# Seminar on Zero Knowledge

## Interactive Proof Systems and Polynomial Space

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# Interactive Proof Systems

**Definition (interactive proof system)**

Pair $(A, B)$

- mapping $A\colon \bigcup_{i\in\mathbb{N}}(\{0,1\}^*)^{1+2i} \to \{0,1\}^*$ (Alice)

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  2. Bob replies with $b_i = B(w, a_1, b_1, \dots, a_{i-1}, b_{i-1}, a_i)$
- final round: Bob accepts or rejects $w$

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- wlog. $|a_i| \leq p(|w|)$ and $|b_i| \leq p(|w|)$ for some polynomial $p$

# Interactive Proof Systems

**Definition (generated language)**

Interactive proof system $(A, B)$ generates language $L \subseteq \{0,1\}^*$ if and only if for all $w \in \{0,1\}^*$:

- Bob rejects $w \in L$ with negligible probability
  (i.e., Bob accepts with probability at least $1 - 2^{-|w|}$)

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$$\mathbf{IP} = \{L \subseteq \{0,1\}^* \mid \exists \text{ interactive proof system generating } L\}$$

Notes:

- introduced by Goldwasser, Micali, and Rackoff in 1985
- Alice is prover and computationally unlimited
- Bob is verifier and restricted to (deterministic) polynomial time

Intuition:

- Completeness:

  If $w \in L$ then designed prover $A$ convinces verifier $B$ almost certainly

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- Completeness:

  If $w \in L$ then designed prover $A$ convinces verifier $B$ almost certainly

- Correctness:

  If $w \notin L$ then any prover convinces $B$ only with negligible probability

**Theorem**

$\mathbf{IP} \subseteq \mathbf{PSPACE}$

Proof (1/5)

- $L \in \mathbf{IP}$ and $(A, B)$ interactive proof system generating $L$
- $w \in \{0,1\}^*$ of length $n$ and polynomial $p(x)$ limiting runtime of $B$
- polynomial $q(x)$ limiting number of rounds

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- polynomial $q(x)$ limiting number of rounds
- wlog. length of $a_1, \ldots, a_{q(n)}$ and $b_1, \ldots, b_{q(n)}$ is $p(n)$
- potential Alice

$$A' : \left( \bigcup_{i=0}^{q(n)-1} \{0,1\}^{n+i\cdot 2p(n)} \right) \to \{0,1\}^{p(n)}$$

Proof (2/5)

- approach: construct optimal Alice in PSPACE
- fix Alice $A'$ and random bit sequence $Z \in \{0,1\}^{q(n)p(n)}$ used by Bob
- protocol $P(A', Z) = a_1 b_1 \cdots a_{q(n)} b_{q(n)}$ completely determined

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$$f(A', P) = \left| \left\{ Z \in \{0,1\}^{q(n)p(n)} \mid P \leq P(A', Z) \text{ and } P(A', Z) \text{ accepting} \right\} \right|$$

$$f(P) = \max \left\{ f(A'', P) \mid A'' \text{ potential Alice} \right\}$$

- $\bar{A}$ optimal for $P$ if $f(\bar{A}, P) = f(P)$

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- $\bar{A}$ optimal for $P$ if $f(\bar{A}, P) = f(P)$
- $w$ accepted with probability $\rho(w) \leq \frac{f(\varepsilon)}{2^{q(n)p(n)}}$

Proof (3/5)

- $\frac{f(\varepsilon)}{2^{q(n)p(n)}} \geq \rho(w) \geq 1 - 2^{-n}$ and $f(\varepsilon) \geq (1 - 2^{-n}) \cdot 2^{q(n)p(n)}$ for all $w \in L$
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1. complete protocol $P = a_1 b_1 \cdots a_{q(n)} b_{q(n)}$
   - if Bob rejects, then $f(P) = 0$.
   - otherwise

$$f(P) = \left| \{ Z \in \{0,1\}^{q(n)p(n)} \mid Z \text{ permits } P \} \right|$$

$Z$ permits $P$ if bit sequence $Z$ yields the responses $b_1, \ldots, b_{q(n)}$ from $B$

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     $Z$ permits $P$ if bit sequence $Z$ yields the responses $b_1, \ldots, b_{q(n)}$ from $B$
   - can be done in polynomial space

Proof (4/5)

2. incomplete protocol $P = a_1 b_1 \cdots a_{i-1} b_{i-1} a_i$ with final message from Alice $1 \leq i \leq q(n)$
   - $f(Pb_i)$ with $b_i \in \{0,1\}^{p(n)}$ known and

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$$
\begin{aligned}
f(P) &= \max\left\{ f(A'', P) \mid \text{potential Alice } A'' \right\} \\
&= \max\left\{ \sum_{b_i \in \{0,1\}^{p(n)}} f(A'', Pb_i) \mid \text{potential Alice } A'' \right\} \\
&= \sum_{b_i \in \{0,1\}^{p(n)}} \max\{ f(A''_{b_i}, Pb_i) \mid \text{potential Alice } A''_{b_i} \} \\
&= \sum_{b_i \in \{0,1\}^{p(n)}} f(Pb_i)
\end{aligned}
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   - second-to-last equality follows because $Z$ determines protocol, so a single $Z$ cannot permit $Pb_i$ and $Pb'_i$ with $b_i \neq b'_i$

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   - clearly also polynomial space

Proof (5/5)

3. incomplete protocol $P = a_1 b_1 \cdots a_{i-1} b_{i-1}$
   with final message from Bob $1 \leq i \leq q(n)$
   - $f(Pa_i)$ with $a_i \in \{0,1\}^{p(n)}$ known and Alice can select the response

$$f(P) = \max \left\{ f(Pa_i) \mid a_i \in \{0,1\}^{p(n)} \right\}$$

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Space requirements:

- recursion depth $2q(n)$
- protocol prefix $P$, messages $a_i$ and $b_i$
- currently best value of $f$ and partial sum both limited by $2^{q(n)p(n)}$ $\square$

**Lemma**

**IP** closed under polynomial-time reductions

$$L \in \mathbf{IP} \qquad \text{for all} \qquad L \preceq_{\mathbf{P}} L' \quad \text{and} \quad L' \in \mathbf{IP}$$

<u>Proof</u>

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- $(A, B)$ interactive proof system generating $L'$ and input $w$

### Lemma

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<u>Proof</u>

- $f$ polynomial-time reduction of $L$ to $L'$
- $(A, B)$ interactive proof system generating $L'$ and input $w$
- Alice $A'$ and Bob $B'$ compute $f(w)$ and then simulate $A$ and $B$
- yields answer for $f(w) \overset{?}{\in} L'$ that is correct for $w \overset{?}{\in} L$ $\square$

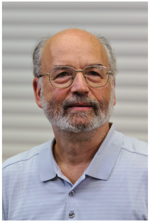
# Interactive Proof Systems

**Theorem (Shamir 1990)**

$\mathbf{IP} = \mathbf{PSPACE}$

**Adi Shamir (* 1952)**

- isra. computer scientist
- professor at Weizmann institute and ENS Paris
- Turing laureate 2002 and 'S' in RSA



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Proof (1/7)

- we know **IP** $\subseteq$ **PSPACE**, so only **PSPACE** $\subseteq$ **IP** remains
- since **IP** is closed under polynomial-time reductions and QBF is **PSPACE**-complete, we just show QBF $\in$ **IP**

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  - $a(x) = x$ and $a(\neg x) = 1 - x$ for all variables $x$

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  - $a(\exists x F_1) = \sum_{x \in \{0,1\}} a(F_1)$ and $a(\forall x F_1) = \prod_{x \in \{0,1\}} a(F_1)$ for all $F_1$

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  - $a(\exists x F_1) = \sum_{x \in \{0,1\}} a(F_1)$ and $a(\forall x F_1) = \prod_{x \in \{0,1\}} a(F_1)$ for all $F_1$
- obviously $a(F) \geq 0$
- $F \in \text{QBF}$ if and only if $a(F) > 0$

Example: $F = \forall x \exists y \Big( (x \vee \neg y) \wedge \exists z(\neg x \wedge z) \Big)$.

$$\begin{aligned}
a(F) &= \prod_{x \in \{0,1\}} \left( \sum_{y \in \{0,1\}} \Big( (x + (1-y)) \cdot \sum_{z \in \{0,1\}} ((1-x) \cdot z) \Big) \right) \\
&= \prod_{x \in \{0,1\}} \left( \sum_{y \in \{0,1\}} \Big( (x + (1-y)) \cdot (1-x) \Big) \right) \\
&= \prod_{x \in \{0,1\}} \Big( (1-x^2) + (x - x^2) \Big) \\
&= 0
\end{aligned}$$

so the formula is “wrong”

Example: For the negation $\neg F = \exists x \forall y \Big( (\neg x \wedge y) \vee \forall z (x \vee \neg z) \Big)$

$$\begin{aligned}
a(\neg F) &= \sum_{x \in \{0,1\}} \left( \prod_{y \in \{0,1\}} \Big( (1-x) \cdot y + \prod_{z \in \{0,1\}} \big( x + (1-z) \big) \Big) \right) \\
&= \sum_{x \in \{0,1\}} \left( \prod_{y \in \{0,1\}} \Big( (1-x) \cdot y + (x^2 + x) \Big) \right) \\
&= \sum_{x \in \{0,1\}} (x^2 + x) \cdot (1 + x^2) \\
&= 4
\end{aligned}$$

so the negated formula is “true”

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# Interactive Proof Systems

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**Lemma**

$a(F) \leq 2^{2^{|F|}}$

Proof

- replace each occurrence of $\exists x G$ by $G[x \mapsto 0] \vee G[x \mapsto 1]$ and each $\forall x G$ by $G[x \mapsto 0] \wedge G[x \mapsto 1]$
- prove $|F'| \leq 2^{|F|}$ for obtained formula $F'$ by induction

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- prove $|F'| \leq 2^{|F|}$ for obtained formula $F'$ by induction
- prove $a(F') \leq 2^{|F'|}$ by induction $\square$

Example: $F = \forall x_1 \cdots \forall x_k \exists y \exists z (y \vee z)$.

$$\begin{aligned}
a(F) &= \prod_{x_1 \in \{0,1\}} \cdots \prod_{x_k \in \{0,1\}} \left( \sum_{y \in \{0,1\}} \sum_{z \in \{0,1\}} (y+z) \right) \\
&= \prod_{x_1 \in \{0,1\}} \cdots \prod_{x_k \in \{0,1\}} \left( \sum_{y \in \{0,1\}} (2y+1) \right) \\
&= \prod_{x_1 \in \{0,1\}} \cdots \prod_{x_k \in \{0,1\}} 4 \\
&= 4^{2^k}
\end{aligned}$$

Notes:

- numbers of size $2^{2^{|F|}}$ require $2^{|F|}$ bits
- cannot be exchanged in protocol round

# Interactive Proof Systems

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- numbers of size $2^{2^{|F|}}$ require $2^{|F|}$ bits
- cannot be exchanged in protocol round
- compute modulo prime

**Lemma [Dietzfelbinger 2004]**

For $n \geq 5$ interval $[2^n, 2^{2n}]$ contains at least $2^n$ primes

Proof (2/7)

- $n = |F|$ and $p_1, \ldots, p_k$ primes between $2^n$ and $2^{2n}$

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- $n = |F|$ and $p_1, \ldots, p_k$ primes between $2^n$ and $2^{2n}$
- $m = \prod_{i=1}^{k} p_i \geq (2^n)^{(2^n)} = 2^{n \cdot 2^n} > 2^{2^n} \geq a(F)$

$$F \notin \mathrm{QBF} \iff a(F) \equiv 0 \bmod m \qquad \text{since } a(F) = 0$$
$$F \in \mathrm{QBF} \iff \exists 1 \leq i \leq k \colon a(F) \not\equiv 0 \bmod p_i$$

because $m = \prod_{i=1}^{k} p_i > a(F)$, so not all $p_i$ can divide $a(F)$

Proof (3/7)

- for $F \in \mathrm{QBF}$ Alice computes smallest prime $p_i \geq 2^n$ with $a(F) \not\equiv 0 \bmod p_i$
- sends $p_i \leq 2^{2n}$ to Bob
- all other computations now modulo $p_i$

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- polynomial $a(F')$ obtained from $a(F)$ by removing first product $\prod$ or sum $\sum$
- $\deg\big(a(F')\big)$ can be exponential in $n$

## Definition (simple formula)

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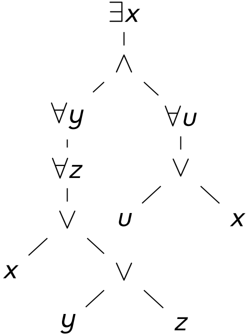



$\exists x \Big(\forall y \forall z \big(x \vee (y \vee z)\big)\Big) \wedge \Big(\forall u \big(u \vee x\big)\Big)$ <u>not</u> simple

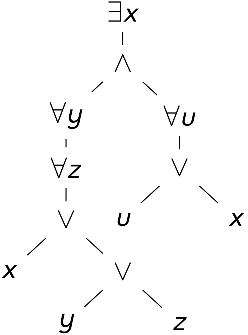
**Lemma**

Each formula can be transformed into equivalent simple formula in polynomial time

Proof

- replace each subformula $\forall y G(x_1, \ldots, x_k, y)$ with free variables $y, x_1, \ldots, x_k$ by

$$\forall y \exists y_1 \cdots \exists y_k \Big( \bigwedge_{i=1}^{k} x_i \leftrightarrow y_i \wedge G(y_1, \ldots, y_k, y) \Big) \qquad \square$$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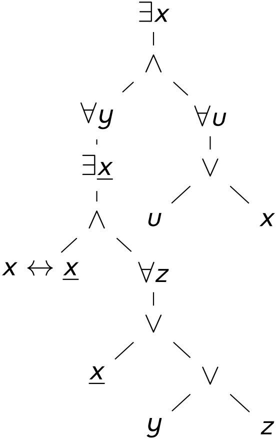
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$\deg\big(a(G')\big) \leq 2|G|$ for simple formula $G = QxG'$ with $Q \in \{\exists, \forall\}$

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- replace in $G'$ each subformula $\forall yH$ in which $x$ occurs freely by $H[y \mapsto 0] \wedge H[y \mapsto 1]$

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- play $\ell$ rounds with $\ell \leq n$ the number of quantifiers in $F$
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- $\rho_1(r_1) = a(G_1)[x_1 \mapsto r_1] \iff a_2 = a(F_2)[x_1 \mapsto r_1]$ provided $c \neq 0$

Proof (7/7)

- $\text{Prob}\big[\rho_1(r_1) \neq a(G_1)[x_1 \mapsto r_1]\big] \geq 1 - \frac{2n}{2^n}$ at start of round 2
- argument repeats with demand for proof of $a_2 = a(F_2)[x_1 \mapsto r_1]$

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- argument repeats with demand for proof of $a_2 = a(F_2)[x_1 \mapsto r_1]$
- at most $\ell$ rounds
- probability of correct answer "$F \notin \text{QBF}$" is at least

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- rerun of protocol lowers it to $\left(\frac{2n^2}{2^n}\right)^2 = \frac{4n^4}{2^{2n}} < 2^{-n}$ for large $n$ $\square$