The spectral mapping property of delay semigroups

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To the memory of G. S. Litvinchuk

Abstract. We offer a new way of proving spectral mapping properties of delay semigroups in L^p-history spaces with finitely many rationally depending delays based on a an explicit construction of approximate eigenvectors. This allows us to provide proper generalizations of the existing spectral mapping theorems.

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1. Introduction

Partial differential equations with delay play an important role in science and engineering and have been studied for many years and by many different methods.

Hale [6] and Webb [11] were among the first to apply semigroup theory to delay equations, and we refer to [1], Diekman et al. [4], or Wu [12] for more recent references on differential equations with delay. By now, it is well-known that fairly general classes of abstract delay equations generate strongly continuous semigroups on appropriate function spaces. However, spectral properties of the delay semigroup are not well understood. In particular, the following basic question still remains open, in general: If the spectral mapping property

$$\sigma\Big(\mathcal{T}(t)\Big) \setminus \{0\} = \mathrm{e}^{t\sigma(\mathcal{A})}, \quad t > 0, \tag{1.1}$$

holds for the spectrum $\sigma(\cdot)$ of the delay semigroup $(\mathcal{T}(t))_{t\geq 0}$ and its generator $\mathcal{A}$.

The spectral mapping property (1.1) is widely discussed in the literature on abstract strongly continuous semigroups, see e.g. the corresponding chapters in Nagel et al. [9], Engel and Nagel [5], van Neerven [10], or [3]. This property is obviously of great importance because in concrete problems we often have a good characterization of the generator but no explicit knowledge of the semigroup itself. The spectral mapping property always holds for many known classes of abstract semigroups, e.g., for eventually norm continuous, in particular, analytic or eventually compact semigroups. In many applications, for example, in the study of invariant manifolds for nonlinear evolution equations, even a more restrictive result,

$$\mathbb{T}\cdot\sigma(\mathcal{T}(t)) = \overline{\mathbb{T}\cdot e^{t\sigma(\mathcal{A})}}, \tag{1.2}$$

where $\mathbb{T} := \{|\lambda| = 1\}$ is the unit circle, is also of great importance. Indeed, it tells us that the hyperbolicity of the semigroup $(\mathcal{T}(t))_{t\geq 0}$ can be characterized via the spectrum of $\mathcal{A}$. Moreover, in the theory of strongly continuous semigroups one is often interested in a consequence of (1.2), that is, in conditions for the equality

$$s(\mathcal{A}) = \omega_0(\mathcal{A}), \tag{1.3}$$

where $s(\mathcal{A}) := \sup\{\mathrm{Re}\lambda \,:\, \lambda \in \sigma(\mathcal{A})\}$ and $\omega_0(\mathcal{A}) = \lim_{t\to\infty} \frac{\log\|\mathcal{T}(t)\|}{t}$ is the spectral bound of the generator and the growth bound of the semigroup, respectively.

Dealing with delay semigroups, results of type (1.1), (1.2) or (1.3) are currently known only in the situations where the delay system is finite dimensional, or under some restrictive compactness assumptions leading to the fact that the delay semigroup becomes eventually norm continuous. Thus, in these situations, properties (1.1), (1.2) and (1.3) follow from general results on strongly continuous semigroups. However, there are many striking examples of abstract semigroups for which the spectral mapping properties fail. Using these examples, we show below that, in general, the spectral mapping properties fail for delay semigroups as well.

In this paper we offer a new approach to properties (1.1), (1.2) and (1.3) that does not require that the delay semigroup is eventually norm continuous and give natural sufficient conditions for which these properties hold. More precisely, we study the spectral mapping property of the semigroup associated to the abstract delay equation of the form

$$\begin{cases} u'(t) = Bu(t) + \Phi u_t, \quad t \geq 0, \\ u(0) = x, \\ u_0 = f, \end{cases} \tag{DE}$$

in a Banach space X, where $(B, D(B))$ is the (unbounded) generator of a strongly continuous semigroup of linear operators on X, $u_t(\cdot) = u(t+\cdot)$ on $[-1,0]$, and the delay operator Φ is assumed to have the form

$$\Phi(f) := \sum\nolimits_{j=1}^{m} C_j f(-\tfrac{j}{m}), \quad f : [-1,0] \to X, \tag{1.4}$$

with given bounded operators $C_j \in \mathcal{L}(X)$, $j = 1,\dots,m$, $m \in \mathbb{N}$. Notice that this setting in fact covers a much broader class of delays than (1.4), namely the case of finitely many rationally depending delays, see remark 2.4.

As a first step one has to choose an appropriate state space for the solution u. One of the possibilities is to work in the space of continuous X-valued functions.

However, the state space $\mathcal{E} := X \times L^p([-1,0],X)$ turns out to be a better choice with regards to certain applications (e.g., in control theory, see Bensoussan et. al. [2], in numerical methods, see Kappel [7]), and will be used in this paper.

To formulate the main result of the current paper, for any $\lambda \in \mathbb{C}$ and $\tau \in [-1,0]$ we let $\epsilon_\lambda(\tau) := \mathrm{e}^{\lambda\tau}$ and define the operator, $\Phi_\lambda \in \mathcal{L}(X)$, as follows:

$$\Phi_\lambda x := \Phi(\epsilon_\lambda \otimes Id)x = \Phi(\mathrm{e}^{\lambda(\cdot)}\, x) \quad \text{for} \quad x \in X. \tag{1.5}$$

Theorem 1.1. *Assume that B and Φ are such that for each $\lambda \in \mathbb{C}$ the strongly continuous semigroup generated by the operator $B + \Phi_\lambda$ has the following spectral mapping property: for all rational $t > 0$,*

$$\sigma\Big(\mathrm{e}^{t(B+\Phi_\lambda)}\Big) \setminus \{0\} = \mathrm{e}^{t\sigma(B+\Phi_\lambda)}\,. \tag{1.6}$$

Then the operator

$$\mathcal{A} := \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \tag{1.7}$$

with the domain

$$D(\mathcal{A}) := \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(B) \times W^{1,p}([-1,0],X) \ : \ f(0) = x \right\} \tag{1.8}$$

generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on the Banach space $\mathcal{E} := X \times L^p([-1,0],X)$ such that the spectral mapping property (1.1) *is satisfied for rational $t > 0$, and, moreover, equality* (1.2) *holds for all $t \geq 0$.*

Corollary 1.2. *Under condition* (1.6)*, the delay semigroup $(\mathcal{T}(t))_{t\geq 0}$ satisfies equality* (1.3)*.*

Condition (1.6) is satisfied in the case where $(B, D(B))$ generates an immediately norm continuous or compact semigroup. As we have mentioned, outside of this class, (1.6) is a nontrivial assumption. We stress again that, in general, neither of the spectral mapping properties (1.1), (1.2) or (1.3) holds for the delay semigroup, see Remark 2.7 below.

2. The delay semigroup

Let us summarize some results from monograph [1], which will be needed later, on the semigroup approach to linear partial differential equations with delay. Consider the general delay equation

$$\text{(GDE)} \qquad \begin{cases} u'(t) = Bu(t) + \Phi u_t, & t \geq 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

where

- $x \in X$, X is a Banach space,
- $B : D(B) \subseteq X \longrightarrow X$ is a linear, closed, and densely defined operator,
- $f \in L^p([-1,0],X)$, $p \geq 1$,

- $\Phi : W^{1,p}([-1,0],X) \longrightarrow X$ is a linear, bounded operator, having the form

$$\Phi(f) := \int_{-1}^{0} d\eta f \tag{2.1}$$

  with a given function $\eta : [-1,0] \to \mathcal{L}(X)$ of bounded variation,
- $u : [-1,\infty) \longrightarrow X$ and $u_t : [-1,0] \longrightarrow X$ is defined by $u_t(\tau) := u(t+\tau)$.

Definition 2.1. *We say that a function $u : [-1,\infty) \longrightarrow X$ is a (classical) solution of (GDE) if*

(i) $u \in C([-1,\infty),X) \cap C^1([0,\infty),X)$,
(ii) $u(t) \in D(B)$ *and* $u_t \in W^{1,p}([-1,0],X)$ *for all* $t \geq 0$, *and*
(iii) u *satisfies (DE) for all* $t \geq 0$.

To solve (GDE) by semigroup methods, we introduce the Banach space

$$\mathcal{E} := X \times L^p([-1,0],X)$$

and the linear operator $\mathcal{A}$ described in (1.7) and (1.8).

Consider now the abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} v'(t) = \mathcal{A}\, v(t), & t \geq 0, \\ v(0) = v_0, \end{cases}$$

associated to the operator matrix $(\mathcal{A}, D(\mathcal{A}))$ on the Banach space $\mathcal{E}$ with initial value $v_0 := \binom{x}{f}$. There is a natural correspondence between the solutions of the two problems, (GDE) and (ACP).

Lemma 2.2. [1, Theorem 3.12]

(i) If u is a solution of (GDE), then $t \mapsto \binom{u(t)}{u_t}$ is a solution of the Cauchy problem (ACP).
(ii) If $t \mapsto \binom{u(t)}{v(t)}$ is a solution of (ACP), then $v(t) = u_t$ for all $t \geq 0$, and u is a solution of (GDE).

As an easy consequence of Lemma 2.2, we have that if $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$, then the solutions u of equation (GDE) are given by the first component of the function $t \mapsto \mathcal{T}(t)\binom{x}{f}$ for $\binom{x}{f} \in D(\mathcal{A})$:

$$u(t) = \begin{cases} x + B\int_0^t u(s)\,ds + \Phi\int_0^t u_s\,ds & \text{for } \ t \geq 0, \\ f(t) & \text{for a.e. } \ t \in [-1,0). \end{cases} \tag{2.2}$$

By means of the perturbation theorem of Miyadera-Voigt, (see [5, Corollary III.3.16]), one can formulate the following sufficient condition of the well-posedness of (GDE). Let us assume that $(B, D(B))$ is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X, $(T_0(t))_{t\geq 0}$ is the nilpotent left shift semigroup on $L^p([-1,0],X)$, and $S_t : X \to L^p([-1,0],X)$ is defined by

$$(S_t\, x)(\tau) := \begin{cases} S(t+\tau)x, & -t < \tau \leq 0, \\ 0, & -1 \leq \tau \leq -t. \end{cases}$$

Theorem 2.3. [1, Theorem 3.29] *Let $(B, D(B))$ be the generator of a strongly continuous semigroup on X and assume that the delay operator Φ is of form (2.1). Then the operator $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a strongly continuous semigroup on $\mathcal{E}$. Thus, (GDE) is well-posed.*

Important special cases are operators Φ defined by

$$\Phi(f) := \sum\nolimits_{k=0}^{n} C_k \delta_{h_k}(f), \quad f \in W^{1,p}([-1,0], X), \tag{2.3}$$

where $\delta_{h_k}(f) = f(h_k)$ are the δ-measures supported at h_k, the operators $C_k \in \mathcal{L}(X)$ are given, and $h_k \in [-1, 0]$ for $k = 0, \dots, n$. In particular, Theorem 2.3 holds for the operators Φ given in (1.4).

Remark 2.4. This seem to be the appropriate point to make a comment on time measurement and the special form (1.4) of the delay in (DE). Assume that we are given an equation of the form

$$v'(t) = Bv(t) + \sum\nolimits_{j=1}^{k} C_j v(t - h_j), t \geq 0, \tag{DE-1}$$

Then, denoting $h := \max\{h_1, \dots, h_k\}$, we can rescale the time making the change of variables $t = hs$ and $u(s) = v(hs)$. Hence we obtain a delay equation where the largest delay equals 1. Further, if all the delays were rationally dependent, meaning that $h_j \in \alpha\mathbb{Q}$ for some $\alpha \in \mathbb{R}$ and all $j = 1, \dots, k$, then the new delays in the rescaled equation will become rational numbers. Hence, there exists $m \in \mathbb{N}$ such that $\tilde{h}_j = \frac{n_j}{m}$ for suitable $n_j \in \mathbb{N}$. Introducing sufficiently many zero operators C_j, we see that (DE-1) could be rewritten as

$$u'(s) = \tilde{B}u(t) + \sum\nolimits_{j=1}^{m} \tilde{C}_j u(t - \tfrac{j}{m}), t \geq 0. \tag{DE-2}$$

Hence, the delay operator in (1.4) actually covers the case of finitely many rationally dependent delays. Notice that the transformation above does not affect the asymptotic behaviour of the solutions (for example, uniform exponential stability or hyperbolicity), but it may affect the speed of the exponential convergence.

Finally, we characterize the resolvent set and the resolvent operator of $\mathcal{A}$ (see [1, Section 3.2]). Let $\epsilon_\lambda(t) := \mathrm{e}^{\lambda t}$ and $\Phi_\lambda \in \mathcal{L}(X)$ be defined by $\Phi_\lambda x := \Phi(\epsilon_\lambda \otimes Id)x = \Phi(\mathrm{e}^{\lambda(\cdot)}\, x)$ for $x \in X$. Further, let $(A_0, D(A_0))$ be the generator of the nilpotent left shift semigroup $(T_0(t))_{t\geq 0}$ in $L^p([-1, 0], X)$. We use notation $\rho(\cdot)$, $\sigma(\cdot)$, $\sigma_p(\cdot)$, $\sigma_a(\cdot)$, $\sigma_r(\cdot)$, and $\sigma_{ess}(\cdot)$, respectively, for the resolvent set, spectrum, point, approximate point, residual, and essential spectrum of an operator $(\cdot)$.

Lemma 2.5. *Let X be a Banach space, $(B, D(B))$ be a linear, closed and densely defined operator, and $\Phi : W^{1,p}([-1, 0], X) \longrightarrow X$ be linear and bounded. Let*

$(\mathcal{A}, D(\mathcal{A}))$ *be the operator matrix defined in (1.7) and (1.8). Then*

$$\lambda \in \sigma(\mathcal{A}) \text{ if and only if } \lambda \in \sigma(B+\Phi_\lambda),$$
$$\lambda \in \sigma_p(\mathcal{A}) \text{ if and only if } \lambda \in \sigma_p(B+\Phi_\lambda),$$
$$\lambda \in \sigma_a(\mathcal{A}) \text{ if and only if } \lambda \in \sigma_a(B+\Phi_\lambda),$$
$$\lambda \in \sigma_r(\mathcal{A}) \text{ if and only if } \lambda \in \sigma_r(B+\Phi_\lambda),$$
$$\lambda \in \sigma_{ess}(\mathcal{A}) \text{ if and only if } \lambda \in \sigma_{ess}(B+\Phi_\lambda).$$

Moreover, for $\lambda \in \rho(\mathcal{A})$ *the resolvent* $R(\lambda, \mathcal{A})$ *is given by*

$$\begin{pmatrix} R(\lambda, B+\Phi_\lambda) & R(\lambda, B+\Phi_\lambda)\Phi R(\lambda, A_0) \\ \epsilon_\lambda \otimes R(\lambda, B+\Phi_\lambda) & [\epsilon_\lambda \otimes R(\lambda, B+\Phi_\lambda)\Phi + Id]R(\lambda, A_0) \end{pmatrix}. \tag{2.4}$$

Using a perturbation argument, one can show the following regularity results for the delay semigroup, see [1, Proposition 4.3], [8].

Theorem 2.6. *Consider the matrix operator* $(\mathcal{A}, D(\mathcal{A}))$ *defined by* (1.7) *and* (1.8).

(a) If $(B, D(B))$ *generates an immediately norm continuous semigroup, then the semigroup generated by* $(\mathcal{A}, D(\mathcal{A}))$ *is norm continuous for* $t > 1$.

(b) If $(B, D(B))$ *generates an immediately compact semigroup, then the semigroup generated by* $(\mathcal{A}, D(\mathcal{A}))$ *is compact for* $t > 1$.

Hence, in these so called regular cases the spectral mapping property for the delay semigroup does hold. It is also known, however, that if $(B, D(B))$ generates an eventually norm continuous or eventually compact semigroup, then this property can be destroyed by the delay term.

Remark 2.7. Formula (2.4) shows that one should not expect the spectral mapping properties (1.1), (1.2) or (1.3) to hold, in general, for delay semigroups. Indeed, consider Φ as in (1.4) with $m = 1$. Choose B and $C = C_1$ such that the spectral mapping property (1.6) does not hold, see examples for such operators in [3, Examples 2.1.5] or in Engel and Nagel [5, Section IV.3.a]. Specifically, let us assume that $2k\pi i \in \rho(B + C)$ for all $k \in \mathbb{Z}$, but $\|(B + C - 2k\pi i)^{-1}\| \to \infty$ as $k \to \infty$. Then $\Phi_{2k\pi i} = C\,\mathrm{e}^{-2k\pi i} = C$, and hence

$$\|R(2k\pi i, B + \Phi_{2k\pi i})\| = \|(B + C - 2k\pi i)^{-1}\| \to \infty, \quad \text{as } k \to \infty. \tag{2.5}$$

By (2.4), we know that $2k\pi i \in \rho(\mathcal{A})$ for all $k \in \mathbb{Z}$, but also by (2.4) and (2.5), we have that $\|R(2k\pi i, \mathcal{A})\| \to \infty$ as $k \to \infty$, and thus $1 \in \sigma(\mathcal{T}(1))$. This shows that assumption (1.6) is a natural assumption on (GDE) if one expects the spectral mapping property for $(\mathcal{T}(t))_{t\geq 0}$.

3. Proof of the main result

We consider from now on delay operators with finitely many rational depending delays, i.e., operators of form (1.4),

$$\Phi := \Sigma_{j=1}^m C_j \delta_{-\frac{j}{m}},$$

for some $m \in \mathbb{N}$. Note that some operators C_j may be zero. Recall, see Remark 2.4, that we rescaled the time so that the greatest time delay is 1. We have the following necessary condition for $\lambda \in \mathbb{C}$ to be in the spectrum of $\mathcal{T}(\frac{1}{m})$.

Proposition 3.1. *Let Φ be as in* (1.4)*, and pick a nonzero $\lambda \in \sigma_a(\mathcal{T}(\frac{1}{m}))$. Then $\lambda \in \sigma_a(\mathrm{e}^{\frac{1}{m}(B+\Phi_\mu)})$ for every μ satisfying $\mathrm{e}^{\frac{\mu}{m}} = \lambda$.*

Proof. By assumption, there exists a sequence $\left\{\binom{x_n}{f_n}\right\}_{n=1}^{\infty} \subset \mathcal{E}$ such that

$$\left\|\binom{x_n}{f_n}\right\|_{\mathcal{E}} = 1, \tag{a}$$

$$\|(\mathcal{T}(\tfrac{1}{m}) - \lambda)\binom{x_n}{f_n}\|_{\mathcal{E}} \to 0, \quad n \to \infty. \tag{b}$$

Our aim is to show that the sequence (x_n) forms an approximate eigenvector corresponding to the eigenvalue λ for the operator $\mathrm{e}^{\frac{1}{m}(B+\Phi_\mu)}$.

Let u be the mild solution of (DE) given by formula (2.2). By Lemma 2.2, assertion (b) means, in terms of u, that

$$\|u(\tfrac{1}{m}) - \lambda x_n\|_X \to 0, \quad n \to \infty, \tag{3.1}$$

$$\|u_{\frac{1}{m}}(\cdot) - \lambda f_n(\cdot)\|_{L^p([-1,0],X)} \to 0, \quad n \to \infty. \tag{3.2}$$

Also by (b), we have, for $j \in \{1, \ldots, m+1\}$,

$$\|[\mathcal{T}(\tfrac{j}{m}) - \lambda\mathcal{T}(\tfrac{j-1}{m})]\binom{x_n}{f_n}\|_{\mathcal{E}} \le \|\mathcal{T}(\tfrac{j-1}{m})\| \cdot \|(\mathcal{T}(\tfrac{1}{m}) - \lambda)\binom{x_n}{f_n}\|_{\mathcal{E}} \to 0, \quad \text{as } n \to \infty$$

and therefore

$$\|u(\tfrac{j}{m}) - \lambda u(\tfrac{j-1}{m})\|_X \to 0, \quad n \to \infty, \quad \forall j = 1, \ldots, m, \tag{3.3}$$

$$\|u_{\frac{j}{m}}(\cdot) - \lambda u_{\frac{j-1}{m}}(\cdot)\|_{L^p([-1,0],X)} \to 0, \quad n \to \infty, \quad \forall j = 1, \ldots, m. \tag{3.4}$$

Take any μ satisfying $\mathrm{e}^{\frac{\mu}{m}} = \lambda$. By representation (2.2) we have for all $t \ge 0$

$$u(t) = x_n + B\int_0^t u(s)ds + \Phi_\mu \int_0^t u(s)ds + g(t),$$

where the function g is given by

$$g(t) := \Phi \int_0^t u_s ds - \Phi_\mu \int_0^t u(s)ds = \Phi\left[\int_0^t u_s ds - \epsilon_\mu \int_0^t u(s)ds\right], \ t \ge 0. \tag{3.5}$$

Then

$$u'(t) = Bu(t) + \Phi_\mu u(t) + g'(t), \quad t \ge 0, \quad u(0) = x_n,$$

and we obtain by the variation of constant formula

$$u(t) = \mathrm{e}^{t(B+\Phi_\mu)}\, x_n + \int_0^t \mathrm{e}^{(t-s)(B+\Phi_\mu)}\, g'(s)ds, \quad t \ge 0. \tag{3.6}$$

First we show that

$$\|g'\|_{L^p([0,1],X)} \to 0, \quad n \to \infty. \tag{3.7}$$

By (3.5) and the form of the delay operator Φ in (1.4), we have:

$$\begin{aligned} g'(t) = \Phi\,[u_t - \epsilon_\mu u(t)] &= \sum_{j=1}^{m} C_j[u(t-\tfrac{j}{m}) - \mathrm{e}^{-j\frac{\mu}{m}}\,u(t)] \\ &= \sum_{j=1}^{m} \lambda^{-j} C_j[\lambda^j u(t-\tfrac{j}{m}) - u(t)]. \end{aligned}$$

We fix $j \in \{1,\dots,m\}$ and show that $\|u(\cdot)-\lambda^j u(\cdot-\frac{j}{m})\|_{L^p([0,1],X)} \to 0,$ as $n \to \infty$. Indeed,

$$\begin{aligned} \|u(\cdot) - \lambda^j u(\cdot - \tfrac{j}{m})\|_{L^p([0,1],X)} &= \|u(1+\cdot) - \lambda^j u(1-\tfrac{j}{m}+\cdot)\|_{L^p([-1,0],X)} \\ = \|u_1 - \lambda^j u_{1-\frac{j}{m}}\|_{L^p([-1,0],X)} &\le \|u_1 - \lambda u_{1-\frac{1}{m}}\|_{L^p([-1,0],X)} \\ &+ |\lambda|\|u_{1-\frac{1}{m}} - \lambda u_{1-\frac{2}{m}}\|_{L^p([-1,0],X)} + \dots \\ &+ |\lambda|^{j-1}\|u_{1-\frac{j-1}{m}} - \lambda u_{1-\frac{j}{m}}\|_{L^p([-1,0],X)}, \end{aligned}$$

and by (3.4) each summand converges to 0 as $n \to \infty$. This proves property (3.7).

Next, we show that $\|\,\mathrm{e}^{\frac{1}{m}(B+\Phi_\mu)}\,x_n - \lambda x_n\|_X \to 0$ as $n \to \infty$. Indeed,

$$\|\,\mathrm{e}^{\frac{1}{m}(B+\Phi_\mu)}\,x_n - \lambda x_n\|_X \le \|\,\mathrm{e}^{\frac{1}{m}(B+\Phi_\mu)}\,x_n - u(\tfrac{1}{m})\|_X + \|u(\tfrac{1}{m}) - \lambda x_n\|_X.$$

The first summand here converges to 0 as $n \to \infty$ by applying the Hölder inequality in (3.6), and using (3.7), while the second by (3.1).

Finally, we show that $\liminf_{n\to\infty} \|x_n\|_X > 0$. Passing to a subsequence, we have from (3.2) and (3.4) as $n \to \infty$:

$$\begin{aligned} \|\lambda f_n\|_{L^p([-1,0],X)} &\le \|u_{\frac{1}{m}}(\cdot) - \lambda f_n\|_{L^p([-1,0],X)} + \|u_{\frac{1}{m}}\|_{L^p([-1,0],X)} \\ &= \|u_{\frac{1}{m}}\|_{L^p([-1,0],X)} + \mathrm{o}(1) \\ &= \|u_{\frac{1}{m}}(\cdot) - \frac{1}{\lambda}u_{2\frac{1}{m}}\|_{L^p([-1,0],X)} + \frac{1}{|\lambda|}\|u_{2\frac{1}{m}}\|_{L^p([-1,0],X)} + \mathrm{o}(1) \\ &= \frac{1}{|\lambda|}\|u_{2\frac{1}{m}}\|_{L^p([-1,0],X)} + \mathrm{o}(1) = \dots \\ &= \frac{1}{|\lambda|^{m-1}}\|u_1\|_{L^p([-1,0],X)} + \mathrm{o}(1) = \frac{1}{|\lambda|^{m-1}}\|u\|_{L^p([0,1],X)} + \mathrm{o}(1). \end{aligned}$$

This implies, by (3.7) and representation (3.6), that

$$\|\lambda^m f_n\|_{L^p([-1,0],X)} \le M\|x_n\|_X + \mathrm{o}(1)$$

for $M := \sup_{t\in[0,1]} \|e^{t(B+\Phi_\mu)}\|$. Thus, $\|x_n\| \to 0$ would contradict (a).

Hence, $\{x_n\}_{n=1}^{\infty}$ is an asymptotic eigenvector for $e^{\frac{1}{m}(B+\Phi_\mu)}$ corresponding to $\lambda \in \sigma_a(\mathrm{e}^{\frac{1}{m}(B+\Phi_\mu)})$. □

Remark 3.2. Let Φ be as in (1.4) and $N \in \mathbb{N}$. Rewriting Φ as $\Phi = \sum_{j=1}^{Nm} C_j\delta_{-\frac{j}{Nm}}$ by adding zero operators C_j, when appropriate and applying Proposition 3.1 with

mN instead of m, we have the following: If a nonzero $\lambda \in \sigma_a(\mathcal{T}(\frac{1}{mN}))$, then $\lambda \in \sigma_a(\mathrm{e}^{\frac{1}{mN}(B+\Phi_\mu)})$ for every μ satisfying $\mathrm{e}^{\frac{\mu}{mN}} = \lambda$.

We will now formulate the following characteristic equation for $\mathcal{T}(\frac{1}{m})$.

Proposition 3.3. *Let* Φ *be as in* (1.4) *and assume that (*1.6*) holds for every* $\lambda \in \mathbb{C}$. *Then the following relation holds for* $\lambda \neq 0$*:*

$$\lambda \in \sigma(\mathcal{T}(\tfrac{1}{m})) \Longleftrightarrow \lambda \in \sigma(e^{\frac{1}{m}(B+\Phi_\mu)}) \textit{ for all/some } \mu \textit{ with } \mathrm{e}^{\frac{1}{m}\mu} = \lambda.$$

Proof. By Proposition 3.1 we only have to prove the implication "$\Leftarrow$". Let $\lambda \in \sigma(\mathrm{e}^{\frac{1}{m}(B+\Phi_\mu)})$ for some μ with $\mathrm{e}^{\frac{\mu}{m}} = \lambda$. By assumption (1.6) there exists $\nu \in \sigma(B+\Phi_\mu)$ with $\lambda = \mathrm{e}^{\frac{\nu}{m}}$. Since $\frac{\nu}{m} = \frac{\mu}{m} + 2\pi ik$ for some $k \in \mathbb{Z}$, we have

$$\Phi_\nu = \sum_{j=1}^{m} C_j e^{-j\frac{\nu}{m}} = \sum_{j=1}^{m} C_j e^{-j\frac{\mu}{m}} = \Phi_\mu.$$

Therefore $\nu \in \sigma(B+\Phi_\nu)$, which means, by Lemma 2.5, that $\nu \in \sigma(\mathcal{A})$. By the spectral inclusion theorem for strongly continuous semigroups, this implies $\lambda = \mathrm{e}^{\frac{\nu}{m}} \in e^{\frac{1}{m}\sigma(\mathcal{A})} \subset \sigma(\mathcal{T}(\frac{1}{m}))$. □

Proof of Theorem 1.1*.* In view of Remark 3.2 and the spectral mapping theorem for polynomials, it is enough to prove (1.1) for $t = \frac{1}{m}$. By the spectral inclusion theorem for strongly continuous semigroups, we only have to prove in (1.1) the inclusion "$\subset$". Fix a nonzero $\lambda \in \sigma(\mathcal{T}(\frac{1}{m}))$. By the spectral mapping theorem for the residual spectrum, we may assume that $\lambda \in \sigma_a(\mathcal{T}(\frac{1}{m}))$. Take any μ satisfying $\mathrm{e}^{\frac{\mu}{m}} = \lambda$. By Proposition 3.1 and assumption (1.6), we have:

$$\lambda \in \sigma(\mathrm{e}^{\frac{1}{m}(B+\Phi_\mu)}) = \mathrm{e}^{\frac{1}{m}\sigma(B+\Phi_\mu)}.$$

Therefore, there exists $\nu \in \sigma(B+\Phi_\mu)$ such that $\mathrm{e}^{\frac{1}{m}\nu} = \lambda$. As in the proof of Proposition 3.3, we have $\frac{\nu}{m} = \frac{\mu}{m} + 2\pi ik$ for some $k \in \mathbb{Z}$ and $\Phi_\mu = \Phi_\nu$. Therefore, $\nu \in \sigma(B+\Phi_\nu)$. By Lemma 2.5, $\nu \in \sigma(\mathcal{A})$ and we thus conclude $\lambda \in \mathrm{e}^{\frac{1}{m}\sigma(\mathcal{A})}$.

Finally, equality (1.2) follows by the fact that if the spectrum of $\mathcal{T}(t)$ for one $t > 0$ does not intersect a circle centered at zero, then it will not intersect the correspondingly rescaled circle for any other $t > 0$, which can be seen by an appropriate modification of [5, Proposition V.1.15]. □

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