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# Pointwise thm for amenable groups
## Part II

**Thm 1** (PET for tempered Følner seq.)

$G$ amenable, $G \curvearrowright (X,\mu)$ by m.p.t., $(F_n)$ tempered Følner seq. Then $\forall f \in L^1(\mu)$ $\exists$ $G$-inv. $\tilde{f} \in L^1$ with

$$\boxed{\lim_{n\to\infty} A(F_n, f)(x) = \tilde{f}(x) \quad \text{a.e.}},$$

where $A(F_n, f) = \frac{1}{|F_n|} \sum_{g \in F_n} \underbrace{(T_g f)(x)}_{f(gx)}$ if $G$ is discrete.

**Thm 2** (Maximal ineq.)

Let $G$ amenable.

Let $(F_n)$ be tempered. Then $\exists\, c>0$ (dep. on $(F_n)$ but indep. of $(X,\mu)$ comp) s.t. $\forall f \in L^1(X,\mu)$

$$\boxed{\mu\left(x : (Mf)(x) > \lambda\right) \le \frac{c}{\lambda}\|f\|_1},$$

where $(Mf)(x) := \sup_n |A(F_n,f)(x)|$.

$\boxed{\text{Prop. 3}}$ Thm 2 $\Rightarrow$ Thm 1

~~Proof~~ Recall: MET (mean erg. thm.) for Følner sets (talk 2):

$$D := \bigcap_g \operatorname{Fix} T_g \oplus \underbrace{\operatorname{lin} \bigcup_g (1-T_g)(L^\infty(X,\mu))}$$

is dense in $L^1(X,\mu)$ and $A(F_n,f)$ conv. a.e. on $D$. (with lim = 0 on the right hand side)

Approximation: Let $f \in \overline{\operatorname{lin}} \bigcup_g (1-T_g)(L^\infty(X,\mu))$

Take $\varepsilon > 0$, decomp. $f = \underset{\in \operatorname{lin} U_g \ldots}{f_1} + \underset{\|f_2\|_1 < \varepsilon}{f_2}$

Maximal ineq. for $|f_2|$:

$$\mu(\underbrace{\{x : M|f_2|(x) > \sqrt{\varepsilon}\}}_{=:R_\varepsilon}) \underset{\text{max. ineq.}}{\leq} \frac{C}{\sqrt{\varepsilon}} \cdot \|f_2\|_1 < \frac{C}{\sqrt{\varepsilon}} \cdot \varepsilon = C\sqrt{\varepsilon},$$

So we have:

$$\overline{\lim_{n\to\infty}} |A(F_n, f)(x)| \leq \underbrace{\lim_{n\to\infty} |A(F_n, f_1)(x)|}_{=0 \ (f_1 \in \operatorname{lin} U_g \ldots)} + \underbrace{M|f_2|(x)}_{< \sqrt{\varepsilon}}$$

$\leq \sqrt{\varepsilon}$ for all $x \notin R_\varepsilon$ and $\mu(R_\varepsilon) < C \cdot \sqrt{\varepsilon}$.

$$\mu(\underbrace{\{x : \overline{\lim_{n\to\infty}} |A(F_n, f)(x)| > \sqrt{\varepsilon}\}}) \leq C\sqrt{\varepsilon}.$$

$\overset{\varepsilon \downarrow 0}{\Longrightarrow}$ $A(F_n, f) \longrightarrow 0$ a.e. $\nearrow_{\varepsilon \downarrow 0}$

∎

<u>Proof of Thm 2 (max. ineq.)</u>

Take $f \geq 0$ (WLOG: $Mf \leq M(|f|)$).

Let $C$ be constant from tempered condition.

Define $c := 2(1+C) = \frac{2}{\gamma(1,C)}$ (recall: $\gamma(\delta, C) = \frac{\delta}{1+\delta C}$)

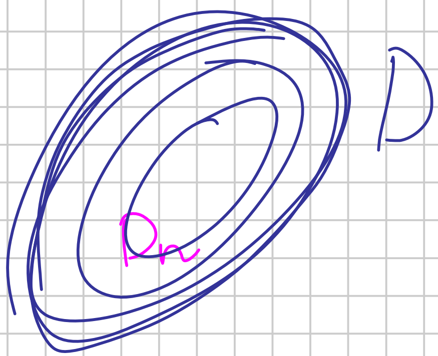
We show: this $c$ works.

Define $(M_n f)(x) := \max_{j \leq n} \mathcal{A}(F_j, f)(x)$. Take $\lambda > 0$

and def. further:

$D_n := \{x : M_n f(x) > \lambda\}$

$D := \{x : Mf(x) > \lambda\}$.




To show: $\mu(D) \leq \frac{c}{\lambda} \|f\|_1$

Take $\varepsilon > 0$ and $n_0$ large enough s.t.

$$\mu(D_{n_0}) \geq \mu(D) - \varepsilon$$

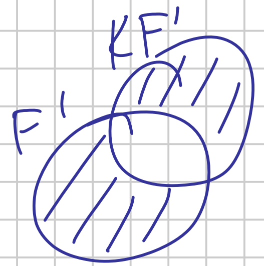
Observation: Define for a comp. $F' \subset G$

$$F := \Big(\bigcup_{j \leq n_0} F_j\Big) \cdot F'$$

Then we can choose $F'$ s.t. $\boxed{|F| < (1+\varepsilon) \cdot |F'|}$

Reason: $\exists (\tilde{F}_n)$ a Følner seq. is $\Big(\bigcup_{j \leq n_0} F_j, \varepsilon\Big)$-inv.: $\exists F'$ s.t.

$$\Big|\bigcup_{j \leq n_0} F_j \cdot F'\Big| \leq \underbrace{\Big|\bigcup_{j \leq n_0} F_j F' \triangle F'\Big|}_{< \varepsilon |F'|} + |F'| < (1+\varepsilon)|F'|.$$




Claim $\forall x \in X$

$$\boxed{\sum_{g \in F'} \mathbb{1}_{D_{n_0}}(gx) \leq \frac{c}{\lambda} \sum_{g \in F} f(gx)}$$

Assume the claim. Then:

$$\mu(D)-\varepsilon \le \mu(D_{n_0}) = \int_X \mathbb{1}_{D_{n_0}}(x)\,d\mu(x) = \frac{1}{|F'|}\sum_{g\in F'}\int_X \mathbb{1}_{D_{n_0}}(gx)\,d\mu(x)$$

does not dep. on $g$ (m.p.t!!)

$$= \frac{1}{|F'|}\int_X \underbrace{\sum_{g\in F'} \mathbb{1}_{D_{n_0}}(gx)}_{\le \frac{c}{\lambda}\sum_{g\in F} f(gx)}$$

- see claim

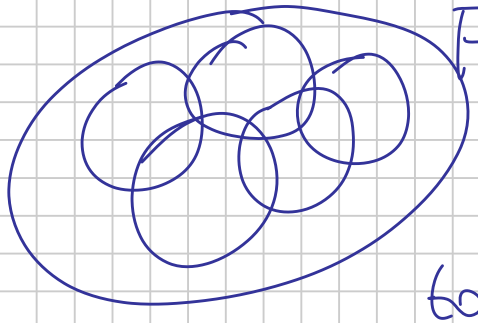
$$\le \frac{c}{\lambda}\cdot\frac{1}{|F'|}\sum_{g\in F}\int_X f(\cancel{g}x)\,d\mu(x) = \frac{c}{\lambda}\underbrace{\frac{|F|}{|F'|}}_{\le(1+\varepsilon)}\|f\|_1$$

$$\le (1+\varepsilon)\cdot\frac{c}{\lambda}\|f\|_1.$$

$\varepsilon\to 0$ - finished.

Proof of the claim: Fix $\cancel{x}\in X$, define for $j\le n_0$

$$\cancel{A}\,\delta_j := \{g\in F' : A(F_j, f)(gx) > \lambda\}$$



$F$ We have $F_j, A_j \subset F_j, F' \subset F$:

$F_j a \subset F \quad \forall a \in A_j$

Apply Lemma 2.1 (Lindenstrauss) - randomised covering lemma

to $F$, $F_j a$, $\delta := 1$:

$\exists (\Omega, \mathbb{P})$ prob. space, $\exists$ map $\begin{cases} \omega \mapsto \mathcal{F}(\omega) \\ \Omega \to \text{mb coll. of } \tilde{\mathcal{F}} := \end{cases}$

$= \{F_j a,\ a \in A_j,\ j = 1, \dots, N\}$

s.t. for the counting fct

$\Lambda : F \to \mathbb{R}$, $\Lambda(g) = \Lambda_\omega(g) := \sum_{B \in \mathcal{F}(\omega)} \mathbb{1}_B g$

one has:

1) $\mathcal{F}(\omega)$ is finite a.e. (autom.)

2) $\forall g \in F$ $\quad \mathbb{E}\left(\Lambda(g) : \Lambda(g) \geq 1\right) \leq 1 + \delta$

3) $\mathbb{E}\left(\sum_{g \in F} \Lambda(g)\right) \geq \gamma(\delta, C) \left| \bigcup_{j=1}^{N} A_j \right|$ for $\gamma(\delta, C) = \frac{\delta}{1 + \delta C}$

Observe:

- $$\mathbb{E}\Big(\sum_{g\in F} \Lambda(g) f(gx)\Big) \le \sum_{g\in F} \underbrace{\mathbb{E}(\Lambda(g))}_{\le 2 \text{ by 2) in Lemma 2.1}} f(gx)$$
$$\le 2 \sum_{g\in F} f(gx)$$

- Since $\forall a \in A_j$: $A(F_j, f)(ax) > \lambda$ (def. of $A_j$):
$$\sum_{g\in F_j a} f(gx) = \sum_{g\in F_j} f(gax) = \underbrace{|F_j|}_{=|F_j a|} \underbrace{A(F_j, f)(ax)}_{>\lambda}$$
$$> \lambda \cdot |F_j a|$$

By 3) from Lemma 2.1:

$$\mathbb{E}\Big(\sum_{g\in F} \Lambda(g) f(gx)\Big) \overset{\Lambda \text{ count. fct}}{=} \mathbb{E}\Big(\sum_{B\in \mathcal{F}} \underbrace{\sum_{g\in B} f(gx)}_{>\lambda \cdot |B|}\Big)$$

$$\geq \lambda \cdot \mathbb{E}\Big(\sum_{g \in F} \Lambda(g)\Big) \underset{3)}{\geq} \lambda \cdot \gamma(1, C) \, \Big|\bigcup_{j=1}^{n_0} A_j\Big| .$$

- $$\Big|\bigcup_{j=1}^{n_0} A_j\Big| = \sum_{g \in F'} \mathbb{1}_{D_{n_0}}(gx)$$

$a \in \bigcup \Leftrightarrow \exists j \in \{1, \dots, n_0\} : \underbrace{a \in A_j}_{A(F_j, f)(ax) > \lambda}$ $\Leftrightarrow (M_{n_0} f)(ax) > \lambda$

Altogether:

$$\sum_{g \in F'} \mathbb{1}_{D_{n_0}}(gx) \leq \frac{1}{\underbrace{\lambda \gamma(1, C)}_{C}} \cdot 2 \sum_{g \in F} f(gx)$$

- claim is proved.

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