SUMMARY OF DELIGNE-MUMFORD THEORY
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Contents

This summary contains some of the results of our upcoming lecture note [13].

1. Riemannian Surfaces and Deligne-Mumford Type Spaces

In this section we recall the results in [13] which we need for the construction of polyfolds in symplectic field theory.

1.1. **Basic Considerations.**

Consider the noded Riemann surface α

$$\alpha := (S, j, M, D),$$

where (S, j) is a closed (not necessarily connected) Riemann surface having the complex structure j, and where $M \subset S$ is a finite subset of points, called marked points. We do not assume the set M to be ordered. The set D consists of finitely many unordered pairs $\{x, y\}$ of points in S satisfying $x \neq y$. Moreover, $\{x, y\} \cap \{x', y'\} \neq \emptyset$ implies the equality of the two sets. Denoting by $|D|$ the collection of all points contained in the pairs of D, we assume, in addition, that $|D|$ and M are disjoint. The 4-tuplet (S, j, M, D) is called connected, if the topological space $\overline{S}$ obtained from S by identifying for every $\{x, y\}$ in D the points x and y, is a connected space. The two 4-tuplets (S, j, M, D) and (S', j', M', D') are called isomorphic if there exists a biholomorphic map

$$\phi : (S, j) \to (S', j')$$

between the Riemann surfaces, i.e., a smooth diffeomorphism $\phi : S \to S'$ satisfying

$$(T\phi) \circ j = j' \circ (T\phi),$$

such that

$$\phi(M) = M'$$

and

$$D' = \phi(D) := \{\{\phi(x), \phi(y)\} \mid \{x, y\} \in D\}\,.$$

In the following we denote by

$$[S, j, M, D]$$

the equivalence class consisting of all noded Riemann surfaces isomorphic to (S, j, M, D). The arithmetic genus g_a of the 4-tuplet (S, j, M, D) is, as usual, the integer

$$g_a = 1 + \sharp D + \sum_C [g(C) - 1].$$

Here, the sum is taken over all connected components C of S and $\sharp D$ denotes the number of pairs in D. Moreover, $g(C)$ is the genus of the component C. The tuplet (S, j, M, D) is called stable if the associated automorphism group G of holomorphic diffeomorphisms is finite.

Proposition 1.1. *If the connected nodal Riemann surface α is stable, then $2g_a + \sharp M \geq 3$. Moreover, the connected α is stable if and only if for every component C of S the following inequality holds,*

$$2g(C) + \sharp M_C \geq 3,$$

where M_C are those points on C which come from points in M and in $|D|$.

In the following we denote by $\overline{\mathcal{N}}$ the collection of all isomorphism classes $[S, j, M, D]$ of connected stable elements. We denote by $\overline{\mathcal{N}}_{g,\mu}$ the subset of $\overline{\mathcal{N}}$ consisting of all elements having arithmetic genus $g_a = g$ and $\mu = \sharp M$ marked points.

We shall write $\mathcal{N}$ for the subset of $\overline{\mathcal{N}}$ consisting of all equivalence classes of stable elements $(S, j, M, \emptyset)$, where S is connected. These spaces carry natural topologies, see also [6]. In order to proceed we introduce the notion of "type" or "nodal pattern". Let $\alpha = (S, j, M, D)$ represent a stable (connected) class $[\alpha] \in \overline{\mathcal{N}}$. We associate with α a graph T decorated with additional data as follows. The vertices of T correspond to the connected components C of S. Each vertex C carries a weight which is its genus $g(C)$, as well as a number $m(C)$ which is the number of marked points from M on C. We draw an edge between C and C' for each un-ordered pair $\{x, y\}$ in D satisfying $x \in C$ and $y \in C'$ (note that $C = C'$ is allowed). Two such graphs with the additional data are isomorphic provided there exists an isomorphism of the underlying graphs preserving the additional data. Let us write τ for the isomorphism class of graphs (T, g, m). We call τ the type or nodal pattern of (S, j, M, D). Fix a type τ and consider the subset $\mathcal{N}_\tau$ of $\overline{\mathcal{N}}$ consisting of all isomorphism classes $[S, j, M, D]$ having type τ. The following result is easily established.

Lemma 1.2. *If $2g+\mu \geq 3$, then the number of stable types τ satisfying $\mathcal{N}_\tau \subset \overline{\mathcal{N}}_{g,\mu}$ is finite.*

1.2. **Cauchy-Riemann Operator and Kodaira Differential.** Abbreviating the nodal Riemann surface by $\alpha := (S, j, M, D)$ we are going to define a distinguished finite-dimensional complex vector space $H^1(\alpha)$. Consider the holomorphic vector bundle $TS \to S$ and denote by $\Gamma(TS; M, D)$ the complex vector space of all smooth sections which vanish at the points in M and the points in D. Denote by $\Omega^{0,1}(TS, TS)$ the space of all smooth sections η of the bundle $\mathrm{Hom}_{\mathbb{R}}(TS, TS)$ so that fibre-wise the maps are complex anti-linear for the structure j. This means that $\eta(z) : T_zS \to T_zS$ satisfies

$$\eta(z)j(z) + j(z)\eta(z) = 0.$$

For convenience of notation we abbreviate for given $\alpha = (S, j, M, D)$ the space of sections by

$$\Omega^{0,1}(\alpha) := \Omega^{0,1}(T(S, j), T(S, j)).$$

and

$$\Gamma(\alpha) := \Gamma(T(S, j); M, D).$$

Note that the points in D and M enter the definition of Γ but not the definition of $\Omega^{0,1}$. The Cauchy-Riemann operator is the complex linear operator

$$\overline{\partial} : \Gamma(TS; M, D) \to \Omega^{0,1}(TS, TS),$$

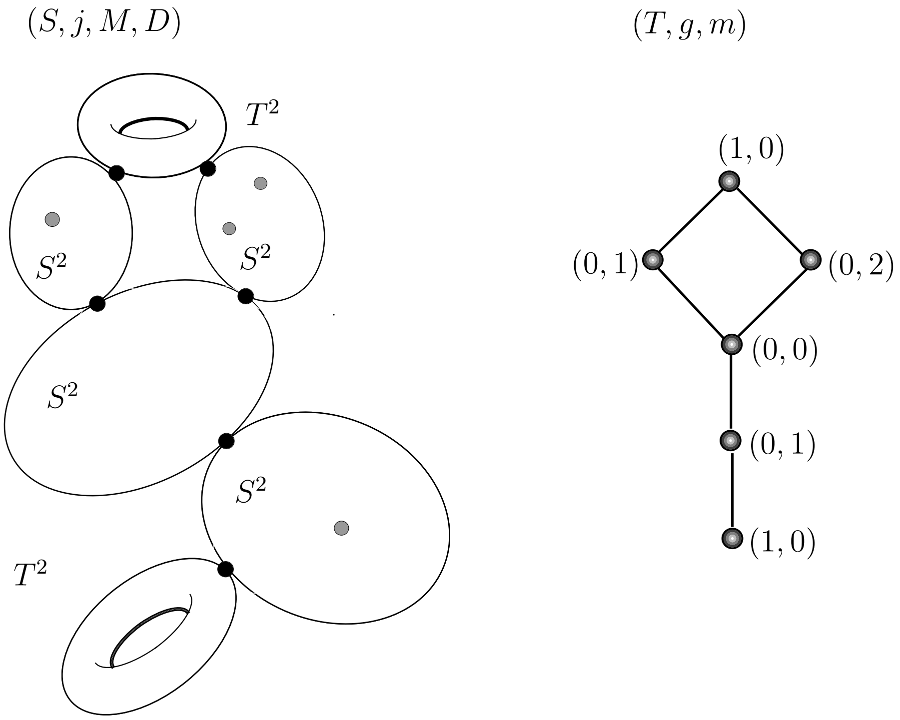

FIGURE 1. A noded Riemann surface (S, j, M, D) and its associated graph (T, g, m)

in holomorphic coordinates given by

$$\overline{\partial} f(z) = \frac{\partial f}{\partial \overline{z}}(z) d\overline{z}.$$

The following proposition is well-known. It can be derived from the Riemann-Roch or the Atiyah-Singer index theorem, together with the positivity of intersections of holomorphic sections and the definition of the first Chern class; see for example [25].

Proposition 1.3. *For a stable noded surface* (S, j, M, D) *the operator* $\overline{\partial}$ *is an injective complex linear Fredholm operator whose index is equal to* $3 - 3g_a - \sharp M - \sharp D$.

If image$(\overline{\partial}) \subset \Omega^{0,1}(TS, TS)$ is the range of the linear operator $\overline{\partial}$, we introduce the quotient space

$$H^1(\alpha) := \Omega^{0,1}(TS, TS)/\text{image}(\overline{\partial}).$$

It is a complex vector space of dimension $3g_a + \#M - \#D - 3$ and can be viewed, as the notation indicates, as a cohomology group. The dual space of $H^1(\alpha)$ denoted by $Q(\alpha)$ can be identified with the vector space of meromorphic quadratic differentials on S with poles of order at most one at the points in $|D| \cup M$.

If G denotes the automorphism group of $\alpha = (S, j, M, D)$, there exists a natural linear representation of G on $H^1(\alpha)$. We denote it by

$$\rho_0 : G \to GL(H^1(\alpha)).$$

Given a connected nodal surface (S, j, M, D) we consider a smooth family of complex structures $v \to j(v)$ satisfying $j(0) = j$ where v belongs to a neighborhood V of 0 in a finite-dimensional complex vector space E. We will call $v \to j(v)$ a deformation of j or of α. Since $j(v)^2 = -1$ for all $v \in V$, the derivative satisfies

$$[Dj(v)\delta v] \circ j(v) = -j(v) \circ [Dj(v)\delta v]$$

for all $(v, \delta v) \in TV$. Hence we may view the derivative $Dj(v)$ at $v \in V$ as a real linear map

$$Dj(v) : E \to \Omega^{0,1}(\alpha_v),$$

where the nodal surface α_v is abbreviated by

$$\alpha_v = (S, j(v), M, D).$$

The map $Dj(v)$ induces a real linear map into the quotient space, the so-called Kodaira differential,

$$[Dj(v)] : E \to H^1(\alpha_v).$$

The following definitions are useful.

Definition 1.4. *Fix a connected and stable $\alpha = (S, j, M, D)$. Let E be a finite dimensional complex vector space and let $V \subset E$ be an open neighborhood of the origin 0. Consider a smooth family $v \mapsto j(v)$ of complex structures on S parameterized by $v \in V$. The family $v \mapsto j(v)$ is called*

- *a deformation of α, if $j(0) = j$.*
- *effective at the point $v_0 \in V$, if the Kodaira differential $[Dj(v_0)]$ is a real linear isomorphism.*
- *effective if it is effective at every point $v \in V$.*
- *complex at the point $v_0 \in V$, if the Kodaira differential $[Dj(v_0)]$ is complex linear.*
- *complex, if it is complex at every point.*

1.3. **Orbifold Structure with Fixed Nodal Type.** Let G be the automorphism group of (S, j, M, D). By the stability condition on (S, j, M, D), the group G is finite. A natural representation of G is a representation $\rho : G \to GL(E)$ isomorphic to ρ_0. We abbreviate the action of G on E by

$$g * v := \rho(g)[v], \quad v \in E.$$

Definition 1.5. *Given a natural representation $\rho : G \to GL(E)$, where E is a finite dimensional complex vector space, and a given family $v \mapsto j(v)$ of complex structures on S defined on a G-invariant neighborhood $V \subset E$ of the origin $0 \in E$, then the pair $(\rho, [v \mapsto j(v)])$ is called a* **symmetric family** *if every automorphism $g \in G$ of the noded surface (S, j, M, D) induces an isomorphism*

$$g : (S, j(v), M, D) \to (S, j(g * v), M, D)$$

for every $v \in V$.

Definition 1.6. *A family $v \to j(v)$ of complex structures on S is called a* **good family**, *if it is smooth, symmetric and effective.*

There always exists such a family according to the next theorem proved [13].

Theorem 1.7. *For every connected stable nodal surface (S, j, M, D) there exists a good family $v \to j(v)$ which is complex.*

The space $\overline{\mathcal{N}}$ and therefore its subsets carry a natural topology $\mathcal{T}$. We will assume in the following that $\overline{\mathcal{N}}$ is equipped with this topology. We also refer the reader to [6]. The restriction of the topology to $\mathcal{N}_\tau$ for a stable type τ can be described as follows. Given $\alpha := (S, j, M, D)$ of type τ, then a typical element of a neighborhood basis of the class $[\alpha]$ consists of all classes $[S, k, M, D]$, where the complex structure k varies in a C^∞-neighborhood U of j. There is no loss of generality in requiring, in addition, that $k = j$ on a sufficiently small open neighborhood of $M \cup |D|$.

The set of isomorphism classes of our objects (S, j, M, D) cannot be described as a manifold due to the presence of automorphism groups of the objects. The structure, however, can be described by an orbifold. Here the concept of a local chart in manifolds is replaced by the notion of a uniformizer which we briefly recall next. We give a definition which is useful for our purposes.

Definition 1.8. *If $\mathcal{U}$ is a connected topological Hausdorff space, then a* **uniformizer** *(of dimension n) for $\mathcal{U}$ is a multiplet*

$$(E, G, V, \mathcal{U}, p),$$

where E is a complex n-dimensional vector space and G is a finite group acting on E via a linear representation by

$$g * e := \rho(g) \cdot e, \quad e \in E, g \in G$$

where $\rho : G \to GL(E)$ is a homomorphism of groups. Moreover, $V \subset E$ is a G-invariant open neighborhood of the origin in E and

$$p : V \to \mathcal{U}$$

is a continuous surjective map which is invariant under the group action, $p(g * e) = p(e)$ for all $e \in V$ and $g \in G$, and which induces a homeomorphism

$$p : G \backslash V \to \mathcal{U}$$

between the space of G-orbits on V and the topological space $\mathcal{U}$[1]

In the case at hand this will look as follows. We take a connected and stable noded Riemann surface $\alpha = (S, j, M, D)$ of type τ. Then the uniformizer for an open neighboorhod $\mathcal{U} \subset \mathcal{N}_\tau$ of the class $[\alpha]$ will be the following multiplet $(E, G, V, \mathcal{U}, p)$. The group G is the finite automorphism group of α. We will have a linear representation $\rho : G \to GL(E)$, where E is a finite dimensional complex vector space which is equivariantly isomorphic to the distinguished complex vector space $H^1(\alpha)$. The topological space V is an open G-invariant neighborhood of $0 \in E$. The map p will be associated with a good family $v \mapsto j(v), v \in V$, of complex structures on S satisfying $j(0) = j$ and a small disk structure (this is essentially a finite union of disks centered around the points in $|D|$, which is invariant under the G-action. The precise definition will be given in Definition 1.12.) belonging to the good family. It is defined by

$$p(v) = [S, j(v), M, D] \in \mathcal{U}.$$

The compatibility conditions for different uniformizers, which replace the transition maps for manifolds, and allow us to define the concept of an orbifold are recalled in the appendix for the convenience of the reader.

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[1] Note that the group acts by orientation preserving maps, since these are complex linear. For the same reason the fixed point set of a nontrivial $\rho(g)$ has at least real co-dimension two.

Definition 1.9. *Let us denote by* $\mathcal{F}_\tau$ *the collection of all uniformizers coming from good families associated with a stable connected* α *of type* τ*, and let us denote by* $\mathcal{F}_\tau^{\mathbb{C}} \subset \mathcal{F}_\tau$ *the subset of uniformizers coming from good families which are, in addition, complex in the sense of Definition 1.4.*

The following result summarizes standard knowledge.

Theorem 1.10. *The collection* $\mathcal{F}_\tau$ *defines a smooth orbifold structure on* $\mathcal{N}_\tau$*. It is called the natural orbifold structure.*

We shall also establish a holomorphic structure on $\mathcal{N}_\tau$.

Theorem 1.11. *The collection* $\mathcal{F}_\tau^{\mathbb{C}}$ *defines a holomorphic orbifold structure on* $\mathcal{N}_\tau$*.*

The following will be important for us. Assume that we are given a good family $v \rightarrow j(v)$ so that $v \rightarrow [S, j(v), Q, D]$ is a uniformizer for an open neighborhood of $[S, j, Q, D]$ in $\mathcal{N}_\tau$. Fix $g \in G$ there exists a uniquely determined smooth map Φ which associates to Q' close to Q and k' C^∞-close to j a diffeomorphism of S close to g and a smooth map $(Q', k') \rightarrow v(Q', k')$ so that

$$\Phi(Q', k') : (S, k', Q', D) \rightarrow (S, j(v(Q', k')), Q, D)$$

is biholomorphic.

1.4. **Gluing Profiles and Plumbing.** As a straightforward consequence consequence of Theorem 1.10, the collection $\overline{\mathcal{N}}$ of all isomorphism classes $[S, j, M, D]$ of connected stable elements admits a stratification

$$\overline{\mathcal{N}} = \bigcup_\tau \mathcal{N}_\tau,$$

so that each stratum has a natural smooth orbifold structure. The main fact is that there it is possible to equip $\overline{\mathcal{N}}$ with a smooth orbifold structure inducing the natural ones on the $\mathcal{N}_\tau$. These global structures are not unique. However, those of interest for us, depend on choices which can be easily described. These global structures are designed to be compatible with SFT-constructions, where we have to look at isomorphism classes of stable maps defined on noded Riemann surfaces.

We need several concepts and begin with the notion of a small disk structure.

Definition 1.12. *Let* (S, j, M, D) *be stable of type* τ *and assume*

$$v \rightarrow j(v), \ v \in V \subset E,$$

is a(perhaps complex) good family having the property that $j(v) = j$ *near* $M \cup |D|$. *A* **small disk structure** *associated to the family* $j(v)$ *assigns to every nodal pair* $\{x, y\}$ *disks* D_x *and* D_y, *having smooth boundaries and centered at* x *and* y, *so that they are mutually disjoint and contain one special point* x *or* y. *Moreover, the union of these disks is invariant under the group action* G. *Moreover,* $j(v) = j$ *on these disks.*

Such a good family which is constant near $M \cup |D|$ can always be chosen.

Proposition 1.13. *Given a good family* $j(v)$ *which is constant on an open neighborhood* U *of* $M \cup |D|$ *there exists a small disk structure associated with* $j(v)$ *such that the disks are all contained in* U.

The proof will follow from Proposition 2.11 together with the following lemma.

Lemma 1.14. *Let* S *be a nodal Riemann surface with the automorphism group* G. *Then given an open neighborhood* U *of* $|D|$, *there exist mutually disks* D_x, $x \in |D|$, *with smooth boundaries such that the union* $\bigcup_{x\in|D|} D_x$ *is invariant under* $g \in G$ *and* $\bigcup_{x\in|D|} D_x \subset U$.

The invariance statement in the lemma implies, with respect to holomorphic polar coordinates in these disks, that the elements in G act (on the union of disks) like a combination of a permutation followed (or preceded) by a rotation. Since G is a finite group we can always find such a disk family of arbitrarily small size. To be more precise, if we fix biholomorphic maps

$$\overline{h}_x : (D, 0) \to (D_x, x)$$

for all x occurring in some pair in D, then

$$\overline{h}_{g(x)}^{-1} \circ g \circ \overline{h}_x : (D, 0) \to (D, 0)$$

is a rotation of the closed standard unit disk D in $\mathbb{C}$. Next we introduce the notion of a nodal identifier.

Definition 1.15. *Given* (S, j, M, D), *a good family* $j(v)$ *and a small disk structure having fixed disk maps* $\overline{h}_x$, $x \in |D|$, *then a* **compatible nodal identifier** *consists of a collection of complex antilinear maps*

$$\varphi_{x,y} : T_yS \to T_xS$$

for every ordered pair (x, y) *and* (y, x) *associated with the set* $\{x, y\} \in D$, *satisfying*

$$\varphi_{x,y} = \varphi_{y,x}^{-1},$$

and such that the complex antilinear map

$$\varphi := T\overline{h}_y(0)^{-1} \circ \varphi_{y,x} \circ T\overline{h}_x(0) : \mathbb{C} \to \mathbb{C}$$

is the complex conjugation $z \mapsto \overline{z}$.

The following proposition is now obvious.

Proposition 1.16. *Given* (S, j, M, D) *and a small disk structure and associated disk maps, there exists a unique compatible nodal identifier.*

Referring in the following to a small disk structure we will always include the choice of the disk maps $\overline{h}_x$ which will fix the nodal identifier. A consequence of these definitions is the following proposition.

Proposition 1.17. *For every nodal pair* $\{x, y\} \in D$ *and every* $g \in G$, *the map*

$$Tg(x)^{-1} \circ \varphi_{g(x),g(y)} \circ Tg(y) \circ \varphi_{y,x} : T_xS \to T_xS$$

is unitary.

Starting with the noded surface (S, j, M, D) and the good family $v \mapsto j(v)$ where $v \in V \subset E$, we fix a compatible nodal identifier. The group G acts linearly on E in the way described before. With every nodal pair $\{x, y\}$ we associate the copy $\mathbb{C}_{\{x,y\}}$ of the complex plane $\mathbb{C}$ and define the complex vector space N as the direct sum

$$N = \bigoplus_{\{x,y\}\in D} \mathbb{C}_{\{x,y\}}.$$

Our first aim is the extension of the G-action ρ from E to the larger vector space $E \times N$. Having fixed nodal identifiers we define, using Proposition 1.17, the **phase function** $\sigma_{\{x,y\}}(g) \in S^1$ for $g \in G$ by

$$\sigma_{\{x,y\}}(g) \cdot h = \varphi_{x,y} \circ Tg(y)^{-1} \circ \varphi_{g(y),g(x)} \circ Tg(x)[h] \in T_xS$$

for all $h \in T_xS$. We have

Proposition 1.18. *Phase functions are well-defined i.e. independent of the order of* x *and* y *in the pair* $\{x, y\}$, *and take values in* S^1.

If $\{x, y\} \in D$ is a nodal pair we introduce the positive holomorphic polar coordinates at x by

$$h_x(s, t) = \overline{h}_x(e^{-2\pi(s+it)}), \quad s \geq 0,$$

and the negative holomorphic polar coordinates at y by

$$h_y(s', t') = \overline{h}_y(e^{2\pi(s'+it')}), \quad s' \leq 0.$$

Assume now $g \in G$. Then $g : D_x \to D_{g(x)}$ and $g : D_y \to D_{g(y)}$ can be expressed in the distinguished holomorphic polar coordinates as follows

$$\begin{aligned} g^+(s,t) &= h_{g(x)}^{-1} \circ g \circ h_x(s,t) = (s, t+\vartheta^+) \\ g^-(s',t') &= h_{g(y)}^{-1} \circ g \circ h_y(s',t') = (s', t'+\vartheta^-), \end{aligned} \tag{1.1}$$

where $s \geq 0$ and $s' \leq 0$. The phase function $\sigma_{x,y}(g)$ and the phases ϑ^+ and ϑ^- are related as follows.

Lemma 1.19.

$$\sigma_{\{x,y\}}(g) = e^{-2\pi i(\vartheta^+ - \vartheta^-)}.$$

Next we define the G-action

$$\sigma_N : G \to GL(N)$$

on N as follows. Let $a = \{a_{\{x,y\}}\}$ and $b = \{b_{\{x,y\}}\}$ be elements of N. Then $\sigma_N(g)$ is defined by

$$b = \sigma_N(g)a,$$

i.e.,

$$b_{\{g(x),g(y)\}} = \sigma_{\{x,y\}}(g) \cdot a_{\{x,y\}}.$$

for all pairs $\{x,y\} \in D$.

Definition 1.20. *We call the action*

$$\sigma = (\sigma_N, \sigma_E) : G \to GL(N) \times GL(E)$$

on $E \times N$, the natural extension of the action $\sigma_E = \rho$ of G associated with a choice of disk structure and disc maps.

In order to describe a full neighborhood of a class $[S,j,M,D]$ of nodal curves in $\overline{\mathcal{N}}$ (and not only in $\mathcal{N}_\tau$) we shall use a gluing technique. It will be crucial for the construction of smooth and holomorphic uniformizers in $\overline{\mathcal{N}}$. We start with the concept of a gluing profile.

Definition 1.21. *A* **gluing profile** *is a smooth diffeomorphism*

$$\varphi : (0,1] \to [0,\infty).$$

Consider a stable nodal surface (S,j,M,D) of type τ. Assume for the following that a gluing profile φ is fixed once and for all. With a nodal pair $\{x,y\}$ we associate a gluing parameter $a_{\{x,y\}} \in D \subset \mathbb{C}$ varying in the closed unit disk, and define the associated object $Z_{a_{\{x,y\}}}^{\{x,y\}}$ as follows.

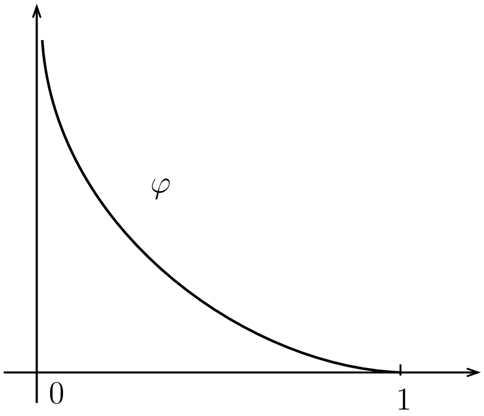

FIGURE 2. A graph of a gluing profile φ

If $a_{\{x,y\}} = 0$, we define $Z_0^{\{x,y\}}$ as the disjoint union of the corresponding disks from the small disk structure

$$Z_0^{\{x,y\}} = (D_x, x) \cup (D_y, y).$$

If $a_{\{x,y\}} \neq 0$, we first take the polar decomposition

$$a_{\{x,y\}} = re^{-2\pi i\vartheta}$$

with $r \in (0,1]$, and define using the gluing profile φ,

$$R := \varphi(r).$$

This way we have associated with the parameter $a_{\{x,y\}} \in D \setminus \{0\}$ the pair $(R, \vartheta) \in \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}$.

Take the positive holomorphic polar coordinates at x by defining

$$h_x(s,t) = \overline{h}_x(e^{-2\pi(s+it)}), \quad s \geq 0$$

and the negative holomorphic polar coordinates at y by defining

$$h_y(s',t') = \overline{h}_y(e^{2\pi(s'+it')}), \quad s' \leq 0.$$

We could of course interchange the role of x and y. Define next the annuli $D_x^R \subset D_x$ and $D_y^{-R} \subset D_y$ by

$$\begin{aligned} D_x^R &: = \{z \in D_x |\, z = h_x(s,t) \text{ for } s \in [0,R] \text{ and } t \in \mathbb{R}\} \\ D_y^{-R} &: = \{z' \in D_y |\, z' = h_y(s',t') \text{ for } s' \in [-R,0] \text{ and } t' \in \mathbb{R}\}. \end{aligned}$$

Now we identify the two annuli D_x^R and D_y^{-R} as follows. The points z and z' are equivalent if

$$s - s' = R \quad \text{and} \quad t - t' = \vartheta \quad (\text{mod } 1).$$

We denote the equivalence class by $[z]_x^{a_{\{x,y\}}} = [z']_y^{a_{\{x,y\}}}$. The subscripts x and y allow us to keep track where the points z and z' come from. Given the gluing parameter $a = \{a_{\{x,y\}}\}$ we cut out, for every nodal pair $\{x, y\}$, the two disks D_x and D_y. If $a_{\{x,y\}} \neq 0$, we replace the two disks by the holomorphic cylinder $Z_{a_{\{x,y\}}}^{\{x,y\}}$ whose boundaries come from the boundaries of the disks.

If $a_{\{x,y\}} = 0$, we put the disks $D_x \bigcup D_y$ back in, so that in the end we did not do anything. This way we obtain the new, possibly unnoded, Riemann surface (S_a, j_a, M_a, D_a), where M_a and D_a are induced by M and by those pairs in D for which $a_{\{x,y\}} = 0$. If, for example, $a_{\{x,y\}} \neq 0$ for all pairs $\{x, y\} \in D$, then $D_a = \emptyset$ and we obtain a closed Riemann surface, which is unnoded and which carries the marked points M_a. Given a good family $v \mapsto j(v)$ of complex structures on S for v varying in V, the above construction induces the family

$$(a, v) \mapsto (S_a, j(a, v), M_a, D_a).$$

We should remark that given a nodal pair, say $P = \{x, y\}$, we have chosen positive holomorphic polar coordinates for one and negative ones for the other element of P. So there are two choices (apart from the choice of angles coming from the biholomorphic maps $\overline{h}_x$ and $\overline{h}_y$). This ambiguity results in harmless modifications. Indeed, by making different choices, the resulting family for the same data is naturally isomorphic.

Abbreviate the group action of G on $N \times E$ by

$$g * (a, v) = (\sigma_N(g)a, \sigma_E(g)v)$$

and introduce the notation

$$g(a) = \sigma_N(g)a.$$

Then the next result follows from our definitions. We will assume throughout the following that $j(v) = j$ near the points in $M \cup |D|$ and that the same holds true on the disks of the small disk structure.

Proposition 1.22. *If $v \mapsto j(v)$ is a good family on S, then there exists, for every automorphism $g \in G$ of the noded surface (S, j, M, D),*

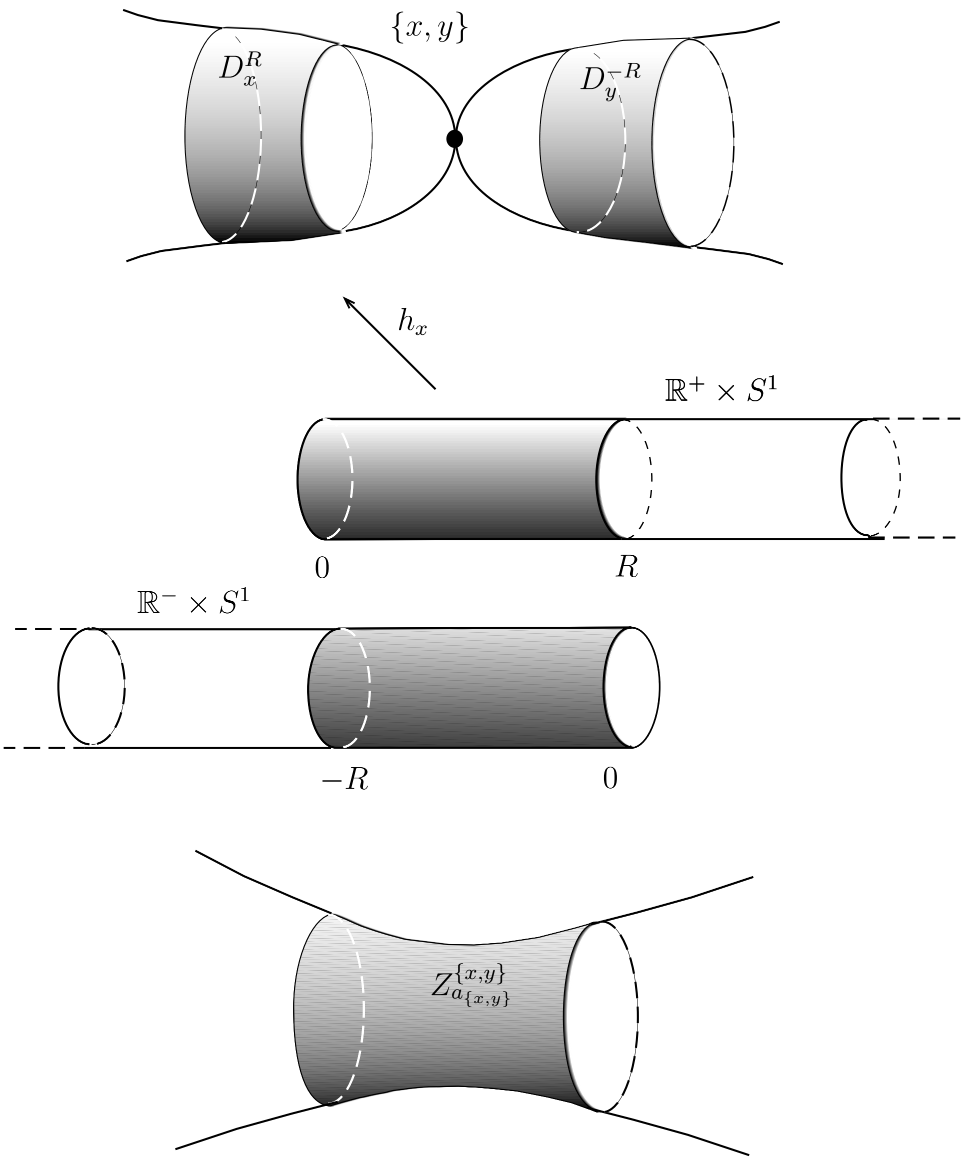

Figure 3. Gluing

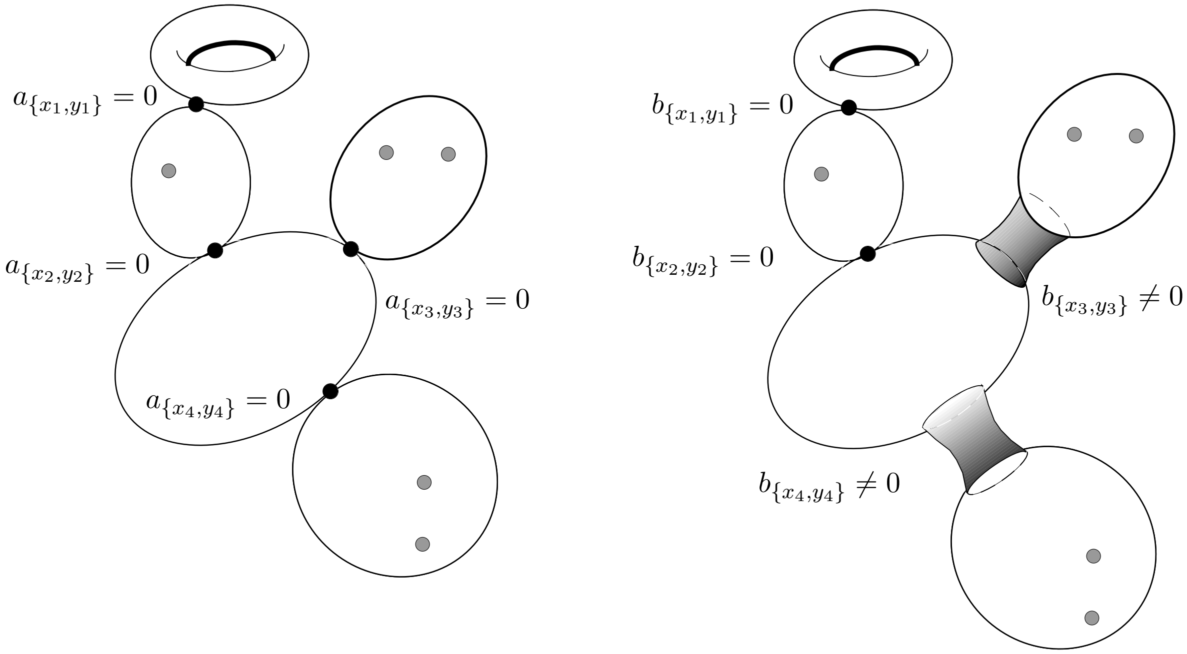

FIGURE 4. Gluing

a natural family g_a *of isomorphisms* ,

$$g_a : (S_a, j(a, v), M_a, D_a) \to (S_{g(a)}, j(g * (a, v)), M_{g(a)}, D_{g(a)}).$$

Proof. Recall that the surface S_a is constructed from the surface S by replacing at every node $\{x, y\} \in D$ with $a_{\{x,y\}} \neq 0$ the two disks D_x and D_y of the small disk structure by the cylinder $Z^{\{x,y\}}_{a_{\{x,y\}}}$. Taking an automorphism $g \in G$ and setting $b = g * a$, we construct the surface S_b in the same way. Define the map $g_a : S_a \to S_b$ first by setting

$$g_a = g$$

on $S \setminus \cup_{x\in|D|} D_x$, and between distinguished cylinders by $g_a : Z^{\{x,y\}}_{a_{\{x,y\}}} \to Z^{\{g(x),g(y)\}}_{b_{\{g(x),g(y)\}}}$, $\{x, y\} \in D$,

$$g_a([z]^{a_{\{x,y\}}}_x) = [g(z)]^{b_{\{g(x),g(y)\}}}_{g(x)}.$$

It remains to show that the map g_a is well-defined. We fix one of the nonzero gluing parameters $a_{\{x,y\}} = re^{-2\pi i\vartheta}$. Writing for the phase function $\sigma_{\{x,y\}}(g) = e^{-2\pi i\sigma(g)}$, with $\sigma(g) \in [0, 1)$ (mod 1), we find for the component $g * a$ under consideration

$$b_{\{g(x),g(y)\}} = \sigma_{\{x,y\}}(g) \cdot a_{\{x,y\}} = e^{-2\pi i\sigma(g)} \cdot a = re^{-2\pi i(\vartheta+\sigma(g))}.$$

In the following we abbreviate $\vartheta' = \vartheta + \sigma(g)$ and recall that $R = \varphi(r)$, with the given gluing profile φ. The automorphism $g \in G$ restricted to the disks $D_x \to D_{g(x)}$ and $D_y \to D_{g(y)}$ is a rotation. In the positive resp. negative polar coordinates compatible with the nodal identifiers, we have the representations,

$$g^+(s, t) = h^{-1}_{g(x)} \circ g \circ h_x(s, t) = (s, t + \vartheta^+)$$

for all $s \geq 0$ and $t \in \mathbb{R}$, and

$$g^-(s', t') = h^{-1}_{g(y)} \circ g \circ h_y(s', t') = (s', t' + \vartheta^-)$$

for all $s' \leq 0$ and $t' \in \mathbb{R}$. The proof of our claim amounts to showing that if $h_x(s, t)$ and $h_y(s', t')$ are equivalent, then $g \circ h_x(s, t)$ is equivalent to $g \circ h_y(s', t')$. The condition that $h_x(s, t)$ and $h_y(s', t')$ are equivalent means that $s - s' = R$ and $t - t' = \vartheta$. Then, using the above representations, the condition that $g \circ h_x(s, t)$ is equivalent to $g \circ h_y(s', t')$ requires that $s - s' = R$ and $(t + \vartheta^+) - (t' + \vartheta^-) = \vartheta + \sigma(g)$. In view of Lemma 1.19 we have $\sigma(g) = \vartheta^+ - \vartheta^-$ so that indeed $(t + \vartheta^+) - (t' + \vartheta^-) = (t - t') + (\vartheta^+ - \vartheta^-) = \vartheta + (\vartheta^+ - \vartheta^-) = \vartheta + \sigma(g)$. Doing this at every nodal pair $\{x, y\} \in D$ we obtain a diffeomorphism

$$g_a : (S_a, j(v, a), M_a, D_a) \to (S_{g(a)}, j(g * (a, v)), M_{g(a)}, D_{g(a)}).$$

Since $v \mapsto j(v)$ is a good family and hence in particular symmetric so that $g \in G$ induces an isomorphism $g : (S, j(v), M, D) \to (S, j(g * v), M, D)$, and since moreover, $j(v) = j$ on the disks of a small disk structure, the diffeomorphism g_a is an isomorphism as claimed. The proof of the proposition is complete.

□

At this point we introduce the natural topology on $\overline{\mathcal{N}}$. Start with a stable $\alpha = (S, j, M, D)$. Introduce a small disk structure compatible with the automorphism group. Also introduce holomorphic polar coordinates on the disks of the disk structure and compatible nodal identifiers. Next fix the following data. A set U consisting of all complex structures k on S which lie in an open C^∞-neighborhood U' of j and in addition satisfy $k = j$ on the small disk structure. Then a number $\varepsilon < 1$. We define $O([\alpha], U, \varepsilon)$ to consist of equivalence classes $[S_a, k_a, M_a, D_a]$ with $(k, a) \in U \times B_\varepsilon(0)$. Denote the collection of all such $O([\alpha], U, \varepsilon)$ by $\mathfrak{V}$. We have the following well-known result which follows from the discussion in Hummel, [18].

Theorem 1.23. *The collection $\mathfrak{V}$ defines a second countable Hausdorff topology on $\overline{\mathcal{N}}$. Each connected component is a compact space.*

As it turns out each point in $\overline{\mathcal{N}}$ has an open neighborhood which is homeomorphic to a quotient of an open set in some $\mathbb{C}$ by a finite group action.

The following result is crucial

Theorem 1.24. *Let the noded surface (S, j, M, D) be connected and stable of type τ. Given a gluing profile φ and a good family $v \mapsto j(v)$ on V, then the associated map*

$$U \times V \to \overline{\mathcal{N}} : (a, v) \to (S_a, j(a, v), M_a, D_a)$$

defines a C^0-uniformizer provided it is restricted to a sufficiently small G-invariant neighborhood of $(0, 0) \in N \times E$. Moreover, restricted to the open subset $U \times V$ of un-noded stable surfaces containing only points of the form (a, v) with $a_{\{x,y\}} \neq 0$ for all $\{x, y\} \in D$, the uniformizer is smoothly compatible with all smooth uniformizers of $\mathcal{N}$.

1.5. **Deligne-Mumford and the Logarithmic Gluing Profile.**

Finally we describe the conditions on the gluing profile under which different uniformizers as introduced above are smoothly compatible. We introduce the following definition.

Definition 1.25. *Let φ be a gluing profile. Denote by $\mathcal{F}^\varphi$ the collection of all uniformizers associated with good families $v \to j(v)$ on stable connected noded surfaces (S, j, M, D) which are obtained by gluing at the nodes as described above. We denote by $\mathcal{F}^{\varphi,\mathbb{C}} \subset \mathcal{F}^\varphi$ the subset of all uniformizers obtained from good families which are, in addition, complex.*

Studying the question of smooth and complex structures we shall first recover the following classical case.

Theorem 1.26. *Choose the special gluing profile* φ *defined by*

$$\varphi(r)=-\frac{1}{2\pi}\ln r,\quad r\in(0,1].$$

Then $\mathcal{F}^{\varphi}$ *defines a smooth orbifold structure on* $\overline{\mathcal{N}}$ *and* $\mathcal{F}^{\varphi,\mathbb{C}}$ *defines a holomorphic orbifold structure on* $\overline{\mathcal{N}}$*.*

Our construction indicates that the gluing profile $\varphi(r)=-\frac{1}{2\pi}\ln r$ is the only profile producing a complex structure. The above construction slightly modified, gives holomorphic uniformizers for the space of connected noded Riemann surfaces with ordered marked points, denoted by $\overline{\mathcal{M}}$. If $2g+m\geq 3$, the subspace $\overline{\mathcal{M}}_{g,m}$ (a connected component) is a compact holomorphic orbifold. This is a classical result. Starting with this result we can recover the Deligne-Mumford result as outlined by Wolpert. He computed in [35] the cohomology class of the Weil-Petersson class and showed that it is rational. Then using a generalization of the Kodaira theorem to V-manifolds by Baily in [1, 2], Wolpert proved that $\overline{\mathcal{M}}_{g,m}$ admits an embedding into projective space. He actually proved it only for $m=0$, but the method should generalize. The results in Theorem 1.26 are, of course, all well known and we refer to [9, 20, 26]. Our presentation and proofs might, however, be non-standard being more in the spirit of our general constructions in SFT.

1.6. **The Gluing Profile for Symplectic Field Theory.** Next we are interested in smooth structures obtained by taking different gluing profiles. The use of such structures is needed in our study of symplectic field theory in order to construct smooth structures on the moduli spaces of stable holomorphic curves. We point out that the gluing constructions in SFT are, in particular, sensitive to the the asymptotic behavior of solutions near punctures. For our studies in SFT we choose another gluing profile which is different from the gluing profile in Theorem 1.26. One of the main results of this paper is the following theorem.

Theorem 1.27. *For the special gluing profile* φ *defined by*

$$\varphi(r)=e^{1/r}-e,\ \ r\in(0,1],$$

the set $\mathcal{F}^{\varphi}$ *defines a smooth orbifold structure on* $\overline{\mathcal{N}}$*.*

There are many gluing profiles which produce smooth or C^k-structures. We do not attempt to classify all the gluing profiles but invite the reader to check the proofs in order to find out the conditions on φ needed. It turns out that suitable growth conditions on the derivatives of φ are required.

Finally we compare the two different orbifold structures on $\overline{\mathcal{N}}$. We write $\overline{\mathcal{N}}^1$ for the one associated with the gluing profile $r \rightarrow e^{1/r} - e$ and $\overline{\mathcal{N}}^2$ for the one associated with $r \rightarrow -\frac{1}{2\pi} \ln r$. We will prove the following statement.

Theorem 1.28. *The identity map*

$$\overline{\mathcal{N}}^1 \rightarrow \overline{\mathcal{N}}^2$$

is induced by a smooth orbifold map. This is not true for the inverse.

The same result is also true in case the marked points are ordered. This will be discussed in Chapter **??**. The following result will be important for our symplectic field theory.

Theorem 1.29. *For the gluing profile φ, defined by*

$$\varphi(r) = e^{1/r} - e, \ \ r \in (0, 1],$$

consider the smooth orbifold structure on $\overline{\mathcal{N}}$ defined by $\mathcal{F}^\varphi$. Then $\overline{\mathcal{N}}$ has a natural homotopy class of almost complex structures.

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