-\pi,\pi),w)$.

Proof. The proof consists out of several steps. In the first two steps we show that (18) is satisfied, and in the last step we prove (19).
Step 1. Define the mapping

$$(M_n f)(x) = e^{inx}f(x)$$

Since e^{inx} has absolute value one, it is easy to see that M_n is a bounded linear mapping on $L_2((-\pi,\pi),w)$ with norm one.

The conjugate function $\tilde{f}$ is defined as

$$\tilde{f}(y) := \frac{1}{2\pi}\lim_{\varepsilon\downarrow 0}\int_{\varepsilon\leq|y|\leq\pi}\frac{f(x-y)}{\tan(\frac{y}{2})}dy. \tag{20}$$

We denote $Hf := \tilde{f}$. It is not hard to see (for example using induction) that

$$\left(He^{inx}\right)(y) = \begin{cases} -ie^{iny}, & n>0; \\ 0, & n=0; \\ ie^{iny}, & n<0. \end{cases} \tag{21}$$

Theorem 1 of [9] shows that H is a bounded linear mapping on $L_2((-\pi,\pi),w)$ whenever w satisfies condition (17).

Step 2. For $f = \sum_k \alpha_k e^{ik\cdot} \in \mathrm{span}_{k\in\mathbb{Z}}\{e^{ikx}\}$ we consider the following mapping

$$\begin{aligned} -iM_nHM_{-n}f + f + \alpha_n e^{inx} &= -iM_nH\left(\sum_k \alpha_k e^{i(k-n)x}\right) + f + \alpha_n e^{inx} \\ &= 2\sum_{k\leq n} \alpha_k e^{ikx} = 2P_n f, \end{aligned}$$

where we have used (21). So we see that the right combination of M's and H equals the projection P_n. Using the bounds on M_n, M_{-n}, H from step 1, we see that this projection is uniformly bounded if and only if the norm of $\alpha_n e^{inx}$ is bounded by some constant (independent of n and f) times the norm of f. Using (17), we see that

$$\begin{aligned} \|\alpha_n e^{inx}\|_w^2 &= |\alpha_n|^2 \int_{-\pi}^{\pi} |e^{inx}|^2 w(x)dx \\ &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right|^2 \int_{-\pi}^{\pi} w(x)dx \\ &\leq \frac{1}{4\pi^2}\int_{-\pi}^{\pi} |f(x)|^2 w(x)dx \int_{-\pi}^{\pi} |e^{-inx}|^2 w(x)^{-1}dx \int_{-\pi}^{\pi} w(x)dx \\ &= c_1\|f\|_w^2, \end{aligned}$$

where $c_1 = \frac{1}{4\pi^2}\int_{-\pi}^{\pi} w(x)^{-1}dx \int_{-\pi}^{\pi} w(x)dx$, which is finite by (17). So we have shown that

$$\|P_n f\|_w \leq c_w \|f\|_w$$

for all $f \in \mathrm{span}_{k\in\mathbb{Z}}\{e^{ikx}\}$. Since by Theorem 8 of [9] this span is dense in $L_2((-\pi,\pi),w)$, we have proved (18).

Step 3. Like in the previous step we first choose $f \in \mathrm{span}_{k\in\mathbb{Z}}\{e^{ikx}\}$. Since $f = \sum_{k\in\mathbb{Z}} \alpha_k e^{ikx}$ with all but finitely many α_k's equal to zero, it is easy to see that (19) holds. Since the projection are uniformly bounded and since the finite span is dense in $L_2((-\pi,\pi),w)$, we have that (19) holds for all $f \in L_2((-\pi,\pi),w)$. ∎

Since the exponential function will be used a lot, we simplify notation a little bit. For $n \in \mathbb{Z}$ we define $\phi_n(x) = e^{inx}, x \in [-\pi,\pi]$. A consequence of the above lemma is the following result.

Lemma 4.2. *Assume that w is a positive weight satisfying the condition (17) and that $\{\beta_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is a sequence with $\beta_n \leq \beta_{n+1}$ for all $n \in \mathbb{Z}$.*

1. *If* $\sup|\beta_n| < \infty$*, then the operator* Q *defined as* $Q\phi_n = i\beta_n\phi_n$*,* $n \in \mathbb{N}$ *extends to a bounded linear operator on* $L_2((-\pi,\pi),w)$*. Furthermore, we have that*

$$\|Q\| \leq c_w \left[\lim_{n\to\infty}[\beta_n - \beta_{-n}]\right] + \lim_{n\to\infty}|\beta_n|. \tag{22}$$

2. *The operator* $R(s,Q)$ *defined as* $R(s,Q)\phi_n = (s - i\beta_n)^{-1}\phi_n$*,* $n \in \mathbb{N}$ *extends for all* $s \in \mathbb{C}$ *with* $Re(s) \neq 0$ *to a bounded linear operator on* $L_2((-\pi,\pi),w)$*. Furthermore, we have that*

$$\|R(s,Q)\| \leq \frac{1+\pi c_w}{|\mathrm{Re}(s)|} \quad \textit{for } \mathrm{Re}(s) \neq 0. \tag{23}$$

3. *If* $\sup|\beta_n| < \infty$*, then every* $s \in \mathbb{C}$ *with* $Re(s) \neq 0$ *is in the resolvent set of* Q*, and* $R(s,Q) = (sI - Q)^{-1}$*.*

In both estimates c_w *is the constant from (18).*

Proof. 1. Let f be an element in the span of ϕ_n, then $f = \sum_{n=-N}^{N}\alpha_n\phi_n$ for some $N > 0$. By the linearity of Q we have

$$\begin{aligned}
Q\left(\sum_{n=-N}^{N}\alpha_n\phi_n\right) &= \sum_{n=-N}^{N}\alpha_n Q\phi_n = \sum_{n=-N}^{N}\alpha_n i\beta_n\phi_n \\
&= \sum_{n=-N}^{N} i\beta_n\left(P_n f - P_{n-1}f\right) \\
&= \sum_{n=-N}^{N} i\beta_n P_n f - \sum_{n=-N}^{N} i\beta_n P_{n-1} f \\
&= \sum_{n=-N}^{N} i\beta_n P_n f - \sum_{n=-N-1}^{N-1} i\beta_{n+1} P_n f \\
&= \sum_{n=-N}^{N-1}\left[i\beta_n - i\beta_{n+1}\right] P_n f + i\beta_N f, \qquad (24)
\end{aligned}$$

where we have used that for our f, $P_N f = f$, and $P_{-N-1} f = 0$. Thus

$$\begin{aligned}
&\left\| Q \left(\sum_{n=-N}^{N} \alpha_n \phi_n \right) \right\|_w \\
&\leq \sum_{n=-N}^{N-1} |i\beta_n - i\beta_{n+1}| \|P_n f\|_w + |\beta_N| \|f\|_w \\
&= \sum_{n=-N}^{N-1} [\beta_{n+1} - \beta_n] \|P_n f\|_w + |\beta_N| \|f\|_w \\
&\leq c_w \sum_{n=-N}^{N-1} [\beta_{n+1} - \beta_n] \|f\|_w + |\beta_N| \|f\|_w \\
&\leq \left[c_w \lim_{N\to\infty} [\beta_N - \beta_{-N}] + \lim_{N\to\infty} |\beta_N| \right] \|f\|_w,
\end{aligned}$$

where we have used twice that $\beta_n \leq \beta_{n+1}$. Since the expression between the square brackets is finite, and since the span of the ϕ_n is dense, we have that Q is a bounded operator. Furthermore, we get that $\|Q\|$ is bounded by the expression within the square brackets, or equivalently that (22) holds.

2. Similar as in (24) we have for $f = \sum_{n=-N}^{N} \alpha_n \phi_n$ that

$$\begin{aligned}
R(s,Q)f &= \sum_{n=-N}^{N-1} \left[(s - i\beta_n)^{-1} - (s - i\beta_{n+1})^{-1} \right] P_n f + \\
&\quad (s - i\beta_N)^{-1} f.
\end{aligned}$$

Since for every real β we have that $|(s - i\beta)^{-1}| \leq |\mathrm{Re}(s)|^{-1}$, we obtain by (18) that

$$\|R(s,Q)f\|_w \leq \left[c_w \sum_{n=-N}^{N-1} \left| \frac{1}{(s - i\beta_n)} - \frac{1}{(s - i\beta_{n+1})} \right| + \frac{1}{|\mathrm{Re}(s)|} \right] \|f\|_w. \quad (25)$$

So it remains to show that $|\mathrm{Re}(s)| \sum_{n=-N}^{N-1} \left| \frac{1}{(s-i\beta_n)} - \frac{1}{(s-i\beta_{n+1})} \right|$ is bounded.

For this we write $s = a + ib$, with a and b real, $a \neq 0$.

$$
\begin{aligned}
&|a| \sum_{n=-N}^{N-1} \left| \frac{1}{(a+ib-i\beta_n)} - \frac{1}{(a+ib-i\beta_{n+1})} \right| \\
&= \sum_{n=-N}^{N-1} \left| \frac{1}{1+i\frac{b-\beta_n}{a}} - \frac{1}{1+i\frac{b-\beta_{n+1}}{a}} \right| \\
&= \sum_{n=-N}^{N-1} \left| \int_{\frac{b-\beta_{n+1}}{|a|}}^{\frac{b-\beta_n}{|a|}} \frac{-i}{(1+i\eta)^2} d\eta \right| \\
&\leq \sum_{n=-N}^{N-1} \int_{\frac{b-\beta_{n+1}}{|a|}}^{\frac{b-\beta_n}{|a|}} \frac{1}{1+\eta^2} d\eta \\
&= \int_{\frac{b-\beta_N}{|a|}}^{\frac{b-\beta_{-N}}{|a|}} \frac{1}{1+\eta^2} d\eta \\
&\leq \int_{-\infty}^{\infty} \frac{1}{1+\eta^2} d\eta = \pi,
\end{aligned} \tag{26}
$$

where we have used the monotonicity of β_n. Combining (25) with (26), we conclude that (23) holds.

3. Since $R(s, Q)$ is the inverse of Q on the basis elements and since $R(s, Q)$ is bounded, the assertion follows immediately. ∎

Note that the above proof is an adaptation of lemma 3.2.5 of Benamara and Nikolski [2] which gives a bound on diagonal operators on a conditional basis.

Let w be a weight which satisfies condition (17). On $L_2((-\pi, \pi), w)$ we introduce the operators which will be used for our counter example. For $N > 0$ we define A_N as

$$
A_N \phi_n := \begin{cases} \frac{in}{N} \phi_n, & |n| \leq N; \\ -i\phi_n, & n < -N; \\ i\phi_n, & n > N. \end{cases} \tag{27}
$$

From Lemma 4.2 the following properties are immediate

- The A_N extend to linear bounded operators on $L_2((-\pi, \pi), w)$ and the norm of these operators is uniformly bounded by $2c_w + 1$.

- For each $s \in \mathbb{C}$ with nonzero real part we have that $sI - A$ is boundedly invertible on $L_2((-\pi,\pi),w)$ and

$$\|(sI - A_N)^{-1}\| \leq \frac{1+\pi c_w}{|\mathrm{Re}(s)|} \qquad \text{for } \mathrm{Re}(s) \neq 0. \tag{28}$$

  Hence the operators A_N satisfy the Kreiss estimate for the same constant.

Since A_N is a bounded operator, it generates the C_0-group $\left(e^{A_N t}\right)_{t\in\mathbb{R}}$. This group has the following property

Lemma 4.3. *For the operator A_N as defined in (27) we have that*

$$e^{A_N t}\phi_n = \begin{cases} e^{i\frac{n}{N}t}\phi_n, & |n| \leq N; \\ e^{-it}\phi_n, & n < -N; \\ e^{it}\phi_n, & n > N, \end{cases} \tag{29}$$

and

$$e^{A_N N\pi}\left(\frac{\sin((N+1/2)x)}{\sin(x/2)}\right) = (-1)^N \frac{\cos((N+1/2)x)}{\cos(x/2)}. \tag{30}$$

Proof. Using the fact that A_N is diagonal, it is not hard to show that (29) holds. So we concentrate on the other equality. First we remark that

$$\frac{\sin((N+1/2)x)}{\sin(x/2)} = \sum_{|n|\leq N} e^{inx} = \sum_{|n|\leq N} \phi_n.$$

So by equation (29) we have that

$$\begin{aligned} e^{A_N N\pi}\frac{\sin((N+1/2)x)}{\sin(x/2)} &= e^{A_N N\pi}\sum_{|n|\leq N}\phi_n \\ &= \sum_{|n|\leq N} e^{A_N N\pi}\phi_n \\ &= \sum_{|n|\leq N} e^{in\pi}\phi_n \\ &= \sum_{|n|\leq N} (-1)^n e^{inx} = (-1)^N \frac{\cos((N+1/2)x)}{\cos(x/2)}. \end{aligned}$$

Hence we have shown (30). ■

With these lemma's we can now construct our example.

Example 4.4. Let γ be a positive number less than 1. We shall construct a bounded operator $\mathcal{A}_\gamma$ such that $e^{\mathcal{A}_\gamma t}$ behaves like $|t|^\gamma$.

We begin by choosing the weight function. Let $0 < \gamma < 1$ be fixed and

$$w(x) = \begin{cases} |x|^\gamma, & -\frac{\pi}{2} \le x \le \frac{\pi}{2}; \\ (\pi - |x|)^{-\gamma}, & \frac{\pi}{2} < |x| < \pi. \end{cases} \tag{31}$$

We proceed as follows. In step 1. we show that this weight satisfies condition (17), and in step 2. we prove that the induced operator norm on $L_2((-\pi,\pi),w)$ of $e^{A_N N\pi}$ is larger than N^γ. In the last step we construct $\mathcal{A}_\gamma$.

Step 1. To prove (17), let us consider first $I = [a,b]$, where $0 < a < b \le \frac{\pi}{2}$. Then

$$\begin{aligned} K_I &:= \frac{1}{|I|^2}\int_I w(x)dx \int_I w(x)^{-1}dx = \frac{1}{(b-a)^2}\int_a^b x^\gamma dx \int_a^b x^{-\gamma}dx \\ &= \frac{1}{(1-\gamma^2)(b-a)^2}(b^{1+\gamma}-a^{1+\gamma})(b^{1-\gamma}-a^{1-\gamma}) = [z := \frac{b}{a}] \\ &= \frac{1}{(1-\gamma^2)(z-1)^2}(z^{1+\gamma}-1)(z^{1-\gamma}-1) \le \frac{1}{1-\gamma^2}. \end{aligned}$$

Since this estimate is independent of a, it holds also for intervals of the form $I = [a,b]$, where $0 \le a < b \le \frac{\pi}{2}$.

Let I be now an arbitrary interval. Notice that if $|I| \ge \delta$ for some $\delta > 0$, then we have immideately that $K_I \le C\delta^{-2}$, where $C := \int_{-\pi}^{\pi} w(s)ds \cdot \int_{-\pi}^{\pi} w(s)^{-1}ds$. So we can assume $|I| \le \frac{\pi}{2}$. By symmetry reason it suffices to consider intervals of the form $J = [-a,b]$ for $0 \le a < b \le \frac{\pi}{2}$. For such intervals we obtain

$$\begin{aligned} (a+b)^2 K_{[-a,b]} &= a^2 K_{[-a,0]} + b^2 K_{[0,b]} + \\ &\quad \int_{-a}^0 w(x)dx \cdot \int_0^b w(x)^{-1}dx + \int_0^b w(x)dx \cdot \int_{-a}^0 w(x)^{-1}dx \\ &\le \frac{a^2+b^2}{1-\gamma^2} + 2\int_0^b w(x)dx \cdot \int_0^b w(x)^{-1}dx \\ &\le \frac{a^2+3b^2}{1-\gamma^2} \le \frac{3}{1-\gamma^2}(a+b)^2. \end{aligned}$$

So the weight w satisfies condition (17).

Step 2. In this step we show that

$$\|e^{A_N N\pi} f_N\|_w^2 \ge N^{2\gamma}\|f_N\|_w^2, \tag{32}$$

where

$$f_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)}.$$

We estimate first the left hand side of (32). For this purpose we need the following trigonometrical facts which are easy to prove:

$$\begin{aligned} \cos(x/2) &\leq \frac{\pi - x}{2}, \qquad x \leq \pi; \\ |\cos((N+1/2)x)| &\geq \frac{2}{\pi}(N+1/2)(\pi - x), \qquad \frac{2N\pi}{2N+1} \leq x \leq \pi. \end{aligned}$$

Using these inequalities and Lemma 4.3 we have

$$\begin{aligned} \|e^{A_N N\pi} f_N\|_w^2 &= 2\int_0^\pi \left| \frac{\cos((N+1/2)x)}{\cos(x/2)} \right|^2 w(x)dx \\ &\geq 2\int_{2N\pi/(2N+1)}^\pi \left| \frac{\cos((N+1/2)x)}{\cos(x/2)} \right|^2 \frac{dx}{(\pi - x)^\gamma} \\ &\geq \frac{8(2N+1)^2}{\pi^2} \int_{2N\pi/(2N+1)}^\pi \frac{dx}{(\pi - x)^\gamma} \\ &= \frac{8(2N+1)^2}{\pi^2} \frac{\pi^{1-\gamma}}{1-\gamma} \left(1 - \frac{2N}{2N+1}\right)^{1-\gamma} \\ &= \frac{8}{(1-\gamma)\pi^{1+\gamma}}(2N+1)^{1+\gamma} \geq \frac{2^{4+\gamma}}{(1-\gamma)\pi^2} N^{1+\gamma}. \end{aligned}$$

To estimate the right hand side of (32) we show first that

$$|f_N(x)| \leq \frac{\pi}{x}, \qquad 0 < x \leq \pi; \tag{33}$$

$$|f_N(x)| \leq \pi(N+1/2), \qquad 0 < x \leq \pi. \tag{34}$$

Really, for $x \in (0, \pi]$ we have

$$\left| \frac{\sin((N+1/2)x)}{\sin(x/2)} \right| \leq \frac{1}{\sin(x/2)} \leq \frac{\pi}{x}$$

and

$$\begin{aligned} \left| \frac{\sin((N+1/2)x)}{\sin(x/2)} \right| &\leq \pi \frac{|\sin((N+1/2)x)|}{x} \\ &= \pi(N+1/2) \frac{|\sin((N+1/2)x)|}{(N+1/2)x} \leq \pi(N+1/2). \end{aligned}$$

Inequalities (33) and (34) are proved. Using them we obtain the following estimate for the left hand side of (32):

$$\begin{aligned}
\|f_N\|_w^2 &= 2\int_0^\pi |f_N(x)|^2 w(x)dx \\
&\le 2\int_0^{1/N} \pi^2(N+1/2)^2 x^\gamma dx + 2\int_{1/N}^{\pi/2} \frac{\pi^2}{x^2}x^\gamma dx + 2\int_{\pi/2}^{\pi} \frac{\pi^2}{x^2}\frac{dx}{(\pi-x)^\gamma} \\
&\le 2\pi^2(N+1/2)^2 \frac{1}{N^{1+\gamma}(1+\gamma)} - \frac{2\pi^2}{\gamma-1}\frac{1}{N^{\gamma-1}} + \frac{8}{1-\gamma}\left(\frac{\pi}{2}\right)^{1-\gamma} \\
&\le \frac{8\pi^2}{\gamma+1}N^{1-\gamma} + \frac{2\pi^2}{1-\gamma}N^{1-\gamma} + \frac{8}{1-\gamma}\left(\frac{\pi}{2}\right)^{1-\gamma} \\
&\le \frac{\pi^2(10-6\gamma)}{1-\gamma^2}N^{1-\gamma} + \frac{3\pi^2}{1-\gamma} \le \frac{16\pi^2}{1-\gamma^2}N^{1-\gamma}.
\end{aligned}$$

So we see that

$$\frac{\|e^{A_N N\pi} f_N\|_w^2}{\|f_N\|_w^2} \ge 2^\gamma(1+\gamma)N^{2\gamma} \ge N^{2\gamma},$$

and therefore $\|e^{A_N N\pi}\| \ge N^\gamma$ holds for all $N \in \mathbb{N}$. Note that the operator $-A$ has the same form as A, so by analogous construction we obtain that the group satisfies

$$\|e^{\pm A_N N\pi}\| \ge N^\gamma. \tag{35}$$

Step 3. Let $0 < \gamma < 1$. Consider the Hilbert space $H := l^2\left(L_2((-\pi,\pi), w)\right)$, where w is given by (31). The inner product on this space is given by

$$\langle (x_n), (y_n)\rangle = \sum_{n=1}^\infty \langle x_n, y_n\rangle_w.$$

If $Q = \mathrm{diag}(Q_n)$ is a (block) diagonal operator on H, then the norm of this operator is given by

$$\|Q\| = \sup_n \|Q_n\|. \tag{36}$$

On H we define $\mathcal{A}_\gamma := \mathrm{diag}(A_n)$. By step 2 and (36), $\mathcal{A}_\gamma$ is a bounded operator on H and it satisfies the Kreiss resolvent condition

$$\|R(s, \mathcal{A}_\gamma)\| \le \frac{1+\pi c_w}{|\mathrm{Re}(s)|}$$

for all s with $\mathrm{Re}(s) \neq 0$. Moreover, by estimate (35) and (36) the group generated by $\mathcal{A}_\gamma$ satisfies

$$\|e^{\mathcal{A}_\gamma N\pi}\| \geq \|e^{A_N N\pi}\| \geq N^\gamma,$$

and the same holds at $t = -N\pi$. So we conclude that $e^{\mathcal{A}_\gamma t}$ grows at least as $|t|^\gamma$. ■

There are several remarks to be made.

Remark 4.5. Concerning the Example 4.4 we have

- Note that it is not clear if there exists an operator on a Hilbert space satisfying Kreiss resolvent condition such that the semigroup generated by this operator grows exactly as t.
- The operator $\mathcal{A}_\gamma$ constructed in the above example is the infinitesimal generator of a unbounded group. If the group on positive time would be bounded, then the Kreiss resolvent condition on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$ implies that the group is bounded on all time, see [4].

Now it remains to construct a matrix which satisfies the Kreiss resolvent condition, but whose exponential function becomes (almost) the dimensions. The example is more or less present in the previous example, but in order to clarify the construction, we shall present the details. Before we do so, we first present a simple lemma.

Lemma 4.6. *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let W be bounded, linear operator on H which is positive and boundedly invertible. With this W we define a new norm on H,*

$$\|f\|_W^2 = \langle f, Wf \rangle.$$

Then for any bounded linear mapping Q on H we have that

$$\|Q\|_W := \sup_{f\neq 0} \frac{\|Qf\|_W}{\|f\|_Q} = \|W^{\frac{1}{2}} Q W^{-\frac{1}{2}}\| := \sup_{f\neq 0} \frac{\|W^{\frac{1}{2}} Q W^{-\frac{1}{2}} f\|}{\|f\|}.$$

Example 4.7. Let $\mathcal{V}_N$ be the $(2N+1)$-dimensional linear subspace of $L_2((-\pi,\pi), w)$ which is spanned by $e^{ik\cdot}$, $k = -N, \dots, N$.

If $f(\cdot) = \sum_{k=-N}^{N} \alpha_k e^{ik\cdot}$, then

$$\|f\|_w^2 = \langle (\alpha_k), W(\alpha_k) \rangle_{\mathbb{C}^{2N+1}}, \tag{37}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^{2N+1}}$ is the standard inner-product on $\mathbb{C}^{2N+1}$, and

$$W = (W_{kl})_{k,l=1,\dots,2N+1} \qquad \text{with} \tag{38}$$

$$W_{kl} = \int_{-\pi}^{\pi} e^{(i(k-1-N)x} e^{(-i(l-1-N)x} w(x)dx. \tag{39}$$

With this W we define a new inner product on $\mathbb{C}^{2N+1}$, namely

$$\|(\alpha_k)\|_W^2 = \langle (\alpha_k), W(\alpha_k)\rangle_{\mathbb{C}^{2N+1}}. \tag{40}$$

Now we can define the matrix on $\mathbb{C}^{2N+1}$ which has growth of the order N^γ. We define the $2N+1$ by $2N+1$ matrix Q_N as

$$Q_N = W^{\frac{1}{2}} \text{diag}\left(i\frac{k}{N}\right) W^{-\frac{1}{2}}, \tag{41}$$

where W is defined in (38), (39) and w is given by (31). First we show that this matrix satisfies the Kreiss condition with a constant independent of N. It is easy to see that

$$(sI - Q_N)^{-1} = W^{\frac{1}{2}} \text{diag}\left((s - i\frac{k}{N})^{-1}\right) W^{-\frac{1}{2}}.$$

Using Lemma 4.6, we find that

$$\begin{aligned} \|(sI - Q_N)^{-1}\|_{\mathbb{C}^{2N+1}} &= \|\text{diag}\left((s - i\frac{k}{N})^{-1}\right)\|_W \\ &= \sup_{(\alpha_k)\neq 0} \frac{\|\left(\text{diag}\left((s - i\frac{k}{N})^{-1}\right)\right)(\alpha_k)\|_W}{\|(\alpha_k)\|_W} \\ &= \sup_{f\in\mathcal{V}_N\setminus\{0\}} \frac{\|(sI - A_N)^{-1} f\|_w}{\|f\|_w}, \end{aligned}$$

where we have used (27) and (37). Since $\mathcal{V}_N$ is a subspace of $L_2(-\pi,\pi), w)$, we have by (28) that

$$\|(sI - Q_N)^{-1}\|_{\mathbb{C}^{2N+1}} \leq \frac{1 + \pi c_w}{|\text{Re}(s)|} \tag{42}$$

for $\text{Re}(s) \neq 0$. Similar one can show that

$$\|e^{Q_N N\pi}\|_{\mathbb{C}^{2N+1}} \geq N^\gamma \tag{43}$$

and thus completing the finite-dimensional example.

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