

# What do you claim when you say you have a proof?

Claus Diem and Christoph Schulze

April 25, 2018

## Prologue

### In the common room

DISCIPLE: Your proof was quite beautiful.

SPEAKER: Thank you!

DISCIPLE: Actually, sometimes, I had a bit a problem following. There was quite a jump from the Fundamental Lemma to Theorem A.

SPEAKER: Well, yes, because the time was short. But I can give the details.

DISCIPLE: Rigorous?

SPEAKER: Yes, of course.

DISCIPLE: And Theorem B?

SPEAKER: The Fundamental Lemma also implies Theorem B; you can adapt the proof for Theorem A.

DISCIPLE: So, you have proven the theorems?

SPEAKER: Yes, I have a proof – otherwise I wouldn't say I have theorems.

DISCIPLE: Of course.

SKEPTIC: I've overheard your conversation. I would like to know what you mathematicians mean when you say that you "have a proof". That does not seem so obvious to me.

DISCIPLE: That we can demonstrate it rigorously.

SKEPTIC: This means in the basis of ZFC?

SPEAKER: Yes, you can say that "by default option" this means on the basis of ZFC.

SKEPTIC: So you say that you have proof of your theorems on the basis of ZFC?

SPEAKER: Yes ...

SKEPTIC: There was nothing remotely similar to proofs on the basis of ZFC on the blackboard.

SPEAKER: I outlined it, as said, I can give the details.

SKEPTIC: Even with more details you won't have proofs on the basis of ZFC, I mean formally in ZFC.

SPEAKER: Yes — no, formally not. But that's just for convenience. I could do it — we all could do it.

DISCIPLE: Of course, that's how it's done.

SKEPTIC: Are you sure you could — could you really? Can you?

SPEAKER: In principle, yes.

SKEPTIC: I am not so sure. I think I couldn't except for very small examples.

DISCIPLE: You know, proofs in ZFC are also abstract objects of mathematics; they can be studied just like finite geometries or groups. You should be familiar with this.

SKEPTIC: I am. But are you sure you want to say that the alleged proof is an abstract mathematical object like a group?

DISCIPLE: Yes, like any other mathematical object.

SKEPTIC: Actually, I am not sure if it is justified to say that abstract objects exist, but of course mathematicians act as if they would.

SPEAKER: I think we don't have to address the existence of abstract objects here. Proofs is what we want to have, we want to be sure that our statements hold — well, that they hold in the right formal framework.

DISCIPLE: Well, maybe, but I don't really see the problem — he can give a proof, well maybe not a formal proof but a proof nonetheless.

SKEPTIC: Different, changing meanings of one word are always problematic, I guess we would have to address this problem first. But I don't want to bother you any longer.

## 1 Statements, objects and statements on objects and statements

Mathematicians reason about abstract objects, or at least they seem to do so. In the course of doing so, they routinely make statements on mathematical statements. In fact, every inference, such as “The Fundamental Lemma also implies Theorem B”, is a statement on mathematical statements. In the conversation the reader will notice several examples, examples of different kinds, including even aesthetic judgments.

An important special case of statements on statements are statements on the existence of proofs and in particular formal proofs, such as the claim “There is a formal proof on the basis of ZFC.” What kind of statement is actually made here? Shall the formal proof “in ZFC” be a mathematical object in the same vein as, say, finite geometries or groups? If so, is the claim then a claim in mathematics? Shall the claim not rather be a claim on mathematics? But what shall then the meaning of the claim be?

Clearly, we would want that a claim that there is a proof, in this case a formal proof, be of a different, “more real” nature than a mere existence claim in the usual set-theoretic mathematics. But how is it possible to give such a “more real” interpretation if the alleged formal proof is not written down, in all likelihood will not be written down and maybe also cannot be written down or even read by humans?

A starting point is here the observation that it seems to be unreasonable to say that humans can establish infinitely many results, in mathematics or otherwise. So, we regard any statement that infinitely many results have been proven to be a priori invalid. To demonstrate this with an easy example: We have the evidently true mathematical statement “Every prime  $p > 2$  is odd.”. We hold here that it is improper to say that for every prime  $p > 2$  the statement “ $p$  is odd” holds, as this would mean that infinitely many statements are made.

More generally, when addressing the task to give a “more real” or “most real” interpretation of claims of alleged formal proofs of mathematical theorems, care has to be taken not to use any unsubstantial a priori claims on the existence of certain “abstract” objects.

With these remarks in mind we now give a brief outline of this note:

**Outline.** We want to be as accurate as possible with respect to statements of existence, avoiding any unsubstantial claim that certain “abstract” objects exist. In order to nonetheless attach meaning to mathematical statements, we start off with an observation of what mathematicians actually do when doing and speaking about mathematics.

Subsequently, we highlight two important ideas of contemporary mathematics: set (and class) theory and formal methods. This leads to a first reflection on mathematical statements as they are usually made and statements on the existence of formal proofs

In the fourth section, we begin to study claims that statements have been proven on the basis of a formal system such as ZFC in comparison with “usual” claims in mathematical texts. A problem which arises is that – contrary to what one might think at first sight – texts as actually written by mathematicians are not “linear” but rather have a complex structure of

reflections upon reflections. In order to even fix a sufficiently precise question to be studied, here we only study rather restricted classes of mathematical texts: One class consists of texts which are free from reflections and meta-statements, another class consists of such texts preceded by introductory statements that the following text can be rewritten in a formal system, for example ZFC. Noticing that such a claim of rewriting is often literally false, we then ask, in a “Central Question”, if there is a possibility to read the latter kind of texts in a “more real” way than the former kind of texts.

To give an answer to this Central Question, we are lead to the consideration of constructive methods, and so in the fifth section we outline how we envision to use them. To avoid any unnecessary claims of existence, we give a strict interpretation of constructive statements as used by us with which we can avoid to talk about infinitely many objects.

In the sixth section, we give possible meanings of statements on the existence of formal proofs as studied in the fourth section via the methods from constructive mathematics via three criteria (which can be found in subsections 6.1 and 6.3).

As already noted, texts as actually written by mathematicians often have a complex structure of reflection. As a preliminary study on such actual texts, briefly study, in the seventh section, other kind of statements on statements in mathematical texts. Here, similarly to the consideration of the establishment of formal proofs, we argue that if a mathematical result is applied as a meta-result in a mathematical argument, this result must be constructive. Again we formulate this as a criterion.

In the final section, we analyze the possible application of mathematical results to meta-mathematics and in particular the application of mathematical results to establish mathematical results (which usually goes under the name model theory).

**An advice.** The text can (and as we would urge: should) be read “as is”, but nonetheless the reader might ask what inspired the authors to consider the questions studied in the text and to write the text in the way it is written. Also, the reader might ask how thoughts expressed, for example, the ones of the next section, relate to previous texts and “schools of thought”. Some information concerning these questions can be found in a supplement which is independent of the text and can be found after at the very end.

## 2 What is mathematics?

According to adherents of Platonic realism, mathematics is (or should be if properly conducted) the study of certain “ideal” objects. Of course this is

challenged. In what sense is, for example the statement

There exists a unique positive real number whose square is  
 $2 \uparrow\uparrow 6 - 1$ ,

in which Donald Knuth's arrow notation (see [coping]) is used, a statement on any kind of "object" after all?

Indeed, one might argue that as there is no such object, the statement is wrong. One might also raise the even deeper objection that notions in the statement (*real number*, *square* and maybe also  $2 \uparrow\uparrow 6 - 1$ ) are void of meaning, and it therefore is not even reasonable to discuss whether the sentence is right or wrong. More generally, one might argue that all or at least a large part of mathematical existence statements which are considered to be true in mathematics are in fact wrong, because there simply are no objects of the asserted kind, or even nonsensical.

Now, independent of metaphysical beliefs, mathematicians do indeed act as if the asserted imagined objects such as the square root of  $2 \uparrow\uparrow 6 - 1$  were real objects. They talk about such objects, they tell themselves stories about the objects and they even get emotionally involved when they have found a new, as mathematicians often say, "beautiful" relationship in their imagined universe. Interestingly, throughout the history of mathematics, aesthetic judgments played a crucial role, maybe more than in any other science.

Let us take the observations of the previous paragraph as a starting point: We shall call mathematics the activity of mathematicians in their role as mathematicians, which is meant as a self-identification. What is then mathematics?

There is then a purely *empirical* answer, following the above descriptions: Mathematicians act as if imagined "abstract" objects in an imagined "abstract" universe were real. For this, they claim to base their arguments on certain "evident" simple assertions called "axioms" and to apply (only) certain "evident" rules of inference. They try to find out what the characteristics of the imagined universe are, what the relationship between objects in the imagined universe are, and they try to convince others of their findings.

We note here that we use the word "imagined" for an empirical description of the activity of mathematicians. Throughout this note we stay neutral towards the fundamental question in how far the content of mathematical statements (in particular of statements in infinitely many objects) should be regarded as being "real" in the sense of being statements on "real abstract objects". It is, however, important to us that statements on infinitely many objects and also statements on large numbers, such as the one mentioned in the beginning of this section, cannot be regarded as referring to anything related human experiences.

The activity of mathematicians is similar to the telling of novels, and indeed one can say that mathematicians tell themselves as an inner thought or each other *stories about imagined abstract objects*. There is, however, a further aspect which is different from pure story telling: Mathematicians do not tell each other arbitrary stories but rather behave like explorers of the imagined universe. They then try to convince themselves and each other that what they perceive to be “true characteristics” of the imagined universe are indeed such.

An interesting aspect of mathematics (the activity of mathematicians) is that certain arguments (which by the nature of mathematics are always relative to “axioms”) are considered to be so rigorous that they are called “proofs”. This is even more remarkable as there is no clear standard as to when an argument given by a mathematician shall be called a “proof”. Indeed, throughout mathematical history, mathematicians have differed greatly in their opinions as to what they consider a “proof”, and if one now rereads older texts (which might be from the 19<sup>th</sup> century), sometimes one obtains the impression that what is called a “proof” does not have the clarity and strictness one wishes to see nowadays. Nonetheless, mathematicians have usually been convinced that it is in praxis obvious what constitutes a “proof” and what not, and in case of a dispute mathematicians have usually been able to find – via discussions and further elaborations – a consensus as to what shall considered to be proven (again relative to particular axioms) and what not. These “proofs” together with discussion on them as well as the expressed conviction that they express a certainty with regards to truth form a major part of the stories told by mathematicians.

The word “proof” is a strong one, but as stated, more often than not in the actual praxis of doing mathematics, one can argue if arguments suffice for a “proof”. Personal judgments play a strong role here. To emphasize this subjective aspect, we usually speak about a *convincing argument* instead of a *proof* when we refer to what is usually considered a proof. Occasionally, we also literally write “*proof*”. This also allows us to distinguish between what is usually called “proofs” by mathematicians (thus convincing arguments in our terminology) and (formal) proofs in formal systems, a distinction which will be of importance later.<sup>1</sup>

---

<sup>1</sup>It is common to call what we call *convincing argument* an *informal proof* (see for example [informal]). Other researchers, who also emphasize the difference between what is actually expressed and formal proofs stick with the word “proof” (see for example [formal-natural]).

## 3 Two crucial developments

Two rather recent developments in mathematics are of particular importance for us: *sets as fundamental objects of mathematics* and the *formalistic approach* to the foundations of mathematics.

### 3.1 Set theory

Set theory is so embedded to our doing of mathematics that it is hard to forget the specific, at one time revolutionary, point of view of this way of thinking: Not only does one regard collections of (imagined) “things” again as one thing, the “things” so obtained are then regarded as things in just the same manner, which makes it possible to repeat this process. Moreover, not just collections of finitely many “things” are considered, but also such of infinitely many “things”.<sup>2</sup>

An early idea of the notion of set (Menge) was that every property should define a set. As this turned out to lead to logical contradictions, a separate notion of *class* was introduced, for which this is indeed the case (with an appropriate definition of “property”) but which cannot necessarily be treated as “thing” in the way a set can.

Amazingly, via the idea of sets, different stories about abstract or ideal objects, numbers, (ideal) geometries, forms, functions and much more can be told from a unified basis, that is, starting with the same set of axioms and rules of inference, and communicated clearly. This unifying set-theoretic point of view has been so successful that nowadays the view is dominant that every object considered in the story of mathematics is (or: should be seen as) a set or a class.

We note that with respect to the use of proper classes, that is, classes which are not sets, the attitude of mathematicians is not always clear. Whereas statements about classes frequently appear in mathematical writings, often these can be reformulated via sets alone. It remains then the question of the use of proper classes is solely meant as a *façon de parler* or if proper classes are really considered as objects in addition to sets.<sup>3</sup> In order to be neutral to this question, we will speak about “set or class theory” rather than only about “set theory” or only about “class theory” (which of course would encompass a theory of sets).

---

<sup>2</sup>Richard Dedekind wrote in [Zahlen]: “Im folgenden verstehe ich unter einem *Ding* jeden Gegenstand unseres Denkens.”, that is: “In the following, I mean by *thing* every object of our thinking.”

<sup>3</sup>We stress that as stated in the previous section we view the activity of mathematicians as storytelling about imagined abstract objects. So we ask here what kind of objects are imagined or should be imagined for a consistent interpretation of the stories. In particular do not ask if sets are “more real” than classes.

One should, however, be aware that there are other kinds of mathematics than set and class theory. We will clarify the notion of “kind of mathematics” and the role of different kinds of mathematics in subsection 7.1. Here we merely give some remarks of importance for the time being:

We first remark that the dominant set-theoretic view changes the intuition in such a way that perspectives which cannot be expressed or are difficult to express set-theoretically vanish nearly completely. To give an example, the usual definition of a function in set theory is such that a function is equal to the graph of itself. To a person already embedded into set-theoretic thinking, this seems to be natural, and such a person might laugh at descriptions involving “rule”, “computation” or “dependent quantities”. But the fact that “function” has an elegant set-theoretic definition does not mean that other ideas are without value, whether they might be also defined (maybe more difficultly) set-theoretically or are not definable set-theoretically at all.

A second remark we demonstrate with an example: A particular axiomatic system for synthetic geometry with basic objects being points and lines leads to a corresponding kind of mathematics. Now, one can define set-theoretically what one means by a geometry corresponding to the given axioms, consisting of points and lines, and then one can argue set-theoretically about these. One can then expect that every story on the basis of the geometric axiomatic system also set-theoretically leads immediately to a set-theoretic story. This does, however, not mean that the opposite also holds, that is, that a set-theoretic story on the given objects leads to a story on the basis of the geometric axiomatic system. We will discuss these aspects in subsection 8.6.

A particular kind of mathematics of importance for us is constructive mathematics. We will consider constructive mathematics as a framework to answer the Central Question on rewriting already mentioned in the introduction. For constructive mathematics it is particularly clear that every story leads immediately to a set-theoretic story, but the opposite is not true.

For the moment, for practical reasons, we will stick to the mentioned dominant view that all mathematics being studied is some set or class theory, whereas constructive mathematics will be seen as an auxiliary tool.

### 3.2 The formalistic approach

By “formalistic approach” we mean the following body of ideas: A formal language to express mathematics is rigorously described via (easy to follow) rules according to which certain expressions are called sentences;<sup>4</sup> certain sentences or bodies of sentences are considered to be interpretable (again via

---

<sup>4</sup>One might also speak of “valid sentences”; we use the term “sentence” in a sense that “valid” would be redundant.



easy to follow rules);<sup>5</sup> certain sentences or bodies of sentences (called “axioms”) are a priori called “true”;<sup>6</sup> there are (easy to follow) rules of inference to derive further “true” or “false” sentences from previously established “true” ones. Calling the language and the rules a *formal system*, a “proof in the formal system” is then a (physically given) (finite) sequence of sentences in the formal language each of the sentences is interpretable and “true” given the previous ones (on the basis of the axioms and the rules).

Rigorously following the formalistic approach would mean that all mathematics would be rewritten in formal systems, and then the outline of the formal systems used and these formal texts written would encompass all of established mathematics. Motivating and interpreting statements could be given, but there would be no need for them, and they would not be part of the body of established mathematics; in particular in this body of established mathematics, there would be no room for “universes of discourse” in which the sentences in the proofs are to be interpreted.

Following this line of thought, ideally then mathematicians would even agree on a single formal system, and there would be an agreement that mathematics is the production of formal proofs in this unique formal system. The rules for the formal system could then be seen as defining criteria for the notion of mathematics, but would themselves lie in meta-mathematics rather than in mathematics. Needless to say that this vision is very different from mathematics as the activity of mathematicians at this point of time.

This rigorous approach should be distinguished from the usual process of doing mathematics, which always involves the effort to find appropriate precise terms and convincing arguments written in a language which keeps possible misunderstandings at a minimum.

We note here that the development and the use of a formalistic approach as described need not go along with a particular attitude with respect to the “nature” of mathematical statements. In particular, it is independent of the acceptance or rejection of Platonic realism.<sup>7</sup>

---

<sup>5</sup>By “interpretability” we mean a formal criterion on texts, i.e. finite sequences of sentences. In propositional logic all sentences are interpretable, but formal languages more in line with conventional languages are conceivable in which this is not the case. An example might be a formal language with a sentence like “Let  $x$  be an element of  $X$ .”

<sup>6</sup>We deliberately write “called ‘true’”, because we want to emphasize that here “true” and “false” are merely expressions assigned to certain statements, which might be replaced by any other expressions. The fact that the emotionally strong words “true” and “false” (which are also used in statements on formal systems in their usual meaning) are used here is of course not without problems. A particular reader of a particular system might be of the opinion that the system with a particular interpretation does really establish true and false statements; but this does not need to be so.

<sup>7</sup>See also the last sentence of footnote 6.

### 3.3 The formalistic set theoretic foundation

The formalistic approach can then in particular be applied to set or class theory, which leads to various (related) formalistic approaches to set or class theory as the foundation of mathematics. With these approaches, it seems that the foundations of mathematics as it is usually conducted have reached a long-lasting nearly stable stage.

There is however another kind of mathematics, constructive mathematics, for which – among others – not the idea of set but the idea of number and algorithm is basic. We will argue in the next section that this kind of mathematics gives (with an appropriate interpretation) the right framework to argue whether results are established via formal proofs.

### 3.4 The praxis of mathematical story-telling and argumentation

Even though the idea of a formalistic approach is now considered to be crucial to mathematics, “mathematical proofs” are very seldom written in a formal system. In fact, the system which is the most accepted one, namely ZFC, itself based on first-order logic, is designed in such a way that it is de facto impossible to tell the stories and arguments mathematicians tell each other in this system – small examples withstanding. One reason for this is that it does not allow for definitions and that all sentences are interpretable without context.

So, rather than actually telling their stories and giving their arguments in a system such as ZFC, mathematicians usually just use usual language augmented with mathematical symbolisms and occasional formal statements.<sup>8</sup> A particular kind of set or class theory is there often not specified. In a more rigorous approach, such as in the books by Bourbaki, first the axioms are outlined in plain language (augmented as described) and then the arguments are also given in plain language with occasional formal statements.

Only seldom, a strict formalistic claim is made, a claim of the following form: A formal system (for example ZFC) is defined in plain (usual) language, the claim: “The following can be rewritten in the formal system in such a way that one obtains a formal proof” is made, and then the usual mathematical arguments are given.

Nonetheless, when asked what actually the phrase “This is proved on the basis of set theory” shall mean, mathematicians (maybe after a discussion that one wants to have a “strict” answer) often reply: “Well it means that there is a formal proof of the statement in ZFC.” What then is stated here?

---

<sup>8</sup>Concerning the language used by mathematicians, the reader might find the introduction to [formalized] interesting.

Is this now again nothing but an existence claim on imaginary mathematical objects or does it mean that such a proof can actually be written? Or is there another meaning one can attach to this phrase? Does it maybe help to say that “it could be written”?

One notices immediately that there is a substantial difference between actually writing in the formal system ZFC or just claiming that it “could be done”. If one reflects what it really would mean to write a proof of a statement in ZFC, one soon realizes that by all human capabilities, it often *cannot be done* by humans. What is more, such a writing (which, as mentioned is anyway not carried out and de facto for many texts even impossible to carry out by any given human) would end in a result which any mathematician would consider a worthless accumulation of symbols – worthless because she or he had the feeling of “not understanding anything”. Indeed, the mental process and the actual work done by mathematicians would if anything go *in the opposite direction*, that is, from the “purely formal” symbols to a high-level text which highlights some underlying structure in the argument.

With these reflections, we can now give a more detailed description of the personal and inter-personal process of mathematical story-telling and arguing in the current state of mathematics: Set or class theory is an agreed upon foundation, even though the references to set or class theory are not always made explicit. An important interest of mathematicians is “results”, which they regard as being “true” on the basis of some (set or class theoretic) axiomatic systems.<sup>9</sup> These are usually informally given but can also be given formally; by default the chosen axiomatic system is Zermelo-Fraenkel set theory with the axiom of choice. In order to be convinced of this, they often accept broad arguments, whose validity they have only superficially or not at all checked – often aesthetic judgments and experience play a strong role here. However, if they are interested in some further arguments on a detail of a longer argument, they want that this is to be explained to them. This process might then lead to a more and more formal argument. Here derivations in formal systems (like ZFC) might play some role. However, mathematicians are usually unsatisfied with purely formal derivations (except maybe for short computations); rather they want to gain an “understanding” or a mental image of the situation which exceeds purely formal considerations.

## 4 The claim of possible rewriting

We now analyze claims of possible rewriting of informal arguments in formal systems such as ZFC and of claims that statements “are proven in ZFC” or a

---

<sup>9</sup>Mathematicians (as mathematicians) also have other interests, for example finding what they consider to be the “right” definitions.

related system.

#### 4.1 The substantive and the formalistic story

Contrary to what one might think at first, there is a great diversity in the kind of statements made in mathematical texts. It is common that mathematical statements are mixed with reflections on the mathematical statements or the process of making such statements.

In this section, for a first study, we restrict ourselves to texts which are more restricted.

As a basis we consider texts consisting of set- and class-theoretic definitions or definitions which can be interpreted set- or class-theoretically, statements based on these definitions and supporting arguments. So far, this corresponds to the usual mathematical praxis. However, in contrast to the way mathematical texts are actually written, we do not allow for statements on what is written. For example, we do not allow for reflections on arguments “from a higher point of view”, as one often finds in introductions, or on judgments that something is “easy” or “easily seen” or on arguments based on analogy (like “the proof is similar to the previous one”). We can say that in such a text a set- or class theoretic story is told in a linear way; correspondingly we speak of a *linearly told substantive set- or class-theoretic story*.<sup>1011</sup>

We now call an outline of the logical and set-theoretic foundations (including schemes of rules of inference and schemes of axioms) a *non-formalistic header*. In such a non-formalistic header there might for example be the (literal) statements “We make use of classical logic, that is, in particular ‘tertium non datur’.” or “We make use of the axiom of separation, that is for every set  $X$  and every property  $p(x)$ , there is a subset of  $X$  consisting of those elements for which  $p(x)$  holds.”. Such statements are not allowed in linearly told substantive stories since these are meta-statements on mathematical statements. But, given such statements in the non-formalistic header, one may apply these statements in each single case to achieve new mathematical statements. Here it is of big importance that one may check immediately if such a mathematical statement arises from some statement in the header (e.g. above it has to be clear what is meant by ‘property’). Apart from such

---

<sup>10</sup>The word “substantive” was chosen as a translation of the German word “inhaltlich” (with regards to content); its meaning in law suggests that it is an appropriate choice.

<sup>11</sup>We note that as one can interpret different mathematical stories set-theoretically, one can also interpret a story on formulas / formal statements set-theoretically. In particular thus it is easy to imagine linearly told substantive set-or class-theoretic stories on formulas, just as one can imagine one on synthetic geometry, say. The restrictions on statements on statements should thus not be interpreted as a limitation to tell such a story; the limitation is only on statements on statements as actually made in such a story.

applications of meta-statements of the non-formalistic header, one may also use definitions and prior proven lemmas or theorems in substantive stories.

Furthermore, we call an outline of a formal set or class theoretic system (such as ZFC) followed by a short claim similar to “The following theorems and proofs<sup>12</sup> can be written in the formal system in such a way that one obtains formal proofs.” a *formalistic header*. A story consisting of a formalistic header and a linearly told substantive set- or class-theoretic story is an example of what we call a *formalistic story*. Note that the claims in such a formalistic story are the claims on the possibility of writing; the claims of the corresponding linearly told substantive set- or class-theoretic story are not claims of the formalistic story. Nevertheless, for a reader the claims of the substantive story have to mentally present since it is not possible to imagine some rewriting if the substantive story is not given in a human understandable way. The general definition of a formalistic story will be given in subsection 6.3, after we have analyzed further claims of rewriting.

Let us call a person who thinks that a particular scheme of rules of inference and axiom scheme of set or class theory is true in the sense that it expresses truth about an abstract universe which indeed exists a *Platonic realist* with respect to the given schemes. If we now consider a particular linearly told substantive set- or class-theoretic story based on particular rule and axiom schemes, we can say: A person who is a Platonic realist with respect to the given axiomatic scheme and who regards the arguments on the basis of this system to be convincing is (unless he or she has strangely inconsistent thoughts) convinced that the statements supported by the arguments are true – again in the sense that they express truth about the abstract universe.

In contrast, the attitude of *formalists* is to consider the claims of many substantive stories to be a priori nonsensical or at least invalid. Some formalists (nominalists) might reject all claims on abstract entities, including numbers, others might reject claims infinitely many objects as nonsensical. Formalists in the Hilbertian tradition take a middle position and allow claims on natural numbers but regard claims on, for example, unaccountably many objects to be nonsensical or invalid.

According to formalism, one should rather argue about whether a result is formally proven. Only for pragmatic reasons, one could (and one would) then behave as if it was reasonable to discuss set- or class theoretic arguments. However, ideally the goal of such a discussion would be no more and no less than to settle the question of the possibility of writing a formal proof in the mentioned formal system.

We have already stated at the beginning of Section 2 that the attitude

---

<sup>12</sup>In our terminology: convincing arguments

of Platonic realists is questionable. Moreover, in subsection 3.4 we have explained that claims that formal proves “can be written” are usually also questionable. In particular, the claims of stories consisting of a formal header followed by a linearly told substantive story are questionable even if the arguments in the corresponding substantive stories are considered to be convincing with respect to the axioms. The central question is then:

**Central Question.** Can stories consisting of a formalistic header followed by a linearly told substantive story somehow be regarded as being “less imaginary” or “less fictional” than the corresponding substantive stories, in particular if the alleged rewriting cannot be carried out by humans? Is there an intermediate realm between true statements and statements which are only true inside the story told by mathematicians?

To address this question, we study ideas inspired by the idea to actually (physically) carry out the process of rewriting. Doing so, we however stay in the realm of some kind of mathematics, that is, we do not address the question if and in how far it is actually (physically) possible to carry out the rewriting.

## 4.2 Ideas for rewriting

An evident idea is to imagine some kind of procedure with which rewriting from a more “human oriented” formal system to a system like ZFC can be done. The claim that there is a rewriting procedure would be part of some (different) story of mathematics (see subsection 7.1), just as the claim that the procedure can be applied in a particular case. We stress this because the idea is not to actually perform the computation and check the resulting first-order text; this would be a different approach which would however be far away from what is actually done by mathematicians.

Note that the problem of rewriting naturally involves a variety of different ways to translate original claims into formulas of the formal system (say ZFC). Thus one may pose the question if all formulas resulting in this way are equivalent. But actually the argument would go in the opposite direction: rewriting procedures have to be compatible in such a way that one mathematical claim does not lead to non-equivalent formulas. This indicates that a mathematician who makes a statement of possible rewriting even believes that there is some (not necessarily unique) proper way to reformulate the stated mathematical claims into formulas (of ZFC, for example).

A first approach to apply a rewriting procedure in some concrete example is to give:

1. a formal system for the input; let us call it  $I$ ;

2. a rewriting procedure from  $I$  to the set-theoretic output system, say ZFC, translating formal proofs in  $I$  to formal proofs in the output system;
3. a convincing argument that outputs are formal proofs if the inputs are in  $I$ ;
4. a formal proof in  $I$  for the statement to be established.

The idea here is that the system  $I$  is “more humanly oriented” and it might be possible to directly write formal proofs in  $I$  which are also humanly understandable.

In subsection 6.2 we will discuss how one might use computers to partially accomplish these goals and what problems occur if one tries to do so. For the moment we just note that humanly understandable texts are usually not written in a formal language, and one can clearly improve the understandability of a text with a less formal writing or strictly speaking (!) just plainly wrong statements.<sup>13</sup>

Rather, one would like to argue that a convincing argument written in natural language can be translated, leading to a formal proof in ZFC, say.

For this reason, it is reasonable to substitute the fourth desideratum by:

- 4'. a convincing argument that the argument in natural language can be written in the input language  $I$  in such a way that one obtains a formal proof in  $I$ .

Interestingly, the input system  $I$  can have a very different “feeling” concerning the allowed constructions than the set-theoretic output system. Let us say that the goal is a rewriting in terms of ZFC, that is, the output system is ZFC. Then even though ZFC is untyped,  $I$  can be typed, and even though ZFC does not allow for classes,  $I$  can allow for a class for each (individual) property  $p(x)$ .

An evident problem concerning the first three points is that all these claims are again non-trivial claims inside some (different) mathematical story. Independently of this problem, we want to stress a further aspect: The claim that “it could be done” has a *constructive meaning*. Now, it might not be possible to perform the computation on interesting examples. But nonetheless, it is still an evident requirement that the procedures are actually constructively given and analyzed and not just claimed “to exist”. This

---

<sup>13</sup>Just to give one example: If  $G$  is a group and  $g$  an element of the set underlying  $G$  (strictly formally !), one writes  $g \in G$ . However, purely set-theoretically,  $G$  might be a tuple  $(X, f)$ , where  $X$  is a set and  $o$  an operation on  $X$ , and  $(X, o)$  might be the set  $\{X, \{X, o\}\}$ . Then, again strictly formally,  $g \in G$  would actually mean that  $g$  is either  $X$  or  $\{X, o\}$ .

suggests that one should apply ideas from constructive mathematics, as for example from [constr-ana], here.

## 5 Interlude: constructive mathematics

As the reader might not be familiar with constructive mathematics, we now first provide some information on it. We then discuss how we intend to use constructive mathematics in arguments on the existence of formal proofs of theorems one wishes to establish. Finally, we give a particular interpretation of statements of constructive mathematics, which might be called “hypothetical interpretation”, and which we view as being particularly suitable for our applications.

### 5.1 What is constructive mathematics?

Constructive mathematics (including its meta-theories) is a body of ideas on what statements and arguments are proper in mathematics, how these statements should be interpreted together with actual mathematical results following these ideas. It emerged from the “Gundlagenstreit” in the beginning of the 20<sup>th</sup> century, and according to this, it is often associated L.E.J. Brouwer’s intuitionism. From the current point of view, constructive mathematics is broader than intuitionism, or to say it differently, intuitionism is one of the schools inside of constructive mathematics. As shall be made clear below, it is not this school we are interested in for our purposes.

As constructive mathematics is not commonly studied, a first question is: What ideas are alerted to by the term “constructive mathematics”?

According to the opening sentence of the entry on constructive mathematics in the Stanford Encyclopedia of Philosophy ([constructivism]), written by the constructive mathematician Douglas Bridges:

Constructive mathematics is distinguished from its traditional counterpart, classical mathematics, by the strict interpretation of the phrase “there exists” as “we can construct”.

For us, it is important to note that the phrase “we can construct” is not taken literally. Rather in constructive mathematics, also stories are told, albeit stories different from the ones in classical mathematics.

In constructive mathematics, the starting point of reflection are the natural numbers, which are a priori assumed to exist.<sup>14</sup> Fundamentally, the mathematical statements are on algorithmic operations on natural numbers

---

<sup>14</sup>See however our interpretation of statements of constructive mathematics in subsection 5.3.



and furthermore on algorithmic operations on algorithms operating on natural numbers and so on. For example, a sequence of natural numbers is an algorithm taking natural numbers and outputting natural numbers too. A function from the sequences of natural numbers to sequences of natural numbers is then an algorithm taking and outputting such an algorithm.

Care has to be taken, however, because constructive mathematics is not just computational mathematics with natural numbers. Indeed, according to the philosophy that everything has to be constructed, the algorithms also have to be constructed. This means that algorithms claimed to exist have to be output from previously defined algorithms.

This still does not give a clear criterion what one must do to establish that an algorithm can be constructed. What kind of arguments are allowed for this?

The solution is centered around a self-limitation already on the level of the underlying logic with restrictions around existence statements and negations. In particular, it is not allowed use implications

$$(\neg\forall x : A(x)) \rightarrow (\exists x : \neg A(x)) . \quad (1)$$

Another aspect of constructivism is: It is emphasized that it is only reasonable to speak about the truth value of a mathematical statement if one can convincingly argue for its truth or falseness. The pragmatic rule is that to make a statement  $A$  shall mean exactly the same as to make the statement “ $A$  can be proven.”

This general principle is then applied when sentences made with logical connectors are to be interpreted.

So a statement  $A \vee B$  shall not only mean that  $A \vee B$  can be proven but also (by applying the rule internally) that  $A$  can be proven or  $B$  can be proven.

Likewise, in constructive mathematics, a statement of the form  $A \rightarrow B$  has the same meaning as “Every (potential) proof of  $A$  can be converted into a proof of  $B$ ”, “One can give a procedure that every proof of  $A$  can be converted into a proof of  $B$ .”. Moreover, in all the three statements, one can add without modifying the meaning of the statement the initial phrase “It is known that” or “One can convincingly argue<sup>15</sup> that”. A negation of a statement, say  $\neg A$  of  $A$ , means then that it is impossible to prove  $A$ , that is, every try of a proof of  $A$  leads to a contradiction. Finally, an existence statement, say  $\exists x : A(x)$  means that one can show that an  $x_0$  can be constructed for which  $A(x_0)$  holds.

We note that it is then clear why the implication (1) shall not be used, just in the same way that  $\neg\neg A \rightarrow A$  and  $A \vee \neg A$  shall not be used.

---

<sup>15</sup>In usual terminology: prove

These intuitive rules or reasoning are made precise in the formal system “Intuitionistic Predicate Calculus” or IQC, developed by Arend Heyting. In up-to-date expositions on constructive mathematics, such as in [constructivism], these logical foundations are stressed. Interestingly, the founder of intuitionism, Brouwer, emphasized the preliminary role of mathematics over logic, but as is often the case, an intellectual system has been created which now has its own philosophy, independently of its historical origins.

Building on intuitionistic logic, there are different schools of constructivism. There is agreement on the use of the natural numbers and the importance of the notion of algorithm. However, there are variations concerning what principles of reasoning are allowed, on the style of presentation and on possible interpretations of statements. Information on the different schools can be found in the books [foundations] (Chapter III) and [constructivism-in-math] (Chapter I, Section 4), which we also recommend independently of this.

The different schools allow ways of reasoning which go beyond a pure algorithmic construction. Most strikingly, Brouwer allowed for the “possibility to use our free will to decide at each state what the next number in the sequence will be”, as expressed by Michael Beeson in [foundations], Chapter III, Section 4. Another principle was used by Andrey Markov Jr. He argued: If it is not true that a particular algorithm does not terminate (which means by the principles of constructivism that one can refute every attempt to prove that the algorithm does not terminate), then it terminates.<sup>16</sup>

An important extension of purely algorithmic constructive mathematics is the introduction of the notion of set. As stated in Chapter VIII of [foundations], there are two approaches: One can add the notion of set (or class) or one can postulate a “real” constructive set theory. Moreover, often a constructive axiom of choice is used, for example by Bishop.<sup>17</sup>

Besides different basic principles, the schools of constructive mathematics differ concerning the style or writing and the way of arguing. There is the “dry” “Russian school” which is essentially recursive function theory with intuitionistic logic.<sup>18</sup> In opposite direction, there is Errett Bishop’s book *Foundations of Constructive Analysis* (with a second edition with Douglas Bridges called *Constructive Analysis* ([constructivism])). These books highlight the spirit of constructive mathematics, that is, the construction, while

---

<sup>16</sup>See [constructivism-in-math], Section 4.6 and note for comparison also [foundations] Chapter III, Section 1 with the exercises.

<sup>17</sup>It might seem that in constructive mathematics there is trivially always a choice function when one desires one because everything that is claimed to exist must come along with a construction. As explained in [constr-theories, I,4.7] this is not so.

<sup>18</sup>This evaluation follows [constructivism].

not putting too much emphasis on foundational questions and ignoring discussions on foundational formal systems altogether. The first book, published in 1967, is of historical importance because it showed that one can really “do” constructive mathematics, and one can speak of a “Bishop school of constructive mathematics”; cf. [constructivism]. For us it is of importance that Bishop’s book is intuitively (in contrast to (overly) formally) written and is based on intuitionistic logic and the claim that the natural numbers exist a priori.

## 5.2 Our use of constructive mathematics

For our purposes, that is, for arguments on the existence of formal proofs of theorems one wishes to establish, there is no place for free-choice sequences. As said above, one can integrate set-theoretic arguments into constructive mathematics. We do not envision this for our applications.

Interesting is now Markov’s principle. We do not want to allow this principle either because otherwise we could argue for the existence of formal proofs of mathematical statements by contradiction. We regard this as being too weak.<sup>19</sup>

One aspect has not been addressed so far: Is it reasonable to really allow all thinkable algorithms? This question can be answered along just as it is usually answered in constructive mathematics. To illustrate the answer, let us first consider constructive elementary number theory. Here, finitely many algorithmically defined functions<sup>20</sup> are used to make statements on natural numbers, and then these statements are analyzed, however, there are no “higher algorithms” producing algorithms. The important aspect for us is that it should be obvious that the domain of the functions is total. This suggests to only consider functions defined by `loop`-algorithms, or, what amounts to the same with respect of equality of functions, functions defined by primitive recursion. This idea is formalized in the formal system of *Heyting arithmetic*, HA. This system is built on IQC with a function term and corresponding axioms for each presentation of a primitive recursive function.

One can then “go up” and iterate the idea of operation by primitive recursive function by considering primitive recursion operating on algorithms. For this, to keep track on what the algorithms actually operate on, one should

---

<sup>19</sup>An abstract version of Markov’s principle says  $(\forall n : (\phi(n) \vee \neg\phi(n)) \wedge \neg\forall n : \neg\phi(n)) \rightarrow \exists n : \phi(n)$ ; cf. [constr-theories, I,7]. We reject this principle in our application for arguments on formal proofs for the reason given.

<sup>20</sup>From the point of view of constructive mathematics, functions are the same as algorithms, however, when we speak about functions, we have an extensional notion of equality in mind and when we speak about algorithms are more refined one. Note here that in contrast to “usual” set theoretic mathematics, “equality” in constructive mathematics is not assumed to be a priori given; rather different notions of equality are used depending on the context.

work with *finite types*.<sup>21</sup> One then naturally obtains the notions of *finite type recursion* and – as a generalization of a primitive recursive function – *primitive recursive functional*.<sup>22</sup> Interestingly, in this way one can also obtain new functions from  $\mathbb{N}$  to  $\mathbb{N}$ : As explained in [constructivism-in-math, 9, 1.4], the premier example of a totally recursive but not primitively recursive function, the Ackermann function, can be given in this way. The ideas presented here have been formalized in a typed formal system called *finite-type arithmetic*,  $\text{HA}^\omega$ .<sup>23</sup>

In another direction, an important question is how formal an argument for the existence of a formal proof of a theorem shall be. Would it, for example, be reasonable to demand that it be written in  $\text{HA}^\omega$  (with meta-variables, particularly for the function symbols)?

We can give a clear “no” to this question. Just as set-theoretic mathematics, constructive mathematics is never done on this level of formality (apart from small calculations), and clearly it is not reasonable to demand that it should be.

Is it then reasonable to demand that at least it “should be possible” to reformulate / rewrite an argument in this system? Again, we can answer this with a clear “no”. As such a rewriting realistically would not be carried out and cannot be carried out either and as our starting point was the question what such a claim of “should” could mean in such a situation, it would be ironically inappropriate to carelessly introduce such a demand here.

There is finally the idea for the even weaker demand that one can argue constructively (as envisioned by us) that one can rewrite the argument in a formal system of constructive mathematics (like  $\text{HA}^\omega$ ). But then this new rewriting procedure would realistically also not be given formally, and one would be required to have a new rewriting procedure, and so on, leading to a never ending regression.

In summary and positively expressed, for arguments on the existence of formal proofs of mathematical theorems, we envision an informal presentation which is “purely algorithmic” along the lines just outlined.<sup>24</sup> The

---

<sup>21</sup>A type is an expression built following these rules: 0 is a type and if  $\sigma, \tau$  is a type then  $\sigma \times \tau$  and  $\sigma \rightarrow \tau$  is a type. Algorithms of type 0 are the algorithms which do not have an input and output a natural number. The function associated to such an algorithm can be identified with the output, that is, one can say that the function is a natural number. For the interpretation of the statements given in the next subsection it is, however, important that an algorithm of type 0 is not a natural number – it is an algorithm producing a natural number.

<sup>22</sup>These ideas were developed by Kurt Gödel in [dialectica].

<sup>23</sup>Strictly speaking there are at least two distinct systems with this name. The “most basic” system is given in [constr-logic&-math], which also gives a nice exposition to constructive mathematics. The definition of  $\text{HA}^\omega$  in [constructivism-in-math] includes a “combinator” which is not present in the definition in [constr-logic&-math].

<sup>24</sup>In the terminology given by Solomon Feferman in [constr-theories] (which is also

presentation might be similar to that of Bishop’s book with possibly a presentation of algorithms given by pseudo-code. (But as said, Bishop uses principles we would not like to use, namely sets and the constructive axiom of choice.)

We note that with this approach has the nice feature that one can always “go up” with algorithmic arguments.<sup>25</sup> This means that if we envision a meta-analysis of arguments on the existence of formal proofs, a meta-analysis of this and so on, we never have to leave the framework outlined.

### 5.3 Our interpretation of constructive statements

We now come to the *interpretation* of constructive statements for our application in arguments on the existence of formal proofs of theorems, along with a corresponding suggestion for the use of language.

Following the citation in the beginning of subsection 5.1, “constructive mathematics is distinguished from [...] classical mathematics by the strict interpretation of the phrase ‘there exists’ as “we can construct”. More precisely, it seems that there is a general consensus of mathematicians who regard themselves in the constructive tradition that one imagines the natural numbers and strives to construct everything from this basis – as was programmatically expressed by Leopold Kronecker (see [on-Kronecker]): “Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.”

For our applications, the evident approach would then be to say that texts based on a fixed alphabet are represented by natural numbers and vice versa. Statements as described in the previous subsection then become statements on transformations of texts via algorithms, transformations of algorithms on such algorithms and so on. Above, we have tacitly already used this identification of natural numbers and texts.

Let us reflect on the meaning of the statements on texts which are then made: We first note that the concept of text is not the same as the concept of a natural number, or to say it differently, the phrases “text” and “natural number” have a prescientific meaning which is clearly not identical. Texts have the evident property that they can be given literally; what one writes down *is* a text and does not just denote a text. In contrast, the idea of a natural number is always abstract. Surely, one can represent numbers by texts, for example using the usual decimal representation or the “elementary” unitary representation. Nonetheless, if for example somebody writes

---

given in [foundations, V,3]), our assessment is that the system  $\text{HA}^\omega$  as defined in [constructivism-in-math] is directly adequate and directly in accordance with the envisioned body of arguments on the existence of formal proofs of mathematical theorems, where texts on a fixed alphabet are identified with natural numbers.

<sup>25</sup>This corresponds to the formal feature of  $\text{HA}^\omega$  that for two types  $\sigma, \tau$ , there is always the type  $\sigma \rightarrow \tau$ .

down “123” she or he surely writes down a unique text independently of the meaning of the symbols, but she or he only gives a number if a meaning is attached to the text.

So, texts can be given physically-literally,<sup>26</sup> but in our application we talk about texts which are constructed by algorithms without this ever being the case physically. So we clearly tell ourselves a story about imaginary texts here.

The goal is now to limit the imaginary aspect of the story to a minimum. Particularly, we do not want to make statements on the transformation of an *infinite amount* of texts via algorithms. Rather we want to make statements on the transformation of *individual texts* via algorithms. Also, because algorithms can also take algorithms as inputs and output algorithms, we want to make statements on the transformation of *individual algorithms*. This will then correspond well with the statement made in the introduction that we regard a statement that infinitely many results have been established to be a priori invalid (see also subsection 6.3 for this point).

For this, we use what we have already used in the previous subsection: We can give algorithms by writing down pseudo-code, and we can iterate this process by using already given algorithms. We can then in particular consider algorithms which take no input. We then say that the output *is algorithmically given or can be constructed*. As a special case of this, we can consider algorithms which output texts. Again, we then say the output text *is algorithmically given or can be constructed*.

We iterate and stress again that when making these statements, we again tell a story which should not (and in many applications cannot reasonably) be interpreted physically. The phrase “is algorithmically given” makes this clearer than the phrase “can be constructed”, but both phrases are unproblematic if one realizes that a story is being told. We also note that any literally given text is also algorithmically given (in the story).

As already stated, we do not want to make statements on infinitely many input instances. Rather, concerning the input instances, we merely want to make *hypothetical* statements of the form “Suppose that some  $x_0$  is given. Then  $A(x_0)$  holds.”.

But what shall it mean that the statement is “hypothetical”? Our idea is that the reader always only imagines a particular algorithmically given  $x_0$ . But this raises again a question as it is not clear what algorithmic constructions shall be allowed. (It does not help to say “all possible constructions”, because we want to discuss what we shall deem as being possible.) The strictest possible solution, the most narrow interpretation which seems to be

---

<sup>26</sup>One can then discuss when two texts should be regarded as being equal and also if a physically given text denotes something like an “abstract text”; in any case it is evident that texts can be given physically and literally.

reasonable, is: Such a hypothetical statement is on all  $x_0$  which have been algorithmically given or will be algorithmically given by any conscious being shall be allowed. Wider interpretations, which go more in the direction of story telling, are possible.

These hypothetical interpretations lead to a corresponding interpretation of the forall-quantifier: We interpret a statement of the form “ $\forall x : A(x)$ ” (including corresponding statements in natural language) as:

Suppose some  $x_0$  is algorithmically given. Then  $A(x_0)$  holds.

Similarly, for “ $\exists x : A(x)$ ” one might then not only say “Some  $x_0$  with  $A(x_0)$  can be constructed, but also:

Some  $x_0$  with  $A(x_0)$  is algorithmically given.

Is this interpretations always possible, that is, can one reasonably interpret statements of constructive mathematics as outlined in the previous subsection in this “hypothetical” way? In particular, is the interpretation consistent with (typed) Intuitionistic Predicate Calculus? The author’s answer is that this is so; after all the only occurrence of the forall-quantifier in the axiom scheme is the scheme  $(\forall x : A(x)) \rightarrow A(t)$ , and this is nicely consistent with our interpretation.

With the outlined “hypothetical” interpretation of forall-statements, there is no need to imagine infinitely many objects. For this reason, by itself it would seem to be reasonable to call this interpretation “finitism”. However, as the term is already used (as far as the authors can tell with rather different meanings) in a similar way as “constructivism” (based on natural numbers), we refrain from doing so. We note that our interpretation is different from ultrafinitism which rejects natural numbers deemed to be “too large”.

In our applications, we want to talk about texts and not natural numbers, but to complete the picture and to deepen the link to constructive statements as they are usually made (including in the previous subsection), let us come back to natural numbers now. As said above, the idea of a natural number is always abstract. Moreover, as it is based on the idea of counting, an algorithmic aspect is already present in the notion of natural number. So it is not at all clear what a “non-algorithmically given” natural number should be. For this reason we suggest to just speak of a “given natural number”, not an “algorithmically given natural number” and to also say that the number “has been constructed”, not just that it “can be constructed”. The “hypothetical” interpretation can then also be applied to natural numbers.

To given an example of our interpretation, by saying that there is the rule or algorithm  $\mathcal{A} : (a, b) \mapsto a \uparrow\uparrow b$ , one says:

Suppose a tuple of natural numbers  $(a, b)$  is given. Then the

algorithm  $\mathcal{A}$  gives / constructs a particular number (which is then denoted by  $a \uparrow\uparrow b$ ).

If the reader then imagines two given numbers  $a$  and  $b$ , say  $82479824 \cdot 3798217$  and  $2418^{92741}$ , she or he can construct the corresponding number  $a \uparrow\uparrow b$ , in this case  $(82479824 \cdot 3798217) \uparrow\uparrow (2418^{92741})$ , and then this number is given / has been constructed. The statement is here that whatever two numbers the reader imagines, she or he can construct the resulting number by application of the algorithm. We note that again that the statement is a story without reasonable physical interpretation.

## 6 Criteria for rewriting

With the ideas of constructive mathematics in mind we now discuss how one might address the Central Question and questions around it.

### 6.1 First criteria

We return to the approach in subsection 4.2, as given in items 1. – 3. This means that an input system  $I$  is given, a rewriting procedure is given, and a convincing argument that the rewriting procedure is correct is given. As already stated, to study and analyze such a system, it is natural to apply ideas from constructive mathematics. We now reflect upon how exactly this might be done:

The procedures should be given in an explicit and concrete way. The level of concreteness required is a matter of judgment, and indeed, exactly this question is always a matter of debate in actual applications of the constructive paradigm, like for example in [constr-ana]. Ideally, in our opinion, a detailed “mathematical” description focusing on ideas and on the basis of constructive mathematics, a description in pseudo-code and an actual, testable implementation should be given, but at least the first point should be satisfied.

A reasonable criterion is then:

**Criterion 1** The procedures for testing membership in  $I$  and the rewriting procedure have to be given in such an explicit way that humans can actually perform the computations on examples and such that an implementation on a physical computer seems to be actually possible. Moreover, the claims for correctness have to be given by following the rules of constructive mathematics as outlined in subsection 5.3.

The criterion is of course vague, but this is unavoidable if we speak about mathematical arguments (here arguments for the procedures) as they



are actually given and not formal arguments or computer programs.

We give three examples of rewritings with respect to ZFC which satisfy this demand and which already make a huge difference in usability.

- Definitions as abbreviations for terms
- Syntactic reformulations to make statements better readable, for example “Let  $x \in X$ . Then ...” instead of “ $\forall x : x \in X \rightarrow \dots$ ”.
- If a pure set theory such as ZFC is used: the introduction of class terms for properties: For each property  $p(x)$ , the term  $\{x \mid p(x)\}$  with the rule

$$\forall y : y \in \{x \mid p(x)\} \leftrightarrow p(y)$$

is introduced, along with subclass relationships and equality of classes with the obvious rules. Furthermore, the statements

$$\exists \{x \mid p(x)\}$$

as abbreviations for

$$\exists y : y = \{x \mid p(x)\}$$

are introduced.

We now consider an application to a particular text, following item 4' of subsection 4.2. The following criterion then seems to be appropriate:

**Criterion 2** Let a text of the following kind be given. The text consists of three parts: A linearly told substantive story that a particular statement  $A$  is proven on the basis of Zermelo-Fraenkel set theory, a constructive story on a rewriting system following the first three points of subsection 4.2 and a header of the form: “Based on the following argument for statement  $A$  and the outlined system for rewriting (which follows the approach of subsection 4), one can give a formal proof of  $A$  on the basis of ZFC.” Then this shall mean that Criterion 1 applies to the story on the rewriting system, the substantive story is convincing on the basis of (informally given) Zermelo-Fraenkel set theory with the axiom of choice and without too much effort, one can obtain a formal reformulation  $A^{\text{formal}}$  of  $A$  in ZFC and a convincing argument on the basis of constructive mathematics that one can construct a valid input instance to the rewriting system.

We make some remarks:

- We distinguish between Zermelo-Fraenkel set theory with the axiom of choice and ZFC: The latter is a formal system whereas the former is an axiomatic set theory as actually used by mathematicians. (The

particular theories serve as examples.) We note in particular that there is a psychological difference between convincing oneself that arguments are valid on the basis of the axioms of Zermelo-Fraenkel set theory and convincing oneself that from the arguments one can construct (in the sense of constructive mathematics) a formal proof. The latter task is usually not considered by mathematicians when they study “proofs”.

- The criterion is clearly vague in several ways. What shall in particular “without too much effort” and “formal reformulation  $A^{\text{formal}}$  of  $A$ ” mean?

One can of course argue about the specific formulations, but there is a fundamental reason why vagueness cannot be avoided here: All statements on mathematical texts *as actually written* are vague, already starting from criteria on “mathematical texts”. For example, in Section 2, we already mentioned that there is no clear standard as to what shall be called a “proof” and what not.

The claim of the reformulation of  $A$  is always present when it is claimed that statements can be expressed formally and cannot be avoided. Concerning “not too much effort” we want to give the rule of thumb that a person who claims to have understood the “proof” can convince him- or herself in less time than needed for the understanding of the “proof” that the criterion is satisfied.

- There are obvious variants for the criterion for other formal set and class theories, such as, for example NBG. In fact, it is reasonable to state an analogous criterion for what we shall define as a “kind of mathematics” in subsection 7.1. The same holds for the following criteria.

## 6.2 Computer backed systems

Again we come back to the idea of subsection 4.2 to have a procedure for rewriting and to the approach consisting of the four requirements outlined there. For a particular suggestion for the first three aspects and for a particular mathematical argument, it remains to argue that it can be written in the formal system for the input and holds in there. This is a non-trivial task for itself, with which one comes back to the original problem of rewriting actual (non-formally given) mathematical arguments.

One idea (corresponding to the original item 4 in subsection 4.2) is here to write mathematics directly in a humanly oriented formal system and to check the arguments with the help of a computer. This idea was propagated in the “QED manifesto” ([QED]), and several formal systems for this have subsequently been developed.

The most successful of these is arguably the **Mizar** project. In this system a great number of theorems have been written and formally checked with the help of computers. This is a great success for formalized mathematics. One should, however, keep the following in mind:

First, such systems are not wildly used and mathematicians most often do judge arguments to be valid or not on a rather intuitive basis without proofs in formal systems. So there is indeed a much larger story of mathematics told outside such formal frameworks than inside.

There is also a fundamental problem with the use of such computer-based systems: The formal proofs are clearly more difficult to understand than usual mathematical arguments; for examples of this see [example]. The difficulty is required because of the rigid structure of the formalized language, which would not be necessary for humans and in fact impairs their understanding. So instead of checking the arguments themselves, humans now turn to computers. Computers are physical objects designed by engineers based on supposed physical laws found and formulated by physicists. Physics and engineering rely themselves on mathematics, and the design and the building of computers has an immense complexity which has accumulated, a complexity which is actually much larger than the complexity of nearly any argument in mathematics. Furthermore, to use a computer for proof checking, one uses diverse programs (including programs one maybe does not have in mind at first, like the operating system), some of which is so large that it cannot even be read by any human. Now, why should one assume that programs on computers run correctly if on the other hand one doubts that humans can check mathematical arguments? There might be good reasons to check supposed proofs by computers as an additional check, but fundamentally, one can question a check by computers (built and programmed by humans) as much as one can question a direct check by humans.

### 6.3 Formalistic stories

We end this section with the promised definition of formalistic stories.

Above Criterion 1 we have outlined how procedures for rewriting should be given and analyzed. We can then combine such rules and their analysis with a particular convincing set- or class-theoretic argument to obtain an argument that a formal proof can be constructed (in the sense of constructive mathematics).

More generally, we define: By a *formalistic story* we mean a story in constructive mathematics as described in subsection 5.3 whose main claim is that a formal proof of a particular result or proofs of *finitely many* results can be constructed (on the basis of some formal system). Nevertheless,

we will still refer to rewriting procedures in the future since this is how mathematicians actually would proceed to establish formalistic stories.

We have set the definition up in such a way that it corresponds to the following criterion which reflects our requirements for arguments for formal proofs.

**Criterion 3** Given a set theoretic statement  $A$ , the claim “ $A$  is proven on the basis of ZFC.” shall mean: On the basis of constructive mathematics, one can argue convincingly that one can construct a formal proof of a formal reformulation of  $A$  in ZFC.

So, by this criterion, to establish that  $A$  is proven on the basis of ZFC it is necessary and sufficient to give a convincing formalistic story for  $A$ . Again, ZFC can be substituted with any other explicitly given set-theoretic formal system.

In subsection 6.1 we considered particular formalistic stories and gave criteria for them. In full generality it is outside of the scope of this note to discuss when formalistic stories shall be regarded as being valid. We can just say that there surely is a great subjective component.

For an application of the criterion, a particular statement  $A$  has to be given. We recall that in the introduction we stated that we regard any statement that infinitely many mathematical results have been established to be invalid. Our definition of formalistic stories and the criterion just given reflects this requirement.

We note that if one follows our interpretation of constructive mathematics, described in subsection 5.3, as a framework to argue that mathematical results are established formally, one cannot argue that infinitely many results are established via formal proofs, just as one cannot argue that there are infinitely many natural numbers.<sup>27</sup> One can just give methods (such as, for example, the ones in subsection 4.2) which might then be used to establish a variety of results.

## 7 Interpreting actual stories

With Mathematical texts as actually written most of the time neither linearly told substantive stories (as defined in subsection 4.1) nor formalistic stories (as defined in subsection 6.3) are told. Furthermore, rewriting procedures do not at all belong to mathematical practice. Although one usually may interpret mathematical texts as a set- or class-theoretic story, this is not

---

<sup>27</sup>Of course, in set-theoretic mathematics, one can still argue on infinitely many objects called “formal proofs”. This should not be confused with our study here whose purpose is to discuss the Central Question in subsection 4.

always the case and there are several kinds of mathematics. All these things lead to and contribute to a variety of interpretations of mathematical texts, some of which will be discussed in the following.

## 7.1 Different kinds of mathematics

Up to now we identified mathematics with some set or class theory which gives a unified foundation of a lot of mathematical practice. In general, when speaking of some *kind of mathematics*, we consider a separate story of mathematics and we always expect that there is some (description of) collections of rules of inference and of axioms.<sup>28</sup> This condition gives a very general setting of mathematics. However, usually mathematicians have a more restricted view which things are really worth being considered as mathematics and opinions may differ in this issue.

Given an informal description of a kind of mathematics, it should be possible to write down a formal system about which it can be said that it gives the kind of mathematics formally. In the following, in order to be able to talk about formalistic stories for the kind of mathematics, we assume that such a formal system is actually given.

A kind of mathematics is not to be confused with a mathematical theory, like number theory or K-theory, considered inside some set or class theory - they do not comprise a separate foundation. Contrary to this, some given theory considered in model theory serves as an example for a kind of mathematics.

Famous examples for set or class theories are Zermelo-Fraenkel and Neumann-Bernays-Gödel set theory, where only the second one uses proper classes. These constitute different kind of mathematics, even though model theory indicates that one should expect basically the same results in these two kinds of mathematics. But there are also weaker, stronger or different set or class theories. For example one may use a set theory without the axiom of choice or some weaker or stronger version of this axiom. As another example, one can demand the existence of inaccessible cardinals, or equivalently of Grothendieck universes. These changes have real consequences on the stories which can be and which are told on the bases of the axiom schemes.

Kinds of mathematics different from set or class theories may be found in mathematical history or, as mentioned before, as a theory studied in model theory. Synthetic (euclidean) geometry, propositional logic, Peano arithmetic, group theory or some theory of (maybe further specified) fields

---

<sup>28</sup>Since this is an informal description of mathematics, we do not restrict the language or the languages for our story in any way. Following what is common, we expect that several natural languages combined with some symbolism are used.

as in model theory or even some theory of Sudokus may serve as examples. However, for these kinds of mathematics it would not be unusual to be formalized inside some set or class theory.

Constructive mathematics is different. Usually, it is not the subject being studied, but it serves as a basis of a mathematical theory. This is caused by the imagination of constructive mathematics as a more basic, tangible and trustworthy foundation than set-theoretic mathematics based on classical logic - constructive algorithms with intuitionistic logic may appear more real than abstract sets and indirect proofs. Therefore, it may be strange to consider constructive mathematics inside some classical set or class theory. Nevertheless, model theory of intuitionistic logic exists.

Given an actual mathematical text, in most cases mathematicians would interpret it inside some set or class theory. However, usually it is considered to be irrelevant to give a precise listing of the axioms. This is a feature of actual mathematics: the claim of rewriting of *one* mathematical text may be stated with respect to different foundations. Without mentioning the claim of rewriting, this is used in actual mathematical papers or books, for instance as “the usual proof applies to the new axiom system”. Emphasizing it differently, this means that usual mathematical language may be a tool to transfer results from one kind of mathematics to a different one.

In the last sections we explained in detail why constructive mathematics is the right framework for the rewriting procedure of mathematical arguments in Zermelo-Fraenkel set theory to ZFC. In fact, these arguments apply to all kinds of mathematics. Also the previous definitions of linearly told substantive story, non-formalistic and formalistic header as well as formalistic story apply to all kinds of mathematics, and it is reasonable to apply the criteria suitably adapted.

In contrast to set or class theory a rewriting procedure may be easier, and therefore of less importance, or even unnecessary in “weaker” kinds of mathematics. Within this context, linearly told stories of constructive mathematics on rewriting and results on the rewriting of such stories in a formal system of constructive mathematics (possibly  $HA^\omega$ ) achieve an outstanding role. As mentioned in subsection 5.2, one cannot realistically demand that one can give a formal basis for rewriting but would rather end up in a never ending regression.

## 7.2 Statements on statements in actual stories

As already mentioned in the introduction, actual mathematical texts contain a variety of statements, in particular reflections on the process of doing mathematics. These reflections include remarks inserted for didactic reasons, expressing for example “ideas” before a “proof” is given, or aesthetic

judgments.

To analyze all these different kinds of statements is outside of the scope of this note; we want to turn attention to a particular kind of statements: statements on mathematical statements in how far as they are used to (allegedly) establish mathematical results. One important kind of such statements we have already considered: Formalistic stories consisting of a formal header and a linearly told substantive story. Given such a text, one can then also say that just the formalistic header is a statement on a statement.

We give some examples of statements on statements which are used establish mathematical results as they might actually occur in mathematical texts.

1. After two sets  $x$  and  $X$  have been given: “Now the statement that  $x$  is contained in  $X$  holds.”
2. After in an argument 27 case distinctions have been made and two of them have been proven: “We have now proved the first two of the 27 cases, proofs of the remaining 25 cases are analogous. The reader should keep the changes outlined in the previous two chapters in mind.”
3. “For every three statements  $A, B, C$ , the statement  $(A \vee B) \wedge C$  is equivalent to  $(A \wedge C) \vee (B \wedge C)$ .”

In the first example we have the word “statement” which can be seen to refer to a particular sentence, namely “ $x$  is contained in  $X$ .”, in the second example, “proof” refers to a body of sentences, and in the third example “sentences” are explicitly mentioned.

When analyzing these examples, one notices that even among the restricted kind of statements we analyze, a variety of kinds of statements are made:

The first example is easy: The statement merely says that statement

$x$  is contained in  $X$ .

holds. This might have just been stated directly.

The second example is more tricky. The idea is here that inside the text written there is an instruction as how to generate more text in order to obtain a more complete argument. A straight-forward answer what shall be done now is: Well, the text shall be rewritten according to the instruction. However, an author who makes such a statement in all likelihood does not expect a reader to really follow his or her instruction. Given this, there again arises the question what kind of statement is actually made here. An answer is that the reader should imagine a linearly told substantive set- or class-theoretic story (and not a formalistic story, as in the previous section).

So the reader should image a story which by itself is a story on imagined (non-physical) objects. The authors of this note do not in principle object to this practice, but just remark that such meta-statements are error-prone and should rather be avoided.

The third example is yet different. One notices first that it is vague: What exactly is meant by “three statements  $A, B, C$ ”? Depending on the context, there are now different types of interpretations.

One type of interpretation is that this shall be a statement of some formal logic, which is the kind of mathematics being studied. For this,  $A, B, C$  should be specified as, for example, propositional symbols when working in propositional logic or 0-ary predicate symbols when working in some first-order logic. In the second case one would tend to allow arbitrary formulas in  $A, B, C$ , which means  $A, B, C$  are placeholders for arbitrary formulas. But this is not allowed within this context since it would lead to an infinite number of mathematical statements. However, if one uses some set or class theory to study formal logic formalizing the corresponding language via sets, this problem disappears since this infinite number of statements may be summarized into one set-theoretical statement. So studying some formal logic within some bigger kind of mathematics leads to another type of interpretations inside some mathematics.

Keeping in mind subsection 5.3 one notes that it is also possible to give a correct interpretation of the third statement if  $A, B, C$  are placeholders for first-order formulas when working in first-order logic. Then the statement is true for any (in terms of 5.3) given first-order formulas. Note that this is a statement in constructive mathematics, thus a part of the rewriting story and not the kind of mathematics being studied.

There are more interpretations of the third statement. For instance one could take the point of view that such a simple statement is obviously a statement in propositional logic and when using this statement it should be interpreted as an application of propositional logic in some kind of mathematics as described in subsection 7.1. Another possibility is that one interprets it as a description how to write down a substantive story or even a formal proof working again outside of the mathematics being studied. However, one has to be careful using such interpretations since they are not as precise as the interpretations before.

A conclusion from these considerations is:

**Criterion 4** Whenever in a story of mathematics a statement on seemingly infinitely many sentences (or expressions) are made, the statement must not be interpreted as giving infinitely many mathematical results. If the statement is later used as a meta-statement on how to obtain an argument or a formal proof, it must be established on the basis of constructive mathematics.



We recall here that rule and axiom schemes on which a linearly told substantive story is based are never part of the story if they contain statements on statements; indeed they are part of non-formalistic or formalistic headers. So there is no contradiction between the Criterion and the fact that in axiom systems, often statements on infinitely many statements are made. Of course, infinitely many statements are then not written down in the header. Nonetheless, as mentioned before, it should be possible to check immediately any claim that a particular statement belongs to the axiom scheme.

## 8 Mathematics and meta-mathematics

There is an interesting relationship between the use and the theory of formal systems: Formal systems were first introduced (for example by Gottlob Frege) to study and clarify mathematical arguments, and as already stated in subsection 3.2, the idea of formal reasonings (formal proofs) is foundational to the present day doing of mathematics. Now, the idea of formal systems by itself invites a mathematical study on formal systems, and quickly the study of formal systems was integrated into mathematics.

On the one hand now, the mathematical study of formal systems has its motivation in meta-mathematical considerations. Conversely, the mathematical study of formal systems has led to mathematical results on formal systems which have an impact on how mathematics is viewed and conducted. One example is the application of (not immediately obvious) statements from propositional logic or first-order logic in the story of mathematics.

The (historically surprising) statements on the foundations of set theory (incompleteness, independence of axioms, etc.) have a particular, one might say philosophical, impact on the way mathematics is seen and is conducted. (Explicitly, they have led to a certain “liberal attitude” towards different axiom schemes, where no such scheme is regarded to be “absolutely true”.)

It is, however, not obvious in what sense such results make statements *on* mathematics (in particular on the limitations of mathematical endeavors) rather than *in* mathematics.

To study this question, more generally, we study the relationship between arguments (“proofs”) as such and arguments on formal proofs. For the latter we consider formalistic stories and model-theoretic formal proofs. We treat the relationship between arguments and formalistic stories and call the corresponding directions of inference “formalization” and “deformalization”. Then we discuss the relationship between arguments on formalistic stories and model-theoretic formal proofs, calling the corresponding directions “going down” and “going up”.

Finally, we study set-theoretic interpretations of kinds of mathematics,

using also model-theoretic results. In order to do so, we need the previous results on formalization, deformalization and going down for all kinds of mathematics. For this reason, in comparison to the previous section of this work, we broaden the scope and start immediately with a kind of mathematics.

As mentioned in subsection 7.1, for a kind of mathematics under consideration, we assume that a corresponding formal system is actually given. We can then speak about formal reformulations of statements for the kind of mathematics and formalistic stories.

Let a formal system, say  $S$ , be given. This means that an alphabet, a grammar, possibly a criterion on bodies of sentences being interpretable<sup>29</sup>, axioms and rules of inference are given. This in turn means that algorithms to judge whether an expression is a sentence, a body of sentences is interpretable, a sentence is an axiom and rules of inference are applied correctly shall be given following our rules of constructive mathematics. Moreover, we demand that the system is so “humanly oriented” that trained mathematicians can – “by hand” – check on small examples whether the rules of the formal system are used correctly and can write down some illustrating examples of formal proofs corresponding to convincing arguments.

We stress that we not want to speak about the language as a set; in fact in line with the previous parts of this work, we do not want to use set theory at all when we do meta-mathematics.

Nonetheless, given such a datum, we can easily tell a corresponding story about sets (in Zermelo-Fraenkel set theory). In this story one introduces set-theoretic objects with the same name as the ones for the formal system (like “alphabet”, “sentence”, “axiom”, “rule”), and of course the idea is that these objects model the corresponding objects defined by the formal system. One should, however, not forget that one (merely) tells a set-theoretic story here.

Concretely, there then is a set  $\Sigma$  called *alphabet*. Attached to this set we have the set  $\Sigma^*$  of tuples of arbitrary length on  $\Sigma$ , the elements of which are called *expressions* on  $\Sigma$ ; we have the *language*  $\mathcal{L}$  (consisting of elements called *sentences*), which is a subset of  $\Sigma^*$ ; we have the set of *axioms*  $\Gamma$ , which is a subset of  $\mathcal{L}$ ; there is a set of *rules of inference* and maybe also a set to define interpretable bodies of sentences. In such a story it is then defined, using these sets and set-theoretic constructions, that certain elements of  $\mathcal{L}^*$  (the set of tuples of arbitrary length on  $\mathcal{L}$ ) are called *model-theoretic formal proofs*.

We shall always only consider a single formal system  $S$ . For this, we use the following terminology and notations: We call a sentence in the language of  $S$  an *S-sentence*. We remark that we only make constructive and never

---

<sup>29</sup>See subsection 3.2.

set-theoretic statements on  $S$ -sentences. When telling a set-theoretic story or when reflecting on such a story we might want to reason about a set associated to a given  $S$ -sentence or, more generally, expression in the language of  $S$ . In this case, for an expression  $E$ , we denote the set by  $E^{set}$ . We apply this in particular to  $S$ -statements  $A^{formal}$  which are formal reformulations of statements  $A$  in a given kind of mathematics. We note that even if  $A$  has a meaning in the given kind of mathematics, when we argue about  $(A^{formal})^{set}$ , we merely tell ourselves a story about a particular set.

## 8.1 Formalization

Recall that in Criterion 3 we said that to claim that a mathematical statement  $A$  is proven in Zermelo-Fraenkel set theory shall mean that on the basis of constructive mathematics one can argue convincingly that one can construct a formal proof of a formal reformulation of  $A$  in ZFC. Now, when a mathematician claims that he or she has proven a result, he or she usually just writes down his arguments, which are by default based on Zermelo-Fraenkel set theory with the axiom of choice, but he or she does not say that there is a formal proof in ZFC on the basis of the arguments. With the terminology of Section 4 we can say that a substantive story is told, not a formalistic one.

But what if a formalistic header was added, so that really the claim was made that one can construct a formal proof on the basis of ZFC? Can one a priori say that this claim is valid?

Given our terminology (in particular again Criterion 3) we can also formulate this in the following brief way: Can one a priori say that every statement convincingly argued for on the basis of Zermelo-Fraenkel set theory with the axiom of choice can be proven on the basis of ZFC? This question is non-trivial, because mathematicians in their judgment if an argument is convincing to them do usually not explicitly consider the question of rewriting.

We answer a the corresponding general question for all kinds of mathematics affirmatively with the following thesis:

**Thesis 1** Let some kind of mathematics with a corresponding formal system  $S$ , an (informal) statement  $A$  in the kind of mathematics and a convincing argument for  $A$  (again in the kind of mathematics) be given. Then one can construct a formal proof of a formal reformulation of  $A$  on the basis of  $S$ .

This thesis is in nature similar to the Church-Turing thesis. We stress that it should not be taken as a definition; in fact, the phrase “convincing mathematical argument” (what other authors call “informal proof”) already has a meaning, so it cannot be taken as a definition.

We do however want to suggest that if somebody ever came up with a counterexample, the mathematical intuition on when a mathematical argument is seen as convincing (a “proof”) or the axiomatic basis would change and afterwards the thesis would again be correct.

We note that a related inference as the one in the thesis is present in Gödel’s argumentation for the First Incompleteness Theorem: He starts with a formal system  $F$  containing Robinson Arithmetic and uses this implication: If a formal sentence  $A$  follows from  $F$ , a sentence whose interpretation is “ $A$  can be proven from  $F$ ” also follows from  $F$ .

Despite this similarity, there are, however, differences between Gödel’s technique and what we argue for here: The argument by Gödel is purely mathematical, it is an inference inside of mathematics. Our thesis is however meta-mathematical, it is about the human endeavor concerning mathematics.

## 8.2 Deformalization

Let us now consider the opposite question which we will study in the general setting from the beginning. So let a mathematical statement  $A$  in some kind of mathematics and a formalistic story showing a formal reformulation of  $A$  be given. Can one always suppose that this formalistic story leads to a convincing argument for  $A$ ? Note that particularly the formalistic stories studied in 4.1, that is, linearly told substantive stories with formalistic headers, automatically lead to convincing arguments by just considering the substantive stories.

Considering formalistic stories in full generality, at first view one may doubt that this is always the case. A formalistic story was defined in a pretty general way, and the imagined “output” of such a story would, as noted in subsection 3.4, lead to an accumulation of symbols which would not be readable by humans. But conversely, the formalistic story has to be readable and understandable by humans in order to be a story itself. Therefore the formalistic story should convince the attentive reader that statement  $A$  holds. This leads to the following thesis:

**Thesis 2** Given a kind of mathematics and a statement  $A$  in the given kind of mathematics, a formalistic story showing a formal reformulation of  $A$  is a convincing argument for  $A$  in the given kind of mathematics.

## 8.3 Going-down

By the theses of the previous subsections 8.1 and 8.2, for every given statement, convincing arguments should lead to formalistic stories and conversely, formalistic stories should immediately be convincing arguments. Recall that

formalistic stories are by definition stories within constructive mathematics which show that there exist formal proofs of the given statement.

Suppose such a formalistic story is given for some statement  $A$  of some kind of mathematics – as always with a given formal system. Let  $A^{\text{formal}}$  be a formal reformulation of  $A$  corresponding to the formalistic story. Now the given formalistic story gives rise to a story in Zermelo-Fraenkel set theory to show that there exists a model-theoretic formal proof for  $(A^{\text{formal}})^{\text{set}}$ . This means that algorithms of the formalistic stories are transferred to set-theoretic constructions which carry out the algorithm. Since classical set theory is more powerful than constructive mathematics as described, such a transformation is always possible. Summarizing, we have:

Any formalistic story for  $A$  leads to a convincing argument that there exists a model-theoretic formal proof for  $(A^{\text{formal}})^{\text{set}}$ .

Assuming Thesis 1 we then obtain the following. As it is potentially weaker than Thesis 1, we formulate it as a separate thesis.

**Thesis 3** Let some kind of mathematics with a corresponding formal system, a statement  $A$  in the kind of mathematics and a convincing argument for  $A$  be given. Then one can construct a formal reformulation  $A^{\text{formal}}$  of  $A$  and a convincing argument for the set-theoretic statement for: There is a model-theoretic formal proof of  $(A^{\text{formal}})^{\text{set}}$ .

## 8.4 Going up for existence of arguments

Suppose now that a kind of mathematics (as always with a corresponding formal system  $S$  for which in turn we fix a set theoretic realization) and a statement  $A$  in the kind of mathematics is given. Using the notation from above, suppose now a mathematician does the following:

1. He or she gives a formal reformulation  $A^{\text{formal}}$  of  $A$ .
2. He or she argues (“proves”) set-theoretically (!) that there exists a model-theoretic formal proof for  $(A^{\text{formal}})^{\text{set}}$  on the basis of the given set-theoretic realization of  $S$ .

One might then be tempted to say that  $A$  “is proven” (has been convincingly argued for) on the basis of Zermelo-Fraenkel especially if the kind of mathematics being considered is Zermelo-Fraenkel set theory itself. This would however contradict our general attitude that in order to say that a result is formally proven (on the basis of some set-theoretic formal system) one has to constructively argue that there is a formal proof. Explicitly, this would contradict Criterion 3.

We will come back to the relationship between the two statements 1. and 2. in subsection 8.6.

## 8.5 Going up for non-existence of arguments

We now consider a statement on the non-existence of model-theoretic formal proofs for statements. Examples for this are many, let us state one: One can argue model-theoretically in Zermelo-Fraenkel set theory that if ZF is consistent, then ZF with  $\neg C$  is also consistent, that is, that the axiom of choice does not follow from ZF. This result is particularly interesting as the known arguments for it rely on non-trivial set theory.

We want to “go up” and conclude something on the limitations for humans and more generally on conscious beings of doing mathematics. Let us stick with the given example. Here we would like to conclude: Any convincing argument on the basis of Zermelo-Fraenkel set theory that the axiom of choice holds would lead to a convincing argument that ZF is inconsistent.

It would be a misconception to immediately conclude from a mathematical statement on a (seemingly) corresponding statement of the human future. One can however argue by “going down”:

Assume that someone gives a convincing argument for the axiom of choice on the basis of Zermelo-Fraenkel set theory. Then assuming that Thesis 3 holds for ZF, it would be proven on the basis of ZF that the axiom of choice holds. By Criterion 2 (for ZF) this would mean that one can argue on the basis of constructive mathematics that there exists a formal proof for the axiom of choice from ZF. The conclusion is then that ZF is inconsistent.

## 8.6 Kinds of mathematics interpreted in set theory

We now consider the relationships between statements in kinds of mathematics and corresponding set-theoretic statements. For this, throughout this subsection, we fix some kind of mathematics, based on a formal system  $S$ , and a statement  $A$  inside this kind of mathematics.

### 8.6.1 Convincing arguments and formalistic stories in a kind of mathematics

We have the following statements:

- (Arg) One may give a convincing argument for  $A$ .
- (FS) One may give a formalistic story showing  $A$ .

(Arg) is a statement on human language concerning the given kind of mathematics and (FS) is a statement on human language concerning constructive mathematics (which itself is carried out over the given kind of

mathematics). These statements shall be taken literally and not be interpreted in an ideal way. So we expect in (Arg) and (FS) that such stories are actually given.

Since these statements refer to different kinds of mathematics (if the considered kind of mathematics is not constructive mathematics itself), we do not have a common formal basis to establish formal relationships between them. Nevertheless, by theses 1 and 2 in subsections 8.1 and 8.2, (Arg) and (FS) may be considered as informally equivalent.

### 8.6.2 Models and their underlying structures

We now consider what is called *semantic of formal systems* or *model theory* and with this relationships between statements on models of the given kind of mathematics and similar objects.

We first note that by doing model theory we tell a set-theoretic story. We stress in particular again that we do not want to consider set-theoretic arguments when doing meta-mathematics, and in fact consider it inappropriate to use set theory in meta-mathematics. Nonetheless, we can consider from an outside point of view the doing of set theory and therefore in particular of model theory.

For this, we suppose that the formal system  $S$  on which the kind of mathematics under consideration is based is a system of first-order (classical) logic.<sup>30</sup>

The notation follows what we have introduced in the introduction to this section, so  $\mathcal{L}$  is the set interpreted as the language and  $\Gamma$  is the set of axioms.

Because we want to introduce a particular terminology, we recall the definition of a model: Given are sets of “function symbols” and “relation symbols”.<sup>31</sup> It consists of: First an “underlying set”  $X$  together with functions from a finite power of  $X$  to  $X$  and relations on  $X$  and second an assignment from the set of function symbols to functions and from the set of relation symbols to relations. Here obvious conditions, in particular with respect to the axioms, have to be satisfied. The terminology we introduce is: We call the first part of the description the *underlying structure of a model*.

To give a concrete example: The underlying structure of a model of group theory is nothing but a group.<sup>32</sup>

---

<sup>30</sup>We are confident that the following considerations can be adapted to other kind of formal systems, but for accuracy and because such a generalization would be a topic for itself, we stick to first-order systems.

<sup>31</sup>These so-called symbols are also objects of set theory, that is, sets.

<sup>32</sup>There is a difference between the usual definition of a group and the definition of the structure of a model for  $\Gamma$ : For groups, one writes down the axioms directly, that is, one does not consider a set of axioms. But this does not change the result of the definition. In general, as the axioms are – by assumption – constructing given by the formal system

As models are also sets, one can quantify over all models of a given theory. We now have the following statement inside Zermelo-Fraenkel set theory:

(MT-Truth)  $(A^{\text{formal}})^{\text{set}}$  is logically valid in any model of  $\Gamma$ .

Let us clarify this statement:  $(A^{\text{formal}})^{\text{set}}$  is the set in  $\mathcal{L}$  associated to  $A$ . For the definition of being logically valid in any model, it is necessary to define which statements are “true” and which are “false”. This is defined by a set-theoretic recursive definition over the length of the formulas. That is, one defines in a recursive manner a function on the set of statements with values in a set of cardinality two, whose values are interpreted as “true” and “false”.

Let us for any given  $\mathbf{S}$ -sentence  $S$  consider a ZFC-formal property of models of  $\Gamma$  which corresponds to “ $S$  is logically valid”. Let us denote this property by  $S^{\text{MT}}$ . (The models correspond to a free parameter in the statement.)  $(A^{\text{formal}})^{\text{MT}}$  is then a formal property of models of  $\Gamma$ , and (MT-Truth) can be reformulated as:

(MT-Truth)  $(A^{\text{formal}})^{\text{MT}}$  holds for any model of  $\Gamma$ .

Let us now consider statements on underlying structures: As a starting point we take the formal reformulation  $A^{\text{formal}}$  of  $A$ , from which, as we already discussed, we obtain the property  $(A^{\text{formal}})^{\text{MT}}$  of models of  $\Gamma$ . The close relation between models and their underlying structure gives rise to a ZFC-formal property  $(A^{\text{formal}})^{\text{ST}}$  of the underlying structures. Furthermore, given  $A$ , in mathematical praxis one may observe that one has a corresponding informal property  $A^{\text{ST}}$  of underlying structures of models which may literally coincide with  $A$ . What is meant by this is that  $A^{\text{ST}}$  shall be a informal formulation of  $(A^{\text{formal}})^{\text{ST}}$ .

We illustrate these considerations again by the example of groups. We start with a statement  $A$  on groups which are considered as abstract objects independent of set theory in their proper kind of mathematics. This informal statement  $A$  gives rise to a formal statement  $A^{\text{formal}}$  on groups in the corresponding formal system  $\mathbf{S}$ . Since  $\mathbf{S}$  may be considered in set theory, we obtain a set  $(A^{\text{formal}})^{\text{set}}$  as described above.  $(A^{\text{formal}})^{\text{MT}}$  is a statement on models of groups. Since models of groups and groups, now considered (as usually) in set theory, are closely related, we get a set-theoretical formal statement  $(A^{\text{formal}})^{\text{ST}}$  on (set-theoretic) groups. Furthermore statement  $A$  as a statement on abstract groups may also be formulated as a statement  $A^{\text{ST}}$  on set-theoretic groups. As mentioned above,  $(A^{\text{formal}})^{\text{ST}}$  is now a formal reformulation of  $A^{\text{ST}}$ .

---

$\mathbf{S}$ , one can get rid of the set of axioms if the axiom scheme in  $\mathbf{S}$  is finite.



It is then natural to consider:

(Full-ST)  $A^{ST}$  holds in the underlying structure of any model of  $\Gamma$ .

Due to the close relationship between models and their underlying structures, it is now clear that (MT-Truth) and (Full-ST) are equivalent.

### 8.6.3 Set-theoretic realizations

Our motivation to consider underlying structures of models lies in the observation that realizations of kinds of mathematics in set theory usually differ from simply using the models - underlying structures are very simple and frequently observed examples of realizations. In mathematical practice a huge variety of realizations may be imagined and it lies beyond the scope of this note to consider all such kinds of realizations. However, we will define a notion of realization which captures the essence of what is usually considered as a realization.

Before giving the definition, we consider an example: Let a kind of synthetic geometry be given, for example basis affine geometry. This can be described by a formal system with one relation symbol of arity 1 (“point or line”) and one one of arity 2 (“contained in”) and no function symbol. The underlying set of a model then consists of points and lines, and there is an incidence relation between these.

Now, one does not need to consider such (set-theoretic) geometries in full generality but one can assume that a line is a set of points, and indeed one often does. A statement holds for all these structures if and only if it holds for all models. (The reason for this is that every model is isomorphic to such a model with a appropriate notion of isomorphism, but this is not relevant for the following.) Motivated by this we define:

**Definition.** By a *set-theoretic realization* of a kind of mathematics (based on a formal system) we mean a description, call it  $D$ , of further specified models of the kind of mathematics such that for every statement  $A$  of the kind of mathematics,  $A^{ST}$  holds for the underlying structure of any models of  $\Gamma$  if and only if  $A^{ST}$  holds for the underlying structure of any models described by  $D$ .

We now consider such a realization. We then consider what we call (ST):

(ST)  $A^{ST}$  holds in the underlying structure of any model in the realization.

We might abbreviate this statement as:

(ST)  $A$  holds for the given realization of  $S$ .

By definition, (ST) is equivalent to (Full-ST), which in turn should be equivalent to (MT-Truth).

#### 8.6.4 Arguments in kinds of mathematics and set-theoretic arguments

We can now consider the corresponding statements on convincing arguments and formalistic stories. For (ST) these are:

(Arg-ST) One may give a convincing argument for (ST).

(FS-ST) One may give a formalistic story for (ST).

Here in (FS-ST), one uses the formal reformulation  $(A^{\text{formal}})^{\text{ST}}$  of  $A^{\text{ST}}$ .

The statements (Arg-MT-Truth) and (FS-MT-Truth) and (Arg-full-ST) and (FS-full-ST) are defined in an analogue manner. By the considerations of 8.1 and 8.2, the statements (Arg-ST) and (FS-ST), (Arg-MT-Truth) and (FS-MT-Truth) as well as (Arg-full-ST) and (FS-full-ST) may be considered to be informally equivalent.

As said in the previous subsection, we stipulate that (ST) and (Full-ST) are equivalent such that (Arg-MT-Truth) is equivalent to (Arg-full-ST). We have also seen that (MT-Truth) is equivalent to (Full-ST). The argument was based on the fact that models and their underlying structures correspond to each other. This gives an equivalence between (Arg-MT-Truth) and (Arg-full-ST). But there should also be an equivalence between (FS-MT-Truth) and (FS-full-ST) since it should be possible to implement this correspondence in formalistic stories.

One has two more implications which one usually would accept in mathematical practice: The statement (Arg) implies (Arg-MT-Truth) and (Arg-ST). We illustrate this again by the example of groups. Suppose we have some argument that a statement on abstract groups considered independently of set theory holds. Then usually one would say that this statement also holds for all models of groups and a realization of groups in set theory as defined above. These considerations on arguments carry over to formalistic stories. Given a formalistic story on abstract groups one also expects that one may modify it such that one obtains a formalistic story for models of groups and groups in set theory. This supports that (FS) implies (FS-MT-Truth) and (FS-ST).

We do not consider implications from (Arg) to (Arg-full-ST) or from (FS) to (FS-full-ST) here since (Full-ST) may be seen as a special case of (ST).

### 8.6.5 Model theory

We now consider the well known theorem on soundness and completeness of first-order logic, which are at the center of model theory:

Let  $\mathcal{L}$  be a first-order language. For all sets  $\Gamma$  of sentences (of  $\mathcal{L}$ ) and all sentences  $\varphi$  (of  $\mathcal{L}$ ) the following are equivalent:

- (1) There exists a model-theoretic formal proof (within some proper deductive system) which shows that  $\Gamma$  implies  $\varphi$ .
- (2) The formula  $\varphi$  is logically valid in any model of  $\Gamma$ .

The implication (1)  $\rightarrow$  (2) is called *soundness* and the reverse implication (2)  $\rightarrow$  (1) is called *completeness*. By these equivalences, (MT-Truth) is equivalent to:

(MT-Proof) There exists a model-theoretic formal proof which shows that  $\Gamma$  implies  $(A^{\text{formal}})^{\text{set}}$ .

Here we claim the abstract existence of a formal proof, which – as already stated – is regarded as a set. This is crucial, so in particular one cannot reformulate the statement in a similar way as one can reformulate (MT-Truth) by (ST).

We now consider again the corresponding statements on convincing arguments and formal proofs:

(Arg-MT-Proof) One may give a convincing argument for (MT-Proof).

(FS-MT-Proof) One may give a formalistic story for (MT-Proof).

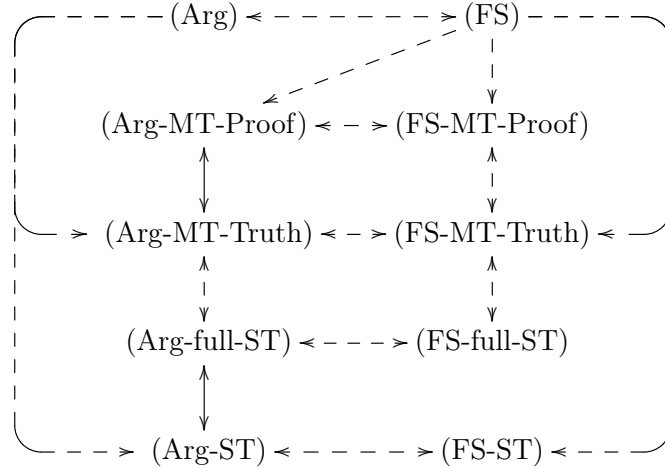
Now, an application of the theorem of soundness and completeness of first-order logic gives an equivalence between (Arg-MT-Truth) and (Arg-MT-Proof). Since this application is a fixed step, not depending on any translations, this equivalence may be regarded as a stronger relationship than most of the former correspondences.

By using subsection 8.6.1, we obtain that (Arg-MT-Proof) and (FS-MT-Proof) should be considered informally equivalent.

Moreover, if one has access to a formalistic story for the (set-theoretic results of) soundness and completeness, one can show that (FS-MT-Truth) and (FS-MT-Proof) are (directly and not just via the theses) equivalent. We argued in subsection 8.3 that clearly (FS) implies (Arg-MT-Proof). In the same vein, (FS) implies (FS-MT-Proof).

### 8.6.6 Summary on kinds of mathematics interpreted in set theory

We sum up the direct implications mentioned so far in the following diagram:



The drawing of a solid arrow line between (Arg-MT-Proof) and (Arg-MT-Truth) reflects that this relies solely on the fact that a particular set-theoretic result has been established. A solid arrow is used between (Arg-full-ST) and (Arg-ST) since this equivalence is stipulated.

For all the other implications we have strong arguments, but we can differentiate here: The strongest dotted lines are arguably from (Arg) to (Arg-MT-Truth) or (Arg-ST). In fact, essentially nothing seems to be necessary here in order to obtain this transformation.

The dotted lines between the (FS...)'s and from (FS) to (Arg-MT-Proof) also seem to be unproblematic, but some work to really write down the algorithms would have to be done. Here the arrows between (FS-MT-Proof) and (FS-MT-Truth), which are based on only one theorem, and the arrows from (FS) to (Arg-MT-Proof) and (FS-MT-Proof) will represent the most easy implications.

The middle dotted lines are the weakest inferences; the statements seem to be equivalent as expressed by Thesis 1 and Thesis 2.

### 8.6.7 Models of ZFC and existence of proofs

We close this section by considering the implications of the previous considerations if applied to ZFC as a formal system.

So now models of ZFC are considered. In this application we have as a special case this informal inference: An argument for  $A$  in Zermelo-Fraenkel set-theory gives an arguments for  $(A^{\text{formal}})^{\text{MT}}$  being logically valid in all models of ZFC and for  $A^{\text{ST}}$  being valid in all underlying structures of models.

Further, to have such an argument is equivalent to having an argument for: There exists a model-theoretical formal proof of  $(A^{\text{formal}})^{\text{set}}$ .

Here we are back at the warning in subsection 8.3. We stress again the importance to not apply the “convenient” implication from the latter three statements to the former.

## 9 On the Central Question

It is now time to come back to the Central Question posed in subsection 4.1. We have argued that the process of rewriting should be based on constructive mathematics, and we have given an interpretation of the statements of constructive mathematics for which one does not need to imagine infinitely many natural numbers or (in our application) texts.

However, the observation of subsection 3.4 that the rewriting process often cannot be done by humans is of course still valid.

The answer then depends on whether one regards the statements of constructive mathematics in our interpretation as being “less imaginary” than set-theoretic statements. Given that both are on the one hand “intuitive” and on the other do not seem to directly correspond to human experiences, the authors do not see this, but the reader is of course invited to have his or her point of view.

In any case, it seems to the authors that with the approach presented, the stories told (on the rewriting) are the closest to actual physical rewriting that one might hope for if one wants to uphold claims of rewriting that cannot be carried out physically at all.

Nonetheless, even if one considers constructive mathematics to be as imaginary as set-theoretic mathematics, in near future the claim of rewriting will still be the answer when mathematicians are being asked about the essence of what they call proofs. Therefore, a further reflection on this philosophical issue may lead to new insights and especially hidden thesis, which get revealed throughout such discussions, possibly leading to further developments.

## References

- [fictional] M. Balaguer, Fictionalism in the Philosophy of Mathematics, Stanford Encyclopedia of Philosophy, 2011
- [foundations] M.J. Beeson, Foundations of Constructive Mathematics, Metamathematical Studies, Springer-Verlag, 1985

- [constr-ana] E. Bishop and D. Bridges, *Constructive Analysis*, Springer-Verlag, 1985
- [constructivism] D. Bridges and E. Palmgren, *Constructive Mathematics*, Stanford Encyclopedia of Philosophy, 1997
- [formal-natural] M. Carl, *Formal and Natural Proof – A phenomenological approach*, 2015, Preprint, <http://www.math.uni-konstanz.de/~carl/Paper/FormalNaturalProofFinal2015.pdf>
- [Zahlen] R. Dedekind, *Was sind und was sollen Zahlen?*, second edition, Vieweg, 1893, <https://archive.org/details/wassindundwasso00dedegoog>
- [constr-theories] S. Feferman, *Constructive theories of sets and classes*, in: M. Boffa, D. van Dalen and K. McAloon (eds.) *Logic Colloquium '78*, pp. 159 – 224, North-Holland, 1979
- [dialectica] K. Gödel, *Über eine bisher noch nicht benutzte Erweiterung des finiten Standpunkts*, *dialectica*, **12**, pp. 280 – 287, 1958
- [formalized] J. Harrison, *Formalized Mathematics*, Technical Report 36, Turku Centre for Computer Science, Technical Report 36, 1996
- [informal] B. Larvor, *How to think about informal proofs*, *Synthese*, **187**, 715 – 730, 2011
- [IUTT] S. Mochizuki, *Inter-universal Teichmüller Theory I – IV*, Preprints, 2016
- [coping] D. Knuth, *Aathematics and Computer Science: Coping with Finiteness*, *Science* **194**, 1235 – 1242, 1976
- [example] A. Naumowicz, *An example of formalizing recent mathematical results in Mizar*, *Journal of Applied Logic* **4**, pp. 396 – 413, 2006
- [constr-logic&-math] T. Streicher, *Introduction to Constructive Logic and Mathematics*, manuscript, 2001, <http://www.mathematik.tu-darmstadt.de/~streicher/CLM/clm.pdf>
- [constructivism-in-math] A. Troelstra and D. van Dalen, *Constructivism in Mathematics, An Introduction*, Volume 1, Springer-Verlag, 1988

- [on-Kronecker] H. Weber, Leopold Kronecker. In: Jahresbericht der Deutschen Mathematiker-Vereinigung **2**, 19 – 25, 1893
- [QED] The QED Manifesto, Automated Deduction - CADE 12, Springer-Verlag, Lecture Notes in Artificial Intelligence, Vol. 814, pp. 238 – 251, 1994

## Supplement: Inspirations and the process of writing

What inspired the authors to write this note and to articulate the perspective on mathematics presented here?

In 2012, Shinichi Mochizuki made four papers about “Inter-universal Teichmüller Theory” ([IUTT]) public which contain allegedly a - yet unconfirmed - proof of the abc conjecture, a central conjecture from number theory. In the last part of the fourth paper, Mochizuki introduces what he calls the “language of species”. The first author assigned to the second author the task for his “Diplomarbeit” to explain “what is going on here”, and the second author agreed to this task.

It turned out that “species” are nothing but formulae describing categories, and there are also “mutations” describing functors. For example, a species defines the underlying class of a category via a property which the objects shall satisfy; this seems to be similar to the possible introduction of classes for individual properties in a higher language which can be rewritten in ZFC as discussed in subsection 4.2. This led to numerous – sometimes hours long discussions. The rather concrete question discussed was: Should the “species” be seen as objects themselves or does Mochizuki merely advocate a particular point of view, maybe a way to do and to write down mathematics? What is or what should be the ontological status of “species”? The authors then realized that to even discuss these questions, they have to consider deeper underlying questions on formalisms in mathematics in general and on the ontological status of mathematics in general.

The second author expressed a formalistic point of view: If mathematicians claim to have proven a result, they actually claim that they can write down such a proof in a formal system, which is by default ZFC, everything else would be imaginary. The first author countered: This does not correspond to what you are doing yourself if you are doing mathematics. In fact, you do not formally write down your “proofs in a formal system”, and you are also not able to do so. So your “formalistic proofs” are also purely imaginary. Moreover, your view on mathematics does not correspond to your behavior: I can see you getting excited when you talk about “mathematical objects”. One can indeed argue that the objects are purely imaginary (as one can also for the “formalistic proofs”), but when expressing their thoughts, mathematicians do indeed act as if the objects were real; this should be expressed.

The first author stressed then the point of view that questions of realism should be sidestepped by simply looking at what mathematicians do, and for this the ideas of mathematical fictionalism should be considered. Mathematical fictionalism (see e.g. [fictional]) expresses the thought that even though mathematical objects are not or might not be real, mathematicians tell themselves a story in which they are real. According to fictionalism this



is similar to the usual process of story telling with novels. The second author emphasize that be that as it may, mathematicians in their quest for certainty want to be sure that their arguments follow formally from clearly outlined axioms.

This stress on the formalistic foundations of mathematics became of importance when the authors wanted to write about the proper interpretation of statements on formal sentences. Here, however, a crucial problem came to light: What does a claim “The argument could be written in ZFC.” actually mean? What if such a writing cannot be done by humans? What if the writing can (or realistically could) be done by computers but then the formal proof could not be checked by humans?

A statement like “There is a formal proof of Fermat’s last theorem in ZFC” does indeed “feel” more real than the very starting point of arithmetic: Counting never stops, or formulated more “platonistically”: Every natural number has a successor, and certainly more real than set theory. The challenge than was to find an appropriate framework which one the one hand is still mathematical (rather than physical) and on the other hand captures the idea of construction. With the idea of construction in mind, the authors then started to study constructive mathematics (which they did not know well before) and found that (with an appropriate interpretation) it provided exactly the kind of framework they searched for.

It still remains the question in what sense, if any, statements on the existence of formal proofs in this framework should be seen as being “less imaginary” than the usual set-theoretic statements. Should such a qualification not at least require some kind of argument in favor of the possibility for humans to obtain such a proof in a physical sense?

In any case, statements of rewriting and on formal proofs in the form of “it could be done” are important in the story and the arguments of mathematics as told today, and it is a challenge to come up with a meaning of such statements. The authors are convinced that for an appropriate answer one should turn to constructive mathematics, whether or not one then regards the claims then as being “less imaginary” than set-theoretic claims.

They invite everybody to reflect on the challenge for themselves.