Reflections on the role of statements on statements in mathematics

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1 Statements, objects and statements on objects and statements

Mathematicians reason about abstract objects, or at least they seem to do so. In the course of doing so, they routinely make statements on mathematical statements. In fact, every inference, such as "From equations (5.17) and (6.2) one obtains the following equation.", is a statement on mathematical statements. Other examples of statements on mathematical statements are: "With the obvious modifications of the proof of Theorem A, one gets a proof of Theorem B.", "This proof is really nice.", "A statement in the language of rings is true in every algebraically closed field of characteristic zero if and only if there are infinitely many prime numbers p such that the statement is true in some algebraically closed field of characteristic p.", "All of our proofs can be formulated in ZFC; for convenience of the reader, as usual we present them informally."

By the above examples, one notices immediately that there are fundamentally different kinds of such statements. An important special case of statements on statements are statements on the existence of proofs and in particular formal proofs, such as the last statements above.

The last statement is the kind of statement we are most concerned with in this note. What kind of statement is actually made here? Shall the formal proof "in ZFC" be a mathematical object in the same vain as, say, finite geometries or groups? If so, is the claim then a claim in mathematics? Shall the claim not rather be a claim on mathematics? But what shall then the meaning of the claim be?

Clearly, we would want that a claim that there is a proof, in this case a formal proof, be of a different, "more real" nature than a mere existence claim in the usual set-theoretic mathematics. But how is it possible to give such a "more real" interpretation if the alleged formal proof is not written down, in all likelihood will not be written down and maybe also cannot be

written down or even read by humans?

A starting point is here the observation that it seems to be unreasonable to say that humans can establish infinitely many results, in mathematics or otherwise. So, we regard any statement that infinitely many results have been proven to be a priori invalid. To demonstrate this with an easy example: We have the evidently true mathematical statement "Every prime $p > 2$ is odd.". We hold here that it is improper to say that for every prime $p > 2$ the statement "p is odd" holds, as this would mean that infinitely many statements are made.

More generally, when addressing the task to give a "more real" or "most real" interpretation of claims of alleged formal proofs of mathematical theorems, care has has to be taken not to use any unsubstantial a priori claims on the existence of certain "abstract" objects.

With these remarks in mind we now give a brief outline of this note:

Outline. We want to be as accurate as possible with respect to statements of existence, avoiding any unsubstantial claim that certain "abstract" objects exist. In order to nonetheless attach meaning to mathematical statements, we start off with an observation of what mathematicians actually do when doing and speaking about mathematics.

Subsequently, we highlight two important ideas of contemporary mathematics: set (and class) theory and formal methods. This leads to a first reflection on mathematical statements as they are usually made and statements on the existence of formal proofs

In the fourth section, we begin to study claims that statements have been proven on the basis of a formal system such as ZFC. For this, we do however not study texts as usually written by mathematicians but only two rather restricted classes of such texts: The first class consists of texts which are free from reflections and meta-statements, and the second class of texts consists of texts consisting of such texts with introductory statements that the following text can be rewritten in a formal system, for example ZFC. Noticing that such a claim of rewriting is often literally false, we then ask, in a "Central Question", if there is a possibility to read the latter kind of texts in a "more real" way than the former kind of texts.

To give an answer to this Central Question, we are lead to the consideration of constructive methods, and so in the fifth section we outline how we envision to use them. To avoid any unnecessary claims of existence, we give a strict interpretation of constructive statements as used by us with which we can avoid to talk about infinitely many objects.

In the sixth section, we give possible meanings of statements on the existence of formal proofs as studied in the forth section via the methods

from constructive mathematics via three criteria (which can be found in subsections 6.1 and 6.3).

We then briefly study other kind of statements on statements in mathematical texts. Here, similarly to the consideration of the establishment of formal proofs, we argue that if a mathematical result is applied as a meta-result in a mathematical argument, this result must be constructive. Again we formulate this as a criterion.

In the final section, we analyze the possible application of mathematical results to meta-mathematics and in particular the application of mathematical results to establish mathematical results (which usually goes under the name model theory).

An advice. The text can (and as we would urge: should) be read "as is", but nonetheless the reader might ask what inspired the authors to consider the questions studied in the text and to write the text in the way it is written. Also, the reader might ask how thoughts expressed, for example, the ones of the next section, relate to previous texts and "schools of thought". Some information concerning these questions can be found in a supplement which is independent of the text and can be found after at the very end.

2 What is mathematics?

According to adherents of Platonic realism, mathematics is (or should be if properly conducted) the study of certain "ideal" objects. Of course this is challenged. In what sense is, for example the statement

> There exists a unique positive real number whose square is $2 \uparrow\uparrow 6 - 1$,

in which Donald Knuth's arrow notation (see [coping]) is used, a statement on any kind of "object" after all?

Indeed, one might argue that as there is no such object, the statement is wrong. One might also raise the even deeper objection that notions in the statement (*real number*, *square* and maybe also $2 \uparrow\uparrow 6 - 1$) are void of meaning, and it therefore is not even reasonable to discuss whether the sentence is right or wrong. More generally, one might argue that all or at least a large part of mathematical existence statements which are considered to be true in mathematics are in fact wrong, because there simply are no objects of the asserted kind, or even nonsensical.

Now, independent of metaphysical believes, mathematicians do indeed act as if the asserted imagined objects such as the square root of $2 \uparrow\uparrow 6 - 1$ were real objects. They talk about such objects, they tell themselves stories

about the objects and they even get emotionally involved when they have found a new, as mathematicians often say, "beautiful" relationship in their imagined universe. Interestingly, throughout the history of mathematics, aesthetic judgments played a crucial role, maybe more than in any other science.

Let us take the observations of the previous paragraph as a starting point: We shall call mathematics the activity of mathematicians in their role as mathematicians, which is meant as a self-identification. What is then mathematics?

There is then a purely *empirical* answer, following the above descriptions: Mathematicians act as if imagined "abstract" objects in an imagined "abstract" universe were real, where the objects and their relationship are described by "evident" simple rules called "axioms". They try to find out what the characteristics of the imagined universe are, what the relationship between objects in the imagined universe are, and they try to convince others of their findings.

We note here that we use the word "imagined" for an empirical description of the activity of mathematicians. Throughout this note we stay neutral towards the fundamental question in how far the content of mathematical statements (in particular of statements in infinitely many objects) should be regarded as being "real" in the sense of being statements on "real abstract objects". It is, however, important to us that statements on infinitely many objects and also statements on large numbers, such as the one mentioned in the beginning of this section, cannot be regarded as referring to anything related human experiences.

The activity of mathematicians is similar to the telling of novels, and indeed one can say that mathematicians tell themselves as an inner thought or each other *stories about imagined abstract objects*. There is, however, a further aspect which is different from pure story telling: Mathematicians do not tell each other arbitrary stories but rather behave like explorers of the imagined universe. They then try to convince themselves and each other that what they perceive to be "true characteristics" of the imagined universe are indeed such.

An interesting aspect of mathematics (the activity of mathematicians) is that certain arguments (which by the nature of mathematics are always relative to "axioms") are considered to be so rigorous that they are called "proofs". This is even more remarkable as there is no clear standard as to when an argument given by a mathematician shall be called a "proof". Indeed, throughout mathematical history, mathematicians have differed greatly in their opinions as to what they consider a "proof", and if one now rereads older texts (which might be from the 19^{th} century), sometimes one obtains the impression that what is called a "proof" does not have the clarity and strictness one wishes

to see nowadays. Nonetheless, mathematicians have usually been convinced that it is in praxis obvious what constitutes a "proof" and what not, and in case of a dispute mathematicians have usually been able to find – via discussions and further elaborations – a consensus as to what shall considered to be proven (again relative to particular axioms) and what not. These "proofs" together with discussion on them as well as the expressed conviction that they express a certainty with regards to truth form a major part of the stories told by mathematicians.

The word "proof" is a strong one, but as stated, more often then not in the actual praxis of doing mathematics, one can argue if arguments suffice for a "proof". Personal judgments play a strong role here. To emphasize this subjective aspect, we usually speak about a *convincing argument* instead of a *proof* when we refer to what is usually considered a proof. Occasionally, we also literally write *"proof"*. This also allows us to distinguish between "proofs" (in our terminology: convincing arguments) as actually expressed by mathematicians and (formal) proofs in formal systems, a distinction which will be of importance later.[1]

3 Two crucial developments

Two rather recent developments in mathematics are of particular importance for us: *sets as fundamental objects of mathematics* and the *formalistic approach* to the foundations of mathematics.

3.1 Set theory

Set theory is so embedded to our doing of mathematics that it is hard to forget the specific, at one time revolutionary, point of view of this way of thinking: Not only does one regard collections of (imagined) "things" again as one thing, the "things" so obtained are then regarded as things in just the same manner, which makes it possible to repeat this process. Moreover, not just collections of finitely many "things" are considered, but also such of infinitely many "things".[2]

An early idea of the notion of set (Menge) was that every property should define a set. As this turned out to lead to logical contradictions, a separate notion of *class* was introduced, for which this is indeed the case (with an

[1] It is common to call what we call *convincing argument* an *informal proof* (see for example [informal]). Other researchers, who also emphasize the difference between what is actually expressed and formal proofs stick with the word "proof" (see for example [formal-natural].

[2] Richard Dedekind wrote in [Zahlen]: "Im folgenden verstehe ich unter einem *Ding* jeden Gegenstand unseres Denkens.", that is: "In the following, I mean by *thing* every object of our thinking."

appropriate definition of "property") but which cannot be treated as "thing" in the way a set can. Whether *proper classes*, i.e. classes which are not sets, should be seen as objects of the (as we said, imagined) mathematical universe or statements on proper classes as façon de parler on sets is under dispute. In order to be neutral to this question, we will speak about "set and class theory" rather than only about "set theory" or only about "class theory" (which of course would include sets).

Amazingly, via the idea of sets, different stories about abstract or ideal objects, numbers, (ideal) geometries, forms, functions and much more can be and were merged into one grand story, which can be communicated clearly. This merger was so successful that nowadays the view is dominant that every object considered in the story of mathematics is (or: can be seen as) a set or a class. Nevertheless, there are more kinds of mathematics (e.g. constructive mathematics), which are usually not considered inside some set and class theory. For the moment, for practical reasons, we will stick to the mentioned dominant view that all mathematics being studied is some set or class theory, whereas constructive mathematics, which we will use later, will be seen as an auxiliary tool. We will discuss and clarify the role of different kinds of mathematics in subsection 7.1.

3.2 The formalistic approach

By "formalistic approach" we mean the following body of ideas: A formal language to express mathematics is rigorously described via (easy to follow) rules according to which certain expressions are called sentences;[3] certain sentences or bodies of sentences are considered to be interpretable (again via easy to follow rules);[4] certain sentences or bodies of sentences (called "axioms") are a priori called "true";[5] there are (easy to follow) rules to derive further "true" or "false" sentences from previously established "true" ones. Calling the language and the rules a *formal system*, a "proof in the formal system" is then a (physically given) (finite) sequence of sentences in the

[3]One might also speak of "valid sentences" we use the term "sentence" in a sense that "valid" would be redundant.

[4]By "interpretability" we mean a formal criterion on texts, i.e. finite sequences of sentences. In propositional logic all sentences are interpretable, but formal languages more in line with conventional languages are conceivable in which this is not the case. An example might be a formal language with a sentence like "Let x be an element of X.".

[5]We deliberately write "called 'true' ", because we want to emphasize that here "true" and "false" are merely expressions assigned to certain statements, which might be replaced by any other expressions. The fact that the emotionally strong words "true" and "false" (which are also used in statements on formal systems in their usual meaning) are used here is of course not without problems. A particular reader of a particular system might be of the opinion that the system with a particular interpretation does really establish true and false statements; but this does not need to be so.

formal language each of the sentences is interpretable and "true" given the previous ones (on the basis of the axioms and the rules).

Rigorously following the formalistic approach would mean that all mathematics would be rewritten in formal systems, and then the outline of the formal systems used and these formal texts written would encompass all of established mathematics. Motivating and interpreting statements could be given, but there would be no need for them, and they would not be part of the body of established mathematics; in particular in this body of established mathematics, there would be no room for "universes of discourse" in which the sentences in the proofs are to be interpreted.

Following this line of thought, ideally then mathematicians would even agree on a single formal system, and there would be an agreement that mathematics is the production of formal proofs in this unique formal system. The rules for the formal system could then be seen as defining criteria for the notion of mathematics, but would themselves lie in meta-mathematics rather than in mathematics. Needless to say that this vision is very different from mathematics as the activity of mathematicians at this point of time.

This rigorous approach should be distinguished from the usual process of doing mathematics, which always involves the effort to find appropriate precise terms and convincing arguments written in a language which keeps possible misunderstandings at a minimum.

We note here that the development and the use of a formalistic approach as described need not go along with a particular attitude with respect to the "nature" of mathematical statements. In particular, it is independent of the acceptance or rejection of Platonic realism.[6]

3.3 The formalistic set theoretic foundation

The formalistic approach can then in particular be applied to set or class theory, which leads to various (related) formalistic approaches to set or class theory as the foundation of mathematics. With these approaches, it seems that the foundations of mathematics as it is usually conducted have reached a long-lasting nearly stable stage.

There is however another kind of mathematics, constructive mathematics, for which – among others – not the idea of set but the idea of number and algorithm is basic. We will argue in the next section that this kind of mathematics gives (with an appropriate interpretation) the right framework to argue whether results are established via formal proofs.

[6] See also the last sentence of footnote 5.

3.4 The praxis of mathematical story-telling and argumentation

Even though the idea of a formalistic approach is now considered to be crucial to mathematics, "mathematical proofs" are very seldomly written in a formal system. In fact, the system which is the most accepted one, namely ZFC, itself based on first-order logic, is designed in such a way that it is de facto impossible to tell the stories and arguments mathematicians tell each other in this system – small examples withstanding. One reason for this is that it does not allow for definitions and that all sentences are interpretable without context.

So, rather than actually telling their stories and giving their arguments in a system such as ZFC, mathematicians usually just use usual language augmented with mathematical symbolisms and occasional formal statements.[7] A particular kind of set or class theory is there often not specified. In a more rigorous approach, such as in the books by Bourbaki, first the axioms are outlined in plain language (augmented as described) and then the arguments are also given in plain language with occasional formal statements.

Only seldom, a strict formalistic claim is made, a claim of the following form: A formal system (for example ZFC) is defined in plain (usual) language, the claim: "The following can be rewritten in the formal system in such a way that one obtains a formal proof" is made, and then the usual mathematical arguments are given.

Nonetheless, when asked what actually the phrase "This is proved on the basis of set theory" shall mean, mathematicians (maybe after a discussion that one wants to have a "strict" answer) often reply: "Well it means that there is a formal proof of the statement in ZFC." What then is stated here? Is this now again nothing but an existence claim on imaginary mathematical objects or does it mean that such a proof can actually be written? Or is there another meaning one can attach to this phrase? Does it maybe help to say that "it could be written"?

One notices immediately that there is a substantial difference between actually writing in the formal system ZFC or just claiming that it "could be done". If one reflects what it really would mean to write a proof of a statement in ZFC, one soon realizes that by all human capabilities, it often *cannot be done* by humans. What is more, such a writing (which, as mentioned is anyway not carried out and de facto for many texts even impossible to carry out by any given human) would end in a result which any mathematician would consider a worthless accumulation of symbols – worthless because he or she had the feeling of "not understanding anything". Indeed, the men-

[7] Concerning the language used by mathematicians, the reader might find the introduction to [formalized] interesting.

tal process and the actual work done by mathematicians would if anything go *in the opposite direction*, that is, from the "purely formal" symbols to a high-level text which highlights some underlying structure in the argument.

With these reflections, we can now give a more detailed description of the personal and inter-personal process of mathematical story-telling and arguing in the current state of mathematics: Set or class theory is an agreed upon foundation, even though the references to set or class theory are not always made explicit. An important interest of mathematicians is "results", which they regard as being "true" on the basis of some (set or class theoretic) axiomatic systems.[8] These are usually informally given but which can also be given formally; by default the chosen axiomatic system is Zermelo-Fraenkel set theory with the axiom of choice. In order to be convinced of this, they often accept broad arguments, whose validity they have only superficially or not at all checked – often aesthetic judgments and experience play a strong role here. However, if they are interested in some further arguments on a detail of a longer argument, they want that this is to be explained to them. This process might then lead to a more and more formal argument. Here derivations in formal systems (like ZFC) might play some role. However, mathematicians are usually unsatisfied with purely formal derivations (except maybe for short computations); rather they want to gain an "understanding" or a mental image of the situation which exceeds purely formal considerations.

4 The claim of possible rewriting

We now analyze further claims of possible rewriting of informal arguments in formal systems such as ZFC and of claims that statements "are proven in ZFC" or a related system.

4.1 The substantive and the formalistic story

Contrary to what one might think at first, there is a great diversity in the kind of statements made in mathematical texts. It is common that mathematical statements are mixed with reflections on the mathematical statements or the process of making such statements; examples of this are given in the beginning of this note.

In this section, we restrict ourselves to texts which are more restricted.

As a basis we consider texts consisting of set- and class-theoretic definitions or definitions which can be interpreted set- or class-theoretically, statements based on these definitions and supporting arguments. So far,

[8] Mathematicians (as mathematicians) also have other interests, for example finding what they consider to be the "right" definitions.

this corresponds to the usual mathematical praxis. However, in contrast to the way mathematical texts are actually written, we do not allow for reflections on what is written. For example, we do not allow for reflections on arguments "from a higher point of view", as one often finds in introductions, or on judgments that something is "easy" or "easily seen" or on arguments based on analogy (like "the proof is similar to the previous one"). Summarizing, we can say that in such a text a set- or class theoretic story is told in a linear way; correspondingly we speak of a *linearly told substantive set- or class-theoretic story*.[9]

Furthermore, let us call an outline of a formal set or class theoretic system (such as ZFC) followed by a short claim similar to "The following theorems and proofs[10] can be written in the formal system in such a way that one obtains formal proofs." a *formalistic header*. A story consisting of a formalistic header and a linearly told substantive set- or class-theoretic story is an example of what we call a *formalistic story*. Note that the claims in such a formalistic story are the claims on the possibility of writing; the claims of the corresponding linearly told substantive set- or class-theoretic story are not claims of the formalistic story. Nevertheless, the claims of the substantive story have to exist in the background since it is not possible to speak of some rewriting if the substantive story is not given in a human understandable way. The general definition of a formalistic story will be given in subsection 6.3, after we have analyzed further claims of rewriting.

Let us call a person who thinks that a particular axiom scheme of set or class theory is true in the sense that it expresses truth about an abstract universe which indeed exists a *Platonic realist* with respect to the given axiomatic scheme. If we now consider a particular linearly told substantive set- or class-theoretic story based on a particular axiomatic scheme, we can say: A person who is a Platonic realist with respect to the given axiomatic scheme and who regards the arguments on the basis of this system to be convincing is (unless he or she has strangely inconsistent thoughts) convinced that the statements supported by the arguments are true – again in the sense that they express truth about the abstract universe.

In contrast, the attitude of *formalists* is to consider the claims of many substantive stories to be a priori nonsensical or at least invalid. Some formalists (nominalists) might reject all claims on abstract entities, including numbers, others might reject claims infinitely many objects as nonsensical. Formalists in the Hilbertian tradition take a middle position and allow claims on natural numbers but regard claims on, for example, uncountably many objects to be nonsensical or invalid.

[9]The word "substantive" was chosen as a translation of the German word "inhaltlich" (with regards to content); its meaning in law suggests that it is an appropriate choice.

[10]In our terminology: arguments

According to formalism, one should rather argue about the truth of formalistic stories. Only for pragmatic reasons, one could (and one would) then behave as if it was reasonable to discuss set- or class theoretic arguments, but the goal of such a discussion would be no more and no less than to settle the question of the possibility of writing a formal proof in the mentioned formal system.

We have already stated at the beginning of Section 2 that the attitude of Platonic realists is questionable. Moreover, in subsection 3.4 we have explained that the claims of stories consisting of a formalistic header followed by a linearly told substantial story are often questionable even if the arguments in the corresponding substantive stories are considered to be convincing with respect to the axioms. The central question is then:

Central Question. Can stories consisting of a formalistic header followed by a linearly told substantive story somehow be regarded as being "less imaginary" or "less fictional" than the corresponding substantive stories, in particular if the alleged rewriting cannot be carried out by humans? Is there an intermediate realm between true statements and statements which are only true inside the story told by mathematicians?

To address this question, we study ideas inspired by the idea to actually (physically) carry out the process of rewriting. Doing so, we however stay in the realm of some kind of mathematics, that is, we do not address the question if and in how far it is actually (physically) possible to carry out the rewriting.

4.2 Ideas for rewriting

An evident idea is to imagine some kind of procedure with which rewriting from a more "human oriented" formal system to a system like ZFC can be done. The claim that there is a rewriting procedure would be part of some (different) story of mathematics (see subsection 7.1), just as the claim that the procedure can be applied in a particular case. We stress this because the idea is not to actually perform the computation and check the resulting first-order text; this would be a different approach which would however be far away from what is actually done by mathematicians.

Note that the problem of rewriting naturally involves a variety of different ways to translate the original claim into a formula of the formal system (say ZFC). Thus one may pose the question if all formulas resulting in this way are equivalent. But actually the argument would go in the opposite direction: rewriting procedures have to be compatible in such a way that one mathematical claim does not lead to non-equivalent formulas. This indicates

that a mathematician who makes a statement of possible rewriting even believes that there is some (not necessarily unique) proper way to reformulate the stated mathematical claims into formulas (of ZFC, for example).

A first approach to apply a rewriting procedure in some concrete example is to give:

1. a formal system for the input; let us call it I;
2. a rewriting procedure from I to the set-theoretic output system, say ZFC, translating formal proofs in I to formal proofs in the output system;
3. a convincing argument that outputs are formal proofs if the inputs are in I;
4. a formal proof in I for the statement to be established.

The idea here is that the system I is "more humanly oriented" and it might be possible to directly write formal proofs in I which are also humanly understandable.

In subsection 6.2 we will discuss how one might use computers to partially accomplish these goals and what problems occur if one tries to do so. For the moment we just note that humanly understandable texts are usually not written in a formal language, and one can clearly improve the understandability of a text with a less formal writing or strictly speaking (!) just plainly wrong statements.[11]

Rather, one would like to argue that a convincing argument written in natural language can be translated, leading to a formal proof in ZFC, say.

For this reason, it is reasonable to substitute the fourth desideratum by:

4'. a convincing argument that the argument in natural language to be rewritten can be written in the input language I in such a way that one obtains a formal proof in I.

Interestingly, the input system I can have a very different "feeling" concerning the allowed constructions than the set-theoretic output system. Let us say that the goal is a rewriting in terms of ZFC, that is, the output system is ZFC. Then even though ZFC is untyped, I can be typed, and even though ZFC does not allow for classes, I can allow for a class for each (individual) property $p(x)$.

[11] Just to give one example: If G is a group and g an element of the set underlying G (strictly formally !), one writes $g \in G$. However, purely set-theoretically, G might be a tuple (X, f), where X is a set and o an operation on X, and (X, o) might be the set $\{X, \{X, o\}\}$. Then, again strictly formally, $g \in G$ would actually mean that g is either X or $\{X, o\}$.

An evident problem concerning the first three points is that all these claims are again non-trivial claims inside some (different) mathematical story. Independently of this problem, we want to stress a further aspect: The claim that "it could be done" has a *constructive meaning*. Now, it might not be possible to perform the computation on interesting examples. But nonetheless, it is still an evident requirement that the procedures are actually constructively given and analyzed and not just claimed "to exist". This suggests that one should apply ideas from constructive mathematics, as for example from [constr-ana], here.

5 Interlude: constructive mathematics

As the reader might not be familiar with constructive mathematics, we now first provide some information on it. We then discuss how we intend to use constructive mathematics in arguments on the existence of formal proofs of theorems one wishes to establish. Finally, we give a particular interpretation of statements of constructive mathematics, which might be called "hypothetical interpretation", and which we view as being particularly suitable for our applications.

5.1 What is constructive mathematics?

Constructive mathematics (including its meta-theories) is a body of ideas on what statements and arguments are proper in mathematics, how these statements should be interpreted together with actual mathematical results following these ideas. It emerged from the "Gundlagenstreit" in the beginning of the 20^{th} century, and according to this, it is often associated L.E.J. Brouwer's intuitionism. From the current point of view, constructive mathematics is broader than intuitionism, or to say it differently, intuitionism is one of the schools inside of constructive mathematics. As shall be made clear below, it is not this school we are interested in for our purposes.

As constructive mathematics is not commonly studied, a first question is: What ideas are alerted to by the term "constructive mathematics?

According to the opening sentence of the entry on constructive mathematics in the Stanford encyclopaedia ([constructivism]), written by the constructive mathematician Douglas Bridges:

> Constructive mathematics is distinguished from its traditional counterpart, classical mathematics, by the strict interpretation of the phrase "there exists" as "we can construct".

For us, it is important to note that the phrase "we can construct" is not taken literally. Rather in constructive mathematics, also stories are told,

albeit stories different from the ones in classical mathematics.

In constructive mathematics, the starting point of reflection are the natural numbers, which are a priori assumed to exist.[12] Fundamentally, the mathematical statements are on algorithmic operations on natural numbers and furthermore on algorithmic operations on algorithms operating on natural numbers and so on. For example, a sequence of natural numbers is an algorithm taking natural numbers and outputting natural numbers too. A function from the sequences of natural numbers to sequences of natural numbers is then an algorithm taking and outputting such an algorithm.

Care has to be taken, however, because constructive mathematics is not just computational mathematics with natural numbers. Indeed, according to the philosophy that everything has to be constructed, the algorithms also have to be constructed. This means that algorithms claimed to exist have to be output from previously defined algorithms.

This still does not give a clear criterion what one must do to establish that an algorithm can be constructed. What kind of arguments are allowed for this?

The solution is centered around a self-limitation already on the level of the underlying logic with restrictions around existence statements and negations. In particular, it is not allowed use implications

$$(\neg\forall x : A(x)) \to (\exists x : \neg A(x)) \; . \tag{1}$$

Another aspect of constructivism is: It is emphasized that it is only reasonable to speak about the truth value of a mathematical statement if one can convincingly argue for its truth of falseness. The pragmatic rule is that to make a statement A shall mean exactly the same as to make the statement "A can be proven."

This general principle is then applied when sentences made with logical connectors are to be interpreted.

So a statement $A \vee B$ shall not only mean that $A \vee B$ can be proven but also (by applying the rule internally) that A can be proven or B can be proven.

Likewise, in constructive mathematics, a statement of the form $A \to B$ has the same meaning as "Every (potential) proof of A can be converted into a proof of B", "One can give a procedure that every proof of A can be converted into a proof of B.". Moreover, in all the three statements, one can add without modifying the meaning of the statement the initial phrase "It is known that" or "One can convincingly argue[13] that". A negation

[12] See however our interpretation of statements of constructive mathematics in subsection 5.3.

[13] In usual terminology: prove

of a statement, say $\neg A$ of A, means then that it is impossible to prove A, that is, every try of a proof of A leads to a contradiction. Finally, an existence statement, say $\exists x : A(x)$ means that one can show that an x_0 can be constructed for which $A(x_0)$ holds.

We note that it is then clear why the implication (1) shall not be used, just in the same way that $\neg\neg A \to A$ and $A \vee \neg A$ shall not be used.

These intuitive rules or reasoning are made precise in the formal system "Intuitionistic Predicate Calculus" or IQC, developed by Arend Heyting. In up-to-date expositions on constructive mathematics, such as in [constructivism], these logical foundations are stressed. Interestingly, the founder of intuitionism, Brouwer, emphasized the preliminary role of mathematics over logic, but as is often the case, an intellectual system has been created which now has its own philosophy, independently of its historical origins.

Building on intuitionistic logic, there are different schools of constructivism. There is agreement on the use of the natural numbers and the importance of the notion of algorithm. However, there are variation concerning what principles of reasoning are allowed, on the style of presentation and on possible interpretations of statements. Information on the different schools can be found in the books [foundations] (Chapter III) and [constructivism-in-math] (Chapter I, Section 4), which we also recommend independently of this.

The different schools allow ways of reasoning which go beyond a pure algorithmic construction. Most strikingly, Brouwer allowed for the "possibility to use our free will to decide at each state what the next number in the sequence will be", as expressed by Michael Beeson in [foundations], Chapter III, Section 4. Another principle was used by Andrey Markov Jr. He argued: If it is not true that a particular algorithm does not terminate (which means by the principles of constructivism that one can refute every attempt to prove that the algorithm does not terminate), then it terminates.[14]

An important extension of purely algorithmic constructive mathematics is the introduction of the notion of set. As stated in Chapter VIII of [foundations], there are two approaches: One can add the notion of set (or class) or one can postulate a "real" constructive set theory. Moreover, often a constructive axiom of choice is used, for example by Bishop.[15]

Besides different basic principles, the schools of constructive mathematics differ concerning the style or writing and the way of arguing. There is

[14] See [constructivism-in-math], Section 4.6 and note for comparison also [foundations] Chapter III, Section 1 with the exercises.

[15] It might seem that in constructive mathematics there is trivially always a choice function when one desires one because everything that is claimed to exist must come along with a construction. As explained in [constr-theories, I,4.7] this is not so.

the "dry" "Russian school" which is essentially recursive function theory with intuitionistic logic.[16] In opposite direction, there is Errett Bishop's book Foundations of Constructive Analysis (with a second edition with Douglas Bridges called Constructive Analysis ([constructivism]). These books highlight the spirit of constructive mathematics, that is, the construction, while not putting too much emphasis on foundational questions and ignoring discussions on foundational formal systems altogether. The first book, published in 1967, is of historical importance because it showed that one can really "do" constructive mathematics, and one can speak of a "Bishop school of constructive mathematics"; cf. [constructivism]. For us it is of importance that Bishop's book is intuitively (in contrast to (overly) formally) written and is based on intuitionistic logic and the claim that the natural numbers exist a priori.

5.2 Our use of constructive mathematics

For our purposes, that is, for arguments on the existence of formal proofs of theorems one wishes to establish, there is no place for free-choice sequences. As said above, one can integrate set-theoretic arguments into constructive mathematics. We do not envision this for our applications.

Interesting is now Markov's principle. We do not want to allow this principle either because otherwise we could argue for the existence of formal proofs of mathematical statements by contradiction. We regard this as being too weak.[17]

One aspect has not been addressed so far: Is it reasonable to really allow all thinkable algorithms? This question can be answered along just as it is usally answered in constructive mathematics. To illustrate the answer, let us first consider constructive elementary number theory. Here, finitely many algorithmically defined functions[18] are used to make statements on natural numbers, and then these statements are analyzed, however, there are no "higher algorithms" producing algorithms. The important aspect for us is that it should be obvious that the domain of the functions is total. This suggests to only consider functions defined by loop-algorithms, or, what amounts to the same with respect of equality of functions, functions defined

[16] This evaluation follows [constructivism].

[17] An abstract version of Markov's principle says $(\forall n : (\phi(n) \vee \neg\phi(n)) \wedge \neg\forall n : \neg\phi(n)) \rightarrow \exists n : \phi(n)$; cf. [constr-theories, I,7]. We reject this principle in our application for arguments on formal proofs for the reason given.

[18] From the point of view of constructive mathematics, functions are the same as algorithms, however, when we speak about functions, we have an extensional notion of equality in mind and when we speak about algorithms are more refined one. Note here that in contrast to "usual" set theoretic mathematics, "equality" in constructive mathematics is not assumed to be a priori given; rather different notions of equality are used depending on the context.

by primitive recursion. This idea is formalized in the formal system of *Heyting arithmetic*, HA. This system is built on IQC with a function term and corresponding axioms for each presentation of a primitive recursive function.

One can then "go up" and iterate the idea of operation by primitive recursive function by considering primitive recursion operating on algorithms. For this, to keep track on what the algorithms actually operate on, one should work with *finite types*.[19] One then naturally obtains the notions of *finite type recursion* and – as a generalization of a primitive recursive function – *primitive recursive functional*.[20] Interestingly, in this way one can also obtain new functions from $\mathbb{N}$ to $\mathbb{N}$: As explained in [constructivism-in-math, 9, 1.4], the premier example of a totally recursive but not primitively recursive function, the Ackermann function, can be given in this way. The ideas presented here have been formalized in a typed formal system called *finite-type arithmetic*, HA^ω.[21]

In another direction, an important question is how formal an argument for the existence of a formal proof of a theorem shall be. Would it, for example, be reasonable to demand that it be written in HA^ω (with meta-variables, particularly for the function symbols)?

We can give a clear "no" to this question. Just as set-theoretic mathematics, constructive mathematics is never done on this level of formality (apart from small calculations), and clearly it is not reasonable to demand that it should be.

Is it then reasonable to demand that at least it "should be possible" to reformulate / rewrite an argument in this system? Again, we can answer this with a clear "no". As our starting point was the question what such a claim of "should" could mean, it would be ironically inappropriate to carelessly introduce such a demand here. Note however, that within this argument we already anticipate (see subsection 7.1) that rewriting also makes sense for different kinds of mathematics. Up to now, constructive mathematics was only seen as an auxiliary tool to answer our posed question.

In summary and positively expressed, for arguments on the existence of formal proofs of mathematical theorems, we envision an informal presen-

[19] A type is an expression built following these rules: 0 is a type and if σ, τ is a type then $\sigma \times \tau$ and $\sigma \to \tau$ is a type. Algorithms of type 0 are the algorithms which do not have an input and output a natural number. The function associated to such an algorithm can be identified with the output, that is, one can say that the function is a natural number. For the interpretation of the statements given in the next subsection it is, however, important that an algorithm of type 0 is not a natural number – it is an algorithm producing a natural number.

[20] These ideas were developed by Kurt Gödel in [dialectica].

[21] Stictly speaking there are at least two distinct systems with this name. The "most basic" system is given in [constr-logic&-math], which also gives a nice exposition to constructive mathematics. The definition of HA^ω in [constructivism-in-math] includes a "combinator" which is not present in the definition in [constr-logic&-math].

tation which is "purely algorithmic" along the lines just outlined.[22] The presentation might be similar to that of Bishop's book with possibly a presentation of algorithms given by pseudo-code. (But as said, Bishop uses principles we would not like to use, namely sets and the constructive axiom of choice.)

We note that with this approach has the nice feature that one can always "go up" with algorithmic arguments.[23] This means that if we envision a meta-analysis of arguments on the existence of formal proofs, a meta-analysis of this and so on, we never have to leave the framework outlined.

5.3 Our interpretation of constructive statements

We now come to the *interpretation* of constructive statements for our application in arguments on the existence of formal proofs of theorems, along with a corresponding suggestion for the use of language.

Following the citation in the beginning of subsection 5.1, "constructive mathematics is distinguished from [...] classical mathematics by the strict interpretation of the phrase 'there exists' as "we can construct". More precisely, it seems that there is a general consensus of mathematicians who regard themselves in the constructive tradition that one imagines the natural numbers and strives to construct everything from this basis – as was programmatically expressed by Leopold Kronecker (see [on-Kronecker]): "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk."

For our applications, the evident approach would then be to say that texts based on a fixed alphabet are represented by natural numbers and vice versa. Statements as described in the previous subsection then become statements on transformations of texts via algorithms, transformations of algorithms on such algorithms and so on. Above, we have tacitly already used this identification of natural numbes and texts.

Let us reflect on the meaning of the statements on texts which are then made: We first note that the concept of text is not the same as the concept of a natural number, or to say it differently, the phrases "text" and "natural number" have a prescientific meaning is clearly not identical. Texts have the evident property that they can be given literally; what one writes down *is* a text and does not just denote a text. In contrast, the idea of a natural number is always abstract. Surely, one can represent numbers by texts, for

[22] In the terminology given by Solomon Feferman in [constr-theories] (which is also given in [foundations, V,3]), our assessment is that the system HA^ω as defined in [constructivism-in-math] is directly adequate and directly in accordance with the envisioned body of arguments on the existence of formal proofs of mathematical theorems, where texts on a fixed alphabet are identified with natural numbers.

[23] This corresponds to the formal feature of HA^ω that for two types σ, τ, there is always the type $\sigma \to \tau$.

example using the usual decimal representation or the "elementary" unitary representation. Nonetheless, if for example somebody writes down "123" he surely writes down a unique text independently of the meaning of the symbols, but he only gives a number if a meaning is attached to the text.

So, texts can be given physically-literally,[24] but in our application we talk about texts which are constructed by algorithms without this ever being the case physically. So we clearly tell ourselves a story about imaginary texts here.

The goal is now to limit the imaginary aspect of the story to a minimum. Particularly, we do *not* want to make statements on the transformation of an *infinite amount* of texts via algorithms. Rather we want to make statements on the transformation of *individual texts* via algorithms. Also, because algorithms can also take algorithms as inputs and output algorithms, we want to make statements on the transformation of *individual algorithms*. This will then correspond well with the statement made in the introduction that we regard a statement that infinitely many results have been established to be a priori invalid (see also subsection 6.3 for this point).

For this, we use what we have already used in the previous subsection: We can give algorithms by writing down pseudo-code, and we can iterate this process by using already given algorithms. We can then in particular consider algorithms which take no input. We then say that the output *is algorithmically given* or *can be constructed*. As a special case of this, we can consider algorithms which output texts. Again, we then say the output text is *is algorithmically given* or *can be constructed*.

We iterate and stress again that when making these statements, we again tell a story which should not (and in many applications cannot reasonably) be interpreted physically. The phrase "is algorithmically given" makes this clearer than the phrase "can be constructed", but both phrases are unproblematic if one realizes that a story is being told. We also note that any literally given text is also algorithmically given (in the story).

As already stated, we do not want to make statements on infinitely many input instances. Rather, concerning the input instances, we merely want to make *hypothetical* statements of the form "Suppose that some x_0 is given. Then $A(x_0)$ holds.".

But what shall it mean that the statement is "hypothetical"? Our idea is that the reader always only imagines a particular algorithmically given x_0. But this reases again a question as it is not clear what algorithmic constructions shall be allowed. (It does not help to say "all possible constructions", because we want to discuss what we shall deem as being possible.) The

[24]One can then discuss when two texts should be regarded as being equal and also if a physically given text denotes something like an "abstract text"; in any case it is evident that texts can be given physically and literally.

strictest possible solution, the most narrow interpretation which seems to be reasonable, is: Such a hypothetical statement is on all x_0 which have been algorithmically given or will be algorithmically given by any concious being shall be allowed. More wider interpretations, which go more in the direction of story telling, are possible.

This leads to a corresponding interpretation of the forall-quantifier: We interpret a statement of the form "$\forall x : A(x)$" (including corresponding statements in natural language) as:

> Suppose some x_0 is algorithmically given. Then $A(x_0)$ holds.

Similarly, for "$\exists x : A(x)$" one might then not only say "Some x_0 with $A(x_0)$ can be constructed, but also:

> Some x_0 with $A(x_0)$ is algorithmically given.

Is this interpretation always possible, that is, can one reasonably interpret statements of constructive mathematics as outlined in the previous subsection in this "hypothetical" way? In particular, is the interpretation consistent with (typed) Intuitionistic Predicate Calculus? The author's answer is that this is so; after all the only occurrence of the forall-quantifier in the axiom scheme is the scheme $(\forall x : A(x)) \to A(t)$, and this is nicely consistent with our interpretation.

With the outlined "hypothetical" interpretation of forall-statements, there is no need to imagine infinitely many objects. For this reason, by itself it would seem to be reasonable to call this interpretation "finitism". However, as the term is already used (as far as the authors can tell with rather different meanings) in a similar way as "constructivism" (based on natural numbers), we refrain from doing so. We note that our interpretation is different from ultrafinitism which rejects natural numbers deemed to be "too large".

In our applications, we want to talk about texts and not natural numbers, but to complete the picture and to deepen the link to constructive statements as they are usually made (including in the previous subsection), let us come back to natural numbers now. As said above, the idea of a natural number is always abstract. Moreover, as it is based on the idea of counting, an algorithmic aspect is already present in the notion of natural number. So it is not at all clear what a "non-algorithmically given" natural number should be. For this reason we suggest to just speak of a "given natural number", not an "algorithmically given natural number" and to also say that the number "has been constructed", not just that it "can be constructed". The "hypothetical" interpretation can then also be applied to natural numbers.

To given an example of our interpretation, by saying that there is the rule or algorithm $\mathcal{A} : (a, b) \mapsto a \uparrow\uparrow b$, one says:

> Suppose a tuple of natural numbers (a, b) is given. Then the algorithm $\mathcal{A}$ gives / constructs a particular number (which is then denoted by $a \uparrow\uparrow b$).

If the reader then imagines two given numbers a and b, say $82479824 \cdot 3798217$ and 2418^{92741}, she or he can construct the corresponding number $a \uparrow\uparrow b$, in this case $(82479824 \cdot 3798217) \uparrow\uparrow (2418^{92741})$, and then this number is given / has been constructed. The statement is here that whatever two numbers the reader imagines, she or he can construct the resulting number by application of the algorithm. We note that again that the statement is a story without reasonable physical interpretation.

6 Criteria for rewriting

With the ideas of constructive mathematics in mind we now discuss how one might address the Central Question and questions around it.

6.1 First criteria

We return to the approach in subsection 4.2, as given in items 1. – 3. This means that an input system I is given, a rewriting procedure is given, and a convincing argument that the rewriting procedure is correct is given. As already stated, to study and analyze such a system, it is natural to apply ideas from constructive mathematics. We now reflect upon how exactly this might be done:

The procedures should be given in an explicit and concrete way. The level of concreteness required is a matter of judgment, and indeed, exactly this question is always a matter of debate in actual applications of the constructive paradigm, like for example in [constr-ana]. Ideally, in our opinion, a detailed "mathematical" description focusing on ideas and on the basis of constructive mathematics, a description in pseudo-code and an actual, testable implementation should be given, but at least the first point should be satisfied.

A reasonable criterion is then:

Criterion 1 The procedures for testing membership in I and the rewriting procedure have to be given in such an explicit way that humans can actually perform the computations on examples and such that an implementation on a physical computer seems to be actually possible. Moreover, the claims for correctness have to be given by following the rules of constructive mathematics as outlined in subsection 5.3.

The criterion is of course vague, but this is unavoidable if we speak about mathematical arguments (here arguments for the procedures) as they are actually given and not formal arguments or computer programs.

We give three examples of rewritings with respect to ZFC which satisfy this demand and which already make a huge difference in usability.

- Definitions as abbreviations for terms
- Syntactic reformulations to make statements better readable, for example "Let $x \in X$. Then ..." instead of "$\forall x : x \in X \to \ldots$".
- If a pure set theory such as ZFC is used: the introduction of class terms for properties: For each property $p(x)$, the term $\{x \mid p(x)\}$ with the rule

  $$\forall y : y \in \{x \mid p(x)\} \leftrightarrow p(y)$$

  is introduced, along with subclass relationships and equality of classes with the obvious rules. Furthermore, the statements

  $$\exists\{x \mid p(x)\}$$

  as abbreviations for

  $$\exists y : y = \{x \mid p(x)\}$$

  are introduced.

We now consider an application to a particular text, following item 4' of subection 4.2. The following criterion then seems to be appropriate:

Criterion 2 Let a text of the following kind be given. The text consists of three parts: A linearly told substantive story that a particular statement A is proven on the basis of Zermelo-Fraenkel set theory, a constructive story on a rewriting system following the first three points of subsection 4.2 and a header of the form: "Based on the following argument for statement A and the outlined system for rewriting (which follows the approach of subsection 4), one can give a formal proof of A on the basis of ZFC." Then this shall mean that Criterion 1 applies to the story on the rewriting system, the substantive story is convincing on the basis of (informally given) Zermelo-Fraenkel set theory with the axiom of choice and without too much effort, one can obtain a formal reformulation A' of A in ZFC and a convincing argument on the basis of constructive mathematics that one can construct a valid input instance to the rewriting system.

We make some remarks:

- We distinguish between Zermelo-Fraenkel set theory with the axiom of choice and ZFC: The latter is a formal system whereas the former is an axiomatic set theory as actually used by mathematicians. (The particular theories serve as examples.) We note in particular that there is a psychological difference between convincing oneself that arguments are valid on the basis of the axioms of Zermelo-Fraenkel set theory and convincing oneself that from the arguments one can construct (in the sense of constructive mathematics) a formal proof. The latter task is usually not considered by mathematicians when they study "proofs".

- The criterion is clearly vague in several ways. What shall in particular "without too much effort" and "formal reformulation A' of A" mean?

  One can of course argue about the specific formulations, but there is a fundamental reason why vagueness cannot be avoided here: All statements on mathematical texts *as actually written* are vague, already starting from criteria on "mathematical texts". For example, in Section 2, we already mentioned that there is no clear standard as to what shall be called a "proof" and what not.

  The claim of the reformulation of A is always present when it is claimed that statements can be expressed formally and cannot be avoided. Concerning "not too much effort" we want to give the rule of thumb that a person who claims to have understood the "proof" can convince him- or herself in less time than needed for the understanding of the "proof" that the criterion is satisfied.

- There are obvious variants for the criterion for other formal set and class theories, such as, for example NBG; the reason we only make a statement on ZFC is that we do not want to enter a discussion on what would be a "reasonable set and class theoretic formal system" and how it should be given.

6.2 Computer backed systems

Again we come back to the idea of subsection 4.2 to have a procedure for rewriting and to the approach consisting of the four requirements outlined there. For a particular suggestion for the first three aspects and for a particular mathematical argument, it remains to argue that it can be written in the formal system for the input and holds in there. This is a non-trivial task for itself, with with one comes back to the original problem of rewriting actual (non-formally given) mathematical arguments.

One idea is here to write mathematics directly in a humanly oriented formal system and to check the arguments with the help of a computer. This

idea was propagated in the "QED manifesto" ([QED]), and several formal systems for this have subsequently been developed.

The most successful of these is arguably the Mizar project. In this system a great number of theorems have been written and formally checked with the help of computers. This is a great success for formalized mathematics. One should, however, keep the following in mind:

First, such systems are not wildly used and mathematicians most often do judge arguments to be valid or not on a rather intuitive basis without proofs in formal systems. So there is indeed a much larger story of mathematics told outside such formal frameworks than inside.

There is also a fundamental problem with the use of such computer-based systems: The formal proofs are clearly more difficult to understand than usual mathematical arguments; for examples of this see [example]. The difficulty is required because of the rigid structure of the formalized language, which would not be necessary for humans and in fact impairs their understanding. So instead of checking the arguments themselves, humans now turn to computers. Computers are physical objects designed by engineers based on supposed physical laws found and formulated by physicists. Physics and engineering rely themselves on mathematics, and the design and the building of computers has an immense complexity which has accumulated, a complexity which is actually much larger than the complexity of nearly any argument in mathematics. Furthermore, to use a computer for proof checking, one uses diverse programs (including programs one maybe does not have in mind at first, like the operating system), some of which is so large that it cannot even be read by any human. Now, why should one assume that programs on computers run correctly if on the other hand one doubts that humans can check mathematical arguments? There might be good reasons to check supposed proofs by computers as an additional check, but fundamentally, one can question a check by computers (built and programmed by humans) as much as one can question a direct check by humans.

6.3 Formalistic stories

We end this section with the promised definition of formalistic stories.

Above Criterion 1 we have outlined how procedures for rewriting should be given and analyzed. We can then combine such rules and their analysis with a particular convincing set-theoretic argument to obtain an argument that a formal proof can be constructed (in the sense of constructive mathematics).

More generally, we define: By a *formalistic story* we mean a story in constructive mathematics as described in subsection 5.3 whose main claim

is that a formal proof of a particular result or proofs of *finitely many* results can be constructed (on the basis of some formal system).

We have set the definition up in such a way that it corresponds to the following criterion which reflects our requirements for arguments for formal proofs.

Criterion 3 Given a set theoretic statement A, the claim "A is proven on the basis of ZFC." shall mean: On the basis of constructive mathematics, one can argue convincingly that one can construct a formal proof of a formal reformulation of A in ZFC.

So, by this criterion, to establish that A is proven on the basis of ZFC it is necessary and sufficient to give a convincing formal story for A. Again, ZFC can be substituted with any other explicitly given set-theoretic formal system.

In subsection 6.1 we considered particular formal stories and gave criteria for them. In full generality it is outside of the scope of this note to discuss when formal stories shall be regarded as being valid. We can just say that there surely is a great subjective component.

For an application of the criterion, a particular statement A has to be given. We recall that in the introduction we stated that we regard any statement that infinitely many mathematical results have been established to be invalid. Our definition of formalistic stories and the criterion just given reflects this requirement.

We note that if one follows our interpretation of constructive mathematics, described in subsection 5.3, as a framework to argue that mathematical results are established formally, one cannot argue that infinitely many results are established via formal proofs, just as one cannot argue that there are infinitely many natural numbers.[25] One can just give methods (such as, for example, the ones in subsection 4.2) which might then be used to establish a variety of results.

7 Interpreting actual stories

Mathematical texts as actually written most of the time are neither substantive stories (as defined in subsection 4.1) nor formalistic stories (as defined in subsection 6.3). Furthermore rewriting procedures do not at all belong to mathematical practice. Although one usually may interpret mathematical texts as a set- or class-theoretic story, this is not always the case and there

[25] Of course, in mathematics, be it constructive or set-theoretic, one can still argue on infinitely many objects called "formal proofs". This should not be confused with our study here whose purpose is to discuss the Central Question in subsection 4.

are several kinds of mathematics. All these things lead to and contribute to a variety of interpretations of mathematical texts, some of which will be discussed in the following.

7.1 Different kinds of mathematics

Up to now we identified mathematics with some set or class theory which gives a unified foundation of a lot of mathematical practice. In general, when speaking of some *kind of mathematics*, we consider a separate story of mathematics and we always expect that there is some (for humans understandable) language, a (description of a) collection of axioms and rules to derive conclusions (with respect to some logic). These conditions give a very general setting of mathematics. However, usually mathematicians have a more restricted view which things are really worth being considered as mathematics and opinions may differ in this issue. A kind of mathematics is not to be confused with a mathematical theory, like number theory or K-theory, considered inside some set or class theory - they do not comprise a separate foundation. Contrary to this, some given theory considered in model theory serves as an example for a kind of mathematics.

Famous examples for set or class theories are Zermelo-Fraenkel and von Neumann-Bernays-Gödel set theory, where only the second one uses proper classes. Model theory indicates that one should expect basically the same results in these two kinds of mathematics. But there are also weaker, stronger or different set or class theories. For example one may use a set theory without the axiom of choice or some weaker or stronger version of this axiom. As another example, one can demand the existence of inaccessible cardinals, or equivalently of Grothendieck universes. These changes have real consenquences on the stories which can the and which are told on the bases of the axiom schemes.

Kinds of mathematics different from set or class theories may be found in mathematical history or, as mentioned before, as a theory studied in model theory. Synthetic (euclidean) geometry, propositional logic, Peano arithmetic, group theory or some theory of (maybe further specified) fields as in model theory or even some theory of Sudokus may serve as examples. However, for these kinds of mathematics it would not be unusual to be formalized inside some set or class theory.

Constructive mathematics is different. In comparison to the other kinds of mathematics, constructive mathematics is usually not considered inside a set or class theory. The main reason for this is that intuitionistic logic, which is fundamental for constructive mathematics, is different from the classical logic of usual set or class theory. Since a main motivation and aspect of constructive mathematics lies in the constructiveness of all algorithms, it

would be strange to consider abstract existence claims of classical logic over constructive statements. Nevertheless model theory of intuitionistic logic exists.

Given an actual mathematical text, in most cases mathematicians would interpret it inside some set or class theory. However, usually it is considered to be irrelevant to give a precise listing of the axioms. This is a feature of actual mathematics: the claim of rewriting of *one* mathematical text may be stated with respect to different foundations. Without mentioning the claim of rewriting, this is used in actual mathematical papers or books, for instance as "the usual proof applies to the new axiom system". Emphasizing it differently, this means that usual mathematical language may be a tool to transfer results from one kind of mathematics to a different one.

In the last sections we explained in detail why constructive mathematics is the right framework for the rewriting procedure of mathematical arguments in Zermelo-Fraenkel set theory to ZFC. In fact, these arguments apply to all kinds of mathematics. This means that it should be possible to give rewriting stories for rewriting any convincing mathematical argument to a corresponding formal systems via constructive mathematics. But in contrast to set or class theory this rewriting procedure may be easier, and therefore of less importance, or even unnecessary in "weaker" kinds of mathematics.

Within this context, stories of constructive mathematics and their rewriting achieve an outstanding role. As mentioned before, this problem may not be solved finally due to the problem of infinite regression when rewriting constructive stories. Note that always basically two stories have to be told: a substantive story in the considered kind of mathematics and a story in constructive mathematics which supports the claim of rewriting of the corresponding formalistic story.

7.2 Statements on statements in actual stories

As already mentioned in the introduction, actual mathematical texts contain a variety of statements, in particular reflections on the process of doing mathematics. These reflections include remarks inserted for didactic reasons, expressing for example "ideas" before a "proof" is given, or aesthetic judgments.

To analyze all these different kinds of statements is outside of the scope of this note; we want to turn attention to a particular kind of statements: statements on mathematical statements in how far as they are used to (allegedly) establish mathematical results. One kind of such statements we have already considered: Formalistic stories consisting of a formal header and a linearly told substancive story. Given such a text, one can then also say that just the formalistic header is a statement on a statement. More

generally, whenever in a text a sentence or a body of sentences S occurs in which a statement on on some statement of the text is made, one can say that S constitutes a statement on a statement in the context of S in the text.

We give some examples of statements on statements which are used establish mathematical results as they might actually occur in mathematical texts.

1. After two sets x and X have been given: "Now the statement that x is contained in X holds."

2. After in an argument 27 case distinctions have been made and two of them have been proven: "We have now proved the first two of the 27 cases, proofs of the remaining 25 cases are analogous. The reader should keep the changes outlined in the previous two chapters in mind."

3. "For every three statements A, B, C, the statement $(A \vee B) \wedge C$ is equivalent to $(A \wedge C) \vee (B \wedge C)$."

In the first example we have the word "statement" which can be seen to refer to a particular sentence, namely "x is contained in X.", in the second example, "proof" refers to a body of sentences, and in the third example "sentences" are explicitly mentioned.

When analysing these examples, one notices that even among the restricted kind of statements we analyze, a variety of kinds of statements are made:

The first example is easy: The statement merely says that statement

> x is contained in X.

holds. This might have just been stated directly.

The second example is more tricky. The idea is here that inside the text written there is an instruction as how to generate more text in order to obtain a more complete argument. A straight-forward answer what shall be done now is: Well, the text shall be rewritten according to the instruction. However, an author who makes such a statement in all likelihood does not expect a reader to really follow his or her instruction. Given this, there again arises the question what kind of statement is actually made here. An answer is that the reader should imagine a linearly told substantive set- or class-theoretic story (and not a formalistic story, as in the previous section). So the reader should image a story which by itself is a story on imagined (non-physical) objects. The authors of this note do not in principle object to this practice, but just remark that such meta-statements are error-prone and should rather be avoided.

The third example is yet different. One notices first that it is vague: What exactly is meant by "three statements A, B, C"? Depending on the context, there are now different types of interpretations.

One type of interpretation is that this shall be a statement of some formal logic, which is the kind of mathematics being studied. For this, A, B, C should be specified as, for example, propositional symbols when working in propositional logic or 0-ary predicate symbols when working in some first-order logic. In the second case one would tend to allow arbitrary formulas in A, B, C, which means A, B, C are placeholders for arbitrary formulas. But this is not allowed within this context since it would lead to an infinite number of mathematical statements. However, if one uses some set or class theory to study formal logic formalizing the corresponding language via sets, this problem disappears since this infinite number of statements may be summarized into one set-theoretical statement. So studying some formal logic within some bigger kind of mathematics leads to another type of interpretations inside some mathematics.

Keeping in mind subsection 5.3 one notes that it is also possible to give a correct interpretation of the third statement if A, B, C are placeholders for first-order formulas when working in first-order logic. Then the statement is true for any (in terms of 5.3) given first-order formulas. Note that this is a statement in constructive mathematics, thus a part of the rewriting story and not the kind of mathematics being studied.

There are more interpretations of the third statement. For instance one could take the point of view that such a simple statement is obviously a statement in propositional logic and when using this statement it should be interpreted as an application of propositional logic in some kind of mathematics as described in subsection 7.1. Another possibility is that one interprets it as a description how to write down a substantive story or even a formal proof working again outside of the mathematics being studied. However, one has to be careful using such interpretations since they are not as precise as the interpretations before.

A conclusion from these considerations is:

Criterion 4 Whenever in a story of mathematics a statement on seemingly infinitely many sentences (or expressions) are made, the statement must not be interpreted as giving infinitely many mathematical results. If the statement is later used as a meta-statement on how to obtain an argument or a formal proof, it must be established on the basis of constructive mathematics.

We remark here that axioms schemes on which a linearly told substantive story is based are never part of the story; indeed they are part of formalistic headers. So there is no contradiction between the Criterion and the fact that in axiom systems, often statements on infinitely many statements are made.

Of course they are not written down in the formalistic header but there is the possibility to check in an algorithmic manner if a statement belongs to the axiom scheme.

8 Mathematics and meta-mathematics

There is an interesting relationship between the use and the theory of formal systems: Formal systems were first introduced (for example by Gottlob Frege) to study and clarify mathematical arguments, and as already stated in subsection 3.2, the idea of formal reasonings (formal proofs) is foundational to the present day doing of mathematics. Now, the idea of formal systems by itself invites a mathematical study on formal systems, and quickly the study of formal systems was integrated into mathematics.

On the one hand now, the mathematical study of formal systems has its motivation in meta-mathematical considerations. Conversely, the mathematical study of formal systems has lead to mathematical results on formal systems which have an impact on how mathematics is viewed and conducted. One example is the application of (not immediately obvious) statements from propositional logic or first-order logic in the story of mathematics.

The (historically surprising) statements on the foundations of set theory (incompleteness, independence of axioms, etc.) have a particular, one might say philosophical, impact on the way mathematics is seen and is conducted. (Explicitly, they have lead to a certain "liberal attitude" towards different axiom schemes, where no such scheme is regarded to be "absolutely true".)

It is however not obvious in what sense such results make statements *on* mathematics (in particular on the limitations of mathematical endeavors) rather then *in* mathematics.

To study this question, slightly more generally, we study the relationship between arguments ("proofs") as such and arguments on formal proofs. For this, we consider two directions of inference which we call "going down" and "going up". The direction of "going up" comes in two flavors: "going up for the existence of proofs" and "going up for the non-existence of proofs".

8.1 Going down

Recall that in Criterion 3 we said that to claim that a mathematical statement A is proven in ZFC shall mean that on the basis of construtive mathematics one can argue convincingly that one can construct a formal proof of a formal reformulation of A in ZFC. Now, when a mathematician claims that he or she has proven a result, he usually just writes down his arguments, which are by default based on Zermelo-Fraenkel set theory with the axiom of choice, but he or she does not say that there is a formal proof in ZFC on

the basis of the arguments. With the terminology of Section 4 we can say that a substantive story is told, not a formalistic one.

But what if a formalistic header was added, so that really the claim was made that one can construct a formal proof on the basis of ZFC? Can one a priori say that this claim is valid?

Given our terminology (in particular again Criterion 3) we can also formulate this in the following brief way: Can one a priori say that every statement convincingly argued for on the basis of Zermelo-Fraenkel set theory with the axiom of choice can be proven on the basis of ZFC?[26] This question is non-trivial, because mathematicians in their judgment if an argument is convincing to them do usually not explicitly consider the question of rewriting.

We answer this question affirmatively with the following thesis:

Thesis. For every (informal) statement A on the basis of Zermelo-Fraenkel set theory with the axiom of choice for which a convincing mathematical argument for A on the basis of Zermelo-Fraenkel set theory with the axiom of choice is given, one can construct a proof of a formal reformulation of A on the basis of ZFC.

This thesis is in nature similar to the Church-Turing thesis. We stress that it should not be taken as a definition; in fact, the phrase "convincing mathematical argument" (what other authors call "informal proof") already has a meaning, so it cannot be taken as a definition.

We do however want to suggest that if somebody ever came up with a counterexample, the mathematical intuition on when a mathematical argument is seen as convincing (a "proof") or the axiomatic basis would change and afterwards the thesis would again be correct.

We note that a related inference as the one in the thesis is present in Gödel's argumentation for the First Incompleteness Theorem: He starts with a formal system F containing Robinson Arithmetic and uses this implication: If a formal sentence A follows from F, a sentence whose interpretation is "A can be proven from F" also follows from F.

Despite this similarity, there are, however, differences between Gödel's technique and what we argue for here: The argument by Gödel is purely mathematical, it is an inference inside of mathematics. Our thesis is however meta-mathematical, it is about the human endeavor concerning mathematics.

[26] Just as in the Criteria, we stick here to ZFC, but similar considerations apply to other formal set or class theoretic systems.

8.2 Going up for existence of arguments

Suppose now that for some statement A a mathematician does the following:

1. He or she gives a suitably formalized ZFC-statement A' of A.
2. He or she argues ("proves") set-theoretically (!) that there exists a ZFC-proof of A'

One might then be tempted to say that A "is proven" (has been convincingly argued for) on the basis of Zermelo-Fraenkel. This would however contradict our general attitude that in order to say that a result is formally proven (on the basis of some set-theoretic formal system) one has to constructively argue that there is a formal proof. Explicitly, this would contradict Criterion 3.

8.3 Going up for non-existence of arguments

We now consider a statement on the non-existence of formal proofs for statements. Examples for this are many, let us state one: One can argue on the basis of Zermelo set theory that if ZF is consistent, then ZF with $\neg\mathsf{C}$ is also consistent, that is, that the axiom of choice does not follow from ZF. This result is particularly interesting as the known arguments for it rely on non-trivial set theory.

We want to "go up" and conclude something on the limitations for humans and more generally on conscious beings of doing mathematics. Let us stick with the given example. Here we would like to conclude: Any convincing argument on the basis of Zermelo-Fraenkel set theory that the axiom of choice holds would lead to a convincing argument that ZF is inconsistent.

It would be a misconception to immediately conclude from a mathematical statement on a (seemingly) corresponding statement of the human future. One can however argue by "going down":

Assume that someone gives a convincing argument for the axiom of choice on the basis of Zermelo-Fraenkel set theory. Then assuming the variant of the Thesis in subsection 8.1 holds for ZF, it would be proven on the basis of ZF that the axiom of choice holds. By Criterion 2 (for ZF) this would mean that one can argue on the basis of constructive mathematics that there exists a formal proof for the axiom of choice from ZF. The conclusion is then that ZF is inconsistent.

8.4 Example: theorem of soundness and completeness

It is useful to discuss some example which may be interpreted mathematically and meta-mathematicalyl and may illustrate and even extend the former subsections. Let us consider the theorem of soundness and completeness of

first-order logic as a mathematical theorem in set theory (especially Zermelo-Fraenkel) and statements similar to the statements in this theorem. A usual formulation of the mathematical theorem is the following:

Let $\mathcal{L}$ be a first-order language. For all sets Γ' of sentences (of $\mathcal{L}$) and all sentences φ' (of $\mathcal{L}$) the following are equivalent:

$(1)'$ We can deduce φ' from Γ'.

$(2)'$ The formula φ' is logically valid in any model of Γ'.

The implication $(1)' \to (2)'$ is called *soundness* and the reverse implication $(2)' \to (1)'$ is called *completeness*.

When we consider something as a mathematical theorem in Zermelo-Fraenkel set theory everything has to be a set. So especially $\mathcal{L}$, Γ' and φ' are sets representing a first-order language, a set of sentences or a sentence respectively. Then it seems likely to interpret "Let $\mathcal{L}$ be a first-order language." as "For all sets $\mathcal{L}$ which have the property representing a first-order language ..." – different interpretations will turn out being meta-mathematical versions later. Now $(1)'$ is by definition the set-theoretic existence of a formal proof which shows φ' given Γ' and this formal proof is again a set. The usual definition of $(2)'$ is φ' being "true" in any structure of $\mathcal{L}$ where all the elements of Γ' are "true". Here being "true" or "false" is defined by a set-theoretic recursive definition over the length of the formulas. Since there is no restriction on the length of the formulas φ' and the formulas in Γ', the output of this procedure necessarily has to be a set like 0 $(:= \emptyset)$ for "false" and 1 $(:= \{\emptyset\})$ for "true". On the basis of these interpretations, the usual arguments give rise to the above theorem.

Let us consider the concrete example where $\mathcal{L}$ is the language of groups. So what is meant by this? Since $\mathcal{L}$ is a set (a mathematical object), speaking about it (doing mathematics) is the same as describing a set which is representing the first-order language of groups. Given such a description, one can apply the above theorem. Abstracting from this example leads to a different interpretation and a first meta-mathematical version of the theorem of soundness and completeness where "Let $\mathcal{L}$ be a first-order language." is interpreted as "Let a first-order language $\mathcal{L}$ be algorithmically given." in the sense of subsection 5.3. After interpreting the quantification over $\mathcal{L}$ in this constructive way, one can do the same for the quantifications over Γ' and/or φ'. Here, if only the quantification over φ' and not over Γ' is interpreted in a constructive way, one has to change the order of quantification. This leads to three further meta-mathematical theorems. In each application of these, one has then a special case of the mathematical theorem. For instance, in

our example of the language of groups, one may take Γ' as the group axioms and φ' as some group-theoretic statement.

Sticking to this example, usually one is not mainly interested in models of the set of group axioms but rather in groups themselves. Actually, the starting point is not Γ' and φ' being given as sets; rather one has some mathematical statements Γ describing groups and a statement φ on groups. Subsequently, one describes $\mathcal{L}$, Γ' and φ' corresponding to Γ and φ in a proper way. Note that Γ and φ are given in mathematical language which may correspond to Zermelo-Fraenkel or some proper kind of mathematics in which it makes sense to formalize objects defined by Γ. The smallest such kind of mathematics would be the one using Γ as axioms - in this case group theory. If one abstracts from our example, Γ and φ, that is, the descriptions of mathematical objects, have to be constructively given. Now, we may also consider:

$(1)^+$ One may give a set-theoretic formal proof of φ starting from Γ (in the sense of subsection 5.3).

$(2)^+$ φ holds for the set-theoretic objects defined by Γ.

(1) One may give a formal proof of φ starting from Γ (in the sense of subsection 5.3).

(2) φ holds in the kind of mathematics defined by Γ.

Note that $(1)^+$ and (1) are by definition the claims that one may write down a substantive story and fulfill the claims of the corresponding formalistic stories for $(2)^+$ and (2) respectively. While $(1)'$, $(2)'$ and $(2)^+$ are set-theoretic statements, $(1)^+$ and (1) belong to constructive mathematics and (2) is a statement in the kind of mathematics defined by Γ. In the previous subsections, we considered $(1)'$ and (1), where Γ consists of the axioms of Zermelo-Fraenkel set theory.

We have seen that $(1)'$ and $(2)'$ are equivalent. But $(2)^+$ is also equivalent to $(2)'$ since set-theoretical objects defined by Γ and models of Γ' are basically the same. Nevertheless, this is something one has to prove in each case separately since it is necessary to show that Γ' is a proper way to formalize Γ within set theory.[27] Due to this equivalence, the claims to give set-theoretical formal proofs of $(1)'$ respectively $(2)'$ will lead to the same statement as $(1)^+$.

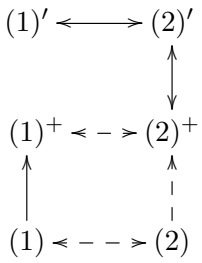
Let us now consider (1) and $(1)^+$ and assume, for the moment, that (1) is true. Then there is a substantive story showing (2) and one may

[27] As Γ are statements in mathematics, these are only finitely many statements. It may still be possible to make Γ' infinite. Consider, for instance, the case of torsion-free groups. Then being torsion-free is one mathematical statement by quantifying over natural numbers which represent the exponents. In the language of groups one would put infinitely many sentences in Γ'.

fulfill the claim of the corresponding formalistic story. Since the language of Zermelo-Fraenkel is able to describe something like "objects defined by Γ", the same substantive story applies to set theory. Here it is not clear if it is again possible to fulfil the claim of the corresponding formalistic story. This is at least partly caused by the difference between the two kinds of mathematics which are studied here within constructive mathematics. Nevertheless, mathematicians usually would accept that $(1)^+$ is proven since there is a proper substantive story. If one assumes that $(1)^+$ is true, this will not directly imply (1) since one could use set-theoretic constructions which are not allowed in convincing arguments for (1).

For further considerations one has to be careful because they belong to different kinds of mathematics. One could use some kind of mathematics which is able to study these different kinds of mathematics. But then, we would end up in some bigger story. To avoid this, we only give some intuitive arguments. Since $(1)^+$ and (1) basically say that there are convincing arguments for $(2)^+$ and (2) respectively, one may say that $(1)^+$ implies $(2)^+$ and (1) implies (2). Furthermore, one will not be able to find cases where $(2)^+$ or (2) is correct and $(1)^+$ respectively (1) is wrong since it is necessary to give a convincing argument to show that something is correct. It is also reasonable to assume that (2) implies $(2)^+$. Indeed, if φ holds for all objects defined by Γ, then it should also hold for all objects defined by Γ inside some set theory.

The following chart sums up the mentioned relations.



9 On the Central Question

It is now time to come back to the Central Question posed in subsection 4.1. We have argued that the process of rewriting should be based on constructive mathematics, and we have given an interpretation of the statements of constructive mathematics for which one does not need to imagine infinitely many natural numbers or (in our application) texts.

However, the observation of subsection 3.4 that the rewriting process often cannot be done by humans is of course still valid.

The answer then depends on whether one regards the statements of con-

structive mathematics in our interpretation as being "less imaginary" than set-theoretic statements. Given that both are on the one hand "intuitive" and on the other do not seem to directly correspond to human experiences, the authors do not see this, but the reader is of course invited to have his or her point of view.

In any case, it seems to the authors that with the approach presented, the stories told (on the rewriting) are the closest to actual physical rewriting that one might hope for if one wants to uphold claims of rewriting that cannot be carried out physically at all.

Nonetheless, even if one considers constructive mathematics to be as imaginary as set-theoretic mathematics, in near future the claim of rewriting will still be the answer when mathematicians are being asked about the essence of what they call proofs. Therefore, a further reflection on this philosophical issue may lead to new insights and especially hidden thesis, which get revealed throughout such discussions, possibly leading to further developments.

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Supplement: Inspirations and the process of writing

What inspired the authors to write this note and to articulate the perspective on mathematics presented here?

In 2012, Shinichi Mochizuki made four papers about "Inter-universal Teichmüller Theory" ([IUTT]) public which contain allegedly a - yet unconfirmed - proof of the abc conjecture, a central conjecture from number theory, public. In the last part of the fourth paper, Mochizuki introduces what he calls the "language of species". The first author assigned to the second author the task for his "Diplomarbeit" to explain "what is going on here", and the second author agreed to this task.

It turned out that "species" are nothing but formulae describing categories, and there are also "mutations" describing functors. For example, a species defines the underlying class of a category via a property which the objects shall satisfy; this seems to be similar to the possible introduction of classes for individual properties in a higher language which can be rewritten in ZFC as discussed in subsection 4.2. This lead to numerous – sometimes hours long discussions. The rather concrete question discussed was: Should the "species" be seen as objects themselves or does Mochizuki merely advocate a particular point of view, maybe a way to do and to write down mathematics? What is or what should be the ontological status of "species"? The authors then realized that to even discuss these question, they have to consider deeper underlying questions on formalisms in mathematics in general and on the ontological status of mathematics in general.

The second author expressed a formalistic point of view: If mathematicians claim to have a proven a result, they actually claim that they can write down such a proof in a formal system, which is by default ZFC, everything else would be imaginary. The first author countered: This does not correspond to what you are doing yourself if you are doing mathematics. In fact, you do not formally write down your "proofs in a formal system", and you are also not able to do so. So your "formalistic proofs" are also purely imaginary. Moreover, your view on mathematics does not correspond to your behavior: I can see you getting excited when you talk about "mathematical objects". One can indeed argue that the objects are purely imaginary (as one can also for the "formalistic proofs"), but when expressing their thoughts, mathematicians do indeed act as it the objects were real; this should be expressed.

The first author stressed then the point of view that questions of realism should be sidestepped by simply looking at what mathematicians do, and for this the ideas of mathematical fictionalism should be considered. Mathematical fictionalism (see e.g. [fictional]) expresses the thought that even though mathematical objects are not or might not be real, mathematicians tell themselves a story in which they are real. According to fictionalism this

is similar to the usual process of story telling with novels. The second author emphasize that be that as it may, mathematicians in their quest for certainty want to be sure that their arguments follow formally from clearly outlined axioms.

This stress on the formalistic foundations of mathematics became of importance when the authors wanted to write about the proper interpretation of statements on formal sentences. Here, however, a crucial problem came to light: What does claim a "The argument could be written in ZFC." actually mean? What if such a writing cannot be done by humans? What if the writing can (or realistically could) be done by computers but then the formal proof could not be checked by humans?

A statement like "There is a formal proof of Fermat's last theorem in ZFC" does indeed "feel" more real than the very starting point of arithmetic: Counting never stops, or formulated more "platonistically": Every natural number has a successor. The first author still remains skeptical of asserting a label of "less imaginary" to the first claim, because, well, the alleged formal proof *is* imaginary. Does it then make sense to label it as "less imaginary"? Such a qualification should at least require some kind of argument in favor of the possibility for humans to obtain such a proof in a physical sense.

In any case, statements of rewriting in the form of "it could be done" are important in the story and the arguments of mathematics as told today, and it is a challenge to come up with a meaning of such statements. The authors have attempted to give such a meaning in this note, following ideas of constructive mathematics. They invite everybody to reflect on this problem for themselves.