Asymptotic behaviour of capillary problems governed by disjoining pressure potentials: corner

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Abstract

To determine the ascent of liquid near a corner is a well-known problem. It is known, that the sum of the opening angle and the contact angle is a dominating parameter. If the sum is equal or greater than 2π the ascent near the corner is bounded, and if the sum is less then 2π the ascent becomes unbounded. But if the liquid has a very low dielectric constant scarcely nothing is known. In that case the arising boundary value problem holds a disturbance term. This disturbance causes a very different behaviour which is a commonly observed result in physical chemistry. The key to solve this boundary value problem is to modify a Comparison Principle of Concus and Finn. The amazing conclusion is, that this modified principle combined with the disturbance term yields an asymptotic result of the problem. Contrary to the classical case the ascent is independent from the opening angle, it only depends on the distance to the boundary.

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1 Introduction

A common chemical problem is to determine the absorbed amount of liquid in porous materials. In this paper we examine the case of low[1] dielectric constant as one has at liquid helium, oxygen or rare gases. Some pores can be modeled as straight cylinders over a domain Ω. The authors of [1] discussed some special cases where Ω has smooth boundary. Here we regard the easiest non-smooth case, that is if Ω is a corner.

Our essential goal is to determine the arising surface S over the domain Ω. The mathematical description of this problem leads to a boundary value problem. The classical procedure is performed in [2, Chapter 1]. In the present case we have to take into consideration the low dielectric constant of the liquid, which leads to the appearance of, namely the so called *disjoining pressure potential*, see [3, Chapter 11]. That is, as described in [4] we are led to the boundary value problem

$$\begin{aligned} \operatorname{div} \mathbf{T}\, u &= \kappa u + P(x,u) + \lambda && \text{in } \Omega, \\ \nu \cdot \mathbf{T}\, u &= 1 && \text{as } x \to \Sigma. \end{aligned} \tag{1}$$

Here u denotes the graph which describes the arising surface and Σ denotes the boundary of Ω. In (1) we have set $\mathbf{T}\, u = \nabla u (1 + |\nabla u|^2)^{-1/2}$. P is the above mentioned disjoining pressure

[1] That is, if the value of the dielectric constant of the liquid lies between the dielectric constants of the vapour and the solid phase.

potential, ν the outer normal on the cylinder wall, κ the (positive) capillary constant and λ the Lagrange multiplier.

The main task of this paper is to examine the behaviour of solutions of (1). Thereto it is necessary to study the term P. It is given by

$$P = P(x, u(x)) = c \int_{\Omega_s} \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2\right]^{-p/2} \mathrm{d}y, \qquad p > 3. \tag{2}$$

Thereby Ω_s denotes the solid outer domain, see Figure 1, and c denotes some *negative* constant, see [1] or [4].

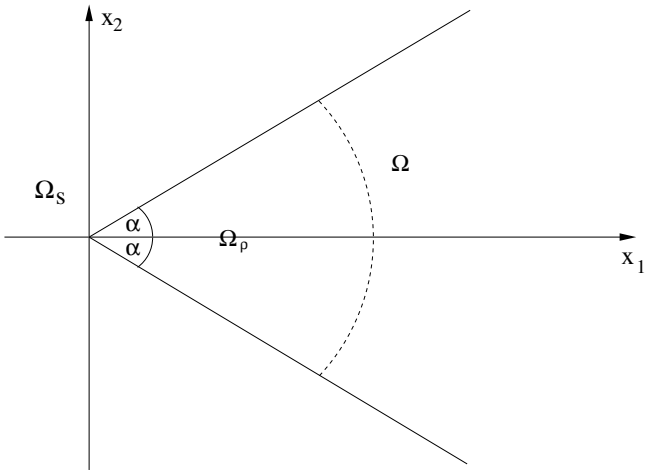

Fig. 1: Wedge domain.

The main tool to study problem (1) is Theorem 1. It is an extension of a Comparison Principle proven by Concus, Finn and Hwang, see [5] and [6]. Here the non-linearity of $\mathbf{T}\,u$ plays a decisive role in the proof of the principle. The actual version of the Comparison Principle, considering the disturbance term P, was proven in [1]. In that paper the monotonicity of P in u was essential for the proof.

Theorem 1
Let $\kappa > 0$ and suppose that $\Sigma = \partial\Omega$ admits a decomposition $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$, where $\Sigma_\beta \in \mathcal{C}^1$ and Σ_0 has one-dimensional Hausdorff measure zero. Let $u, v \in \mathcal{C}^2(\Omega)$ with the properties

(i)	$\operatorname{div} \boldsymbol{T}u - \kappa u - P(x,u)$	$\geq$	$\operatorname{div} \boldsymbol{T}v - \kappa v - P(x,v)$	*in* Ω,
(ii)	u	$\leq$	v	*as* $x \to \Sigma_\alpha$,
(iii)	$\nu \cdot \boldsymbol{T}u$	$\leq$	$\nu \cdot \boldsymbol{T}v$	*as* $x \to \Sigma_\beta$.

Then we have $u \leq v$ in Ω.

As in the classical setting (without disturbance P) we get from Theorem 1 the uniqueness of the solution.

2 Asymptotic results

Formula (2) is given in a general form. In the phyiscally relevant cases we have $p = 6$. Cases with $3 < p < 6$ may arise in small systems without practical importance. Therefore, from now on we consider the special case $p = 6$.

As mentioned before we will examine the situation that Ω is a vertex, that is, a domain bounded by two straight lines. Without loss of generality, we can choose the domain, such that the cusp lies in the origin, symmetric to the x_1-axis. So we have

$$\Omega = \left\{ x \in \mathbb{R}^2 : |x_2| \leq x_1 \tan \alpha \right\},$$

whereby $0 < \alpha < \pi/2$.

To study the behaviour near the cusp we will additionally examine the solution v of the problem (1) over the domain

$$\Omega_\rho = \Omega \cap B_\rho(0).$$

Having attained all these necessary preparations these preparations, we formulate the main theorem.

Theorem 2
Let v be a solution of (1) for $p = 6$ over the domain Ω, $0 < \alpha < \pi/2$. Then there are constants $0 < \rho < 1$, $B \in \mathbb{R}$ and $A \geq 0$, independent of the special solution v considered, so that we have

$$\left| v(r,\theta) - \frac{f(\theta)}{r^3} - \frac{h(\theta)}{r} \right| \leq A \qquad \text{in } \Omega,$$

and

$$\left| v(r,\theta) - \frac{f(\theta)}{r^3} - \frac{h(\theta)}{r} - B \right| \leq Ar \qquad \text{in } \Omega_\rho.$$

Thereby f and h are given by

$$f(\theta) = C \left(3\cot\frac{\alpha+\theta}{2} + 3\cot\frac{\alpha-\theta}{2} + \cot^3\frac{\alpha+\theta}{2} + \cot^3\frac{\alpha-\theta}{2} \right), \qquad C = \frac{-\mathcal{H}}{16\pi\sigma\kappa} > 0,$$

$$h(\theta) = \frac{3}{\kappa} f(\theta) \frac{3f(\theta)f''(\theta) - 9f(\theta)^2 - 4f'(\theta)^2}{\left[9f(\theta)^2 + f'(\theta)^2\right]^{3/2}}.$$

If $\alpha \in (\pi/2, \pi)$, that is, if the solid domain is a wedge, we get the same result as mentioned in Theorem 2.

So we receive from Theorem 2 the following asymptotic result for the solution of (1)

$$\begin{aligned} v \sim \quad & -\frac{\mathcal{H}}{2\pi\sigma\kappa} r^{-3} (\alpha \pm \theta)^{-3} \quad && \text{as } r(\alpha \pm \theta) \to 0 \\ & = -\frac{\mathcal{H}}{2\pi\sigma\kappa} d^{-3} && \text{as } d \to 0, \end{aligned}$$

where d denotes the distance to the boundary of Ω. This is the same qualitative behaviour as gained in other special cases, see [1]. That is the main difference to the classical problem, where the solution essentially depends on the domain Ω.

3 Proof of Theorem 2

A simple consequence of Theorem 1 are the following corollaries.

Corollary 3
Let v be a solution of (1) over the domain Ω and w a suitable comparison function. Then we have:
a) If

$$\begin{array}{lll} \text{(i)} & \operatorname{div} \boldsymbol{T} w - \kappa w - P(x,w) \leq 0 & \text{in } \Omega \\ \text{(ii)} & \nu \cdot \boldsymbol{T} w \geq 1 & \text{on } \Sigma \setminus \{0\}, \end{array}$$

then we have $w \geq v$ in Ω.
b) If

$$\begin{array}{lll} \text{(i)} & \operatorname{div} \boldsymbol{T} w - \kappa w - P(x,w) \geq 0 & \text{in } \Omega \\ \text{(ii)} & \nu \cdot \boldsymbol{T} w \leq 1 & \text{on } \Sigma \setminus \{0\}, \end{array}$$

then we have $w \leq v$ in Ω.

Let be $B_\rho(0) = \{x^2 + y^2 < \rho^2\}$ for a positive constant ρ. Replacing the initial domain Ω with the domain $\Omega_\rho = \Omega \cap B_\rho(0)$ we get an analogous corollary. Thereby we have set $\Gamma_\rho = \Omega \cap \partial B_\rho(0)$ and $\Sigma_\rho = \partial\Omega \cap B_\rho(0)$.

Corollary 4
Let v be a solution of (1) over the domain Ω_ρ, w a suitable comparison function and $0 < \rho < 1$ sufficiently small. Then we have:
a) If

$$\begin{array}{lll} \text{(i)} & \operatorname{div} \boldsymbol{T} w - \kappa w - P(x,w) \leq 0 & \text{in } \Omega_\rho \\ \text{(ii)} & w \geq v & \text{as } x \to \Gamma_\rho \\ \text{(iii)} & \nu \cdot \boldsymbol{T} w \geq 1 & \text{as } x \to \Sigma_\rho \setminus \{0\}, \end{array}$$

then we have $w \geq v$ in Ω_ρ.
b) If

$$\begin{array}{lll} \text{(i)} & \operatorname{div} \boldsymbol{T} w - \kappa w - P(x,w) \geq 0 & \text{in } \Omega_\rho \\ \text{(ii)} & w \leq v & \text{as } x \to \Gamma_\rho \\ \text{(iii)} & \nu \cdot \boldsymbol{T} w \leq 1 & \text{as } x \to \Sigma_\rho \setminus \{0\}, \end{array}$$

then we have $w \leq v$ in Ω_ρ.

These corollaries are the key to determine the asymptotic solution of (1). The idea is to find suitable comparison funtions $w^\pm$ which satisfy conditions (i) and (ii) of Corollary 3 *a*) and *b*), respectively. A convenient choice of $w^\pm$ will lead to an asymptotic result for v. After we recieved the result for the entire domain we modify the comparison functions, $w^\pm$, to apply Corollary 4. This leads to a better result near the cusp.

3.1 Disjoining pressure potential

Here we determine some properties of P which will be required later. After introducing polar coordinates (r,θ) as done in [7, VIII] we obtain $P(x,u(x)) = \tilde{c}\psi(\theta)r^{-3}$, where $\tilde{c} = \mathcal{H}/(48\sigma\pi) < 0$ and

$$\psi(\theta) = 3\cot\frac{\alpha+\theta}{2} + 3\cot\frac{\alpha-\theta}{2} + \cot^3\frac{\alpha+\theta}{2} + \cot^3\frac{\alpha-\theta}{2}.$$

Hence we deduce, that ψ, ψ' and ψ'' are continuous and bounded in $(-\alpha+\varepsilon_0, \alpha+\varepsilon_0)$ for a sufficiently small, positive ε_0.

For later use we state the asymptotic behaviour of ψ and its derivatives as θ tends to $\mp\alpha$. We have

$$\psi(\theta) \sim 8(\alpha\pm\theta)^{-3}, \quad \psi'(\theta) \sim \mp 24(\alpha\pm\theta)^{-4}, \quad \psi''(\theta) \sim 96(\alpha\pm\theta)^{-5} \quad \text{as } \theta \to \mp\alpha. \tag{3}$$

3.2 Requirements in Ω_ρ and Ω

Our next aim is to find an appropriate comparison function to estimate $\operatorname{div}\mathbf{T}\,u - \kappa u - P(x)$. In [1] the divergence of $P(x,u(x))$ was bounded. As we will see in the case of the wedge, the divergence of $\mathbf{T}\,P$ is asymptotically equal to $\kappa h(\theta) r^{-1}$ as r tends to 0 for some non-vanishing function h. So we have to find another suitable comparison function, namely

$$w(\theta,r) = \frac{f(\theta)}{r^3} + \frac{h(\theta)}{r} + A, \tag{4}$$

with

$$f = f(\theta) = -\frac{\tilde{c}}{\kappa}\psi(\theta) \;\text{ and }\; h = h(\theta) = \frac{3}{\kappa}f(\theta)\frac{3f(\theta)f''(\theta) - 9f(\theta)^2 - 4f'(\theta)^2}{\left[9f(\theta)^2 + f'(\theta)^2\right]^{3/2}}, \tag{5}$$

where A and $\tilde{c} = \mathcal{H}/(48\sigma\pi) < 0$ are constants. So we get (see Lemma 5) $h \in \mathcal{C}^2([-\alpha,\alpha])$ and in addition

$$h(\theta) = O([\alpha\pm\theta]^3), \quad h'(\theta) = O([\alpha\pm\theta]^2), \quad h''(\theta) = O([\alpha\pm\theta]^1) \quad \text{as } \theta \to \mp\alpha.$$

Step 1: In this step we consider the behaviour of $\operatorname{div}\mathbf{T}\,w$ in Ω_ρ, for a ρ with $0 < \rho < 1$. That is, we observe the behaviour near the cusp.
From (4) we obtain

$$|\nabla w|^2 = r^{-8}\left\{9f^2 + f'^2 + Q\right\},$$

where we have set

$$Q = 2(3fh + f'h')r^2 + (h^2 + h'^2)r^4 = O([1+[\alpha\pm\theta]^{-2}]r^2). \tag{6}$$

So we obtain the partial derivatives of Q

$$Q_r = 4(3fh + f'h')r + 4(h^2 + h'^2)r^3 = O([1+[\alpha\pm\theta]^{-2}]r) \tag{7}$$

and

$$Q_\theta = 2(3fh + f'h')'r^2 + (h^2 + h'^2)'r^4 = O([1+[\alpha\pm\theta]^{-3}]r^2).$$

Finally, for sufficiently small ρ with $0 < r < \rho < 1$ and $a \in \mathbb{R}$, we have

$$\begin{aligned}\left[r^8 + 9f^2 + f'^2 + Q\right]^a &= \left(9f^2 + f'^2\right)^a \left(1 + \frac{Q + r^8}{9f^2 + f'^2}\right)^a \\ &= \left(9f^2 + f'^2\right)^a \left[1 + O(r^2)\right].\end{aligned} \tag{8}$$

By definition of Q and h, the term $Q/(9f^2 + f'^2)$ is bounded and $9f^2 + f'^2$ is positive. So these transformations are valid.
Recall that the divergence of $\mathbf{T}\, w$ in polar coordinates is given by:

$$\operatorname{div} \mathbf{T}\, w = \frac{1}{r}\left\{\left(\frac{r w_r}{\sqrt{1 + |\nabla w|^2}}\right)_r + \left(\frac{r^{-1} w_\theta}{\sqrt{1 + |\nabla w|^2}}\right)_\theta\right\}.$$

Now we start to compute $\operatorname{div} \mathbf{T}\, w$ beginning with the first term on the right-hand side. There we use the boundedness of h and h' as well as the asymptotic behaviour of f and the relations (6), (7) and (8) to compute

$$\begin{aligned}\left(\frac{r w_r}{\sqrt{1 + |\nabla w|^2}}\right)_r &= \frac{-3f + O(r^2)}{\sqrt{r^8 + 9f^2 + f'^2 + Q}} + \frac{1}{2}\frac{[3fr + O(r^3)] \cdot O([1 + [\alpha \pm \theta]^{-2}]r)}{[r^8 + 9f^2 + f'^2 + Q]^{\frac{3}{2}}} \\ &= \frac{-3f + O(r^2)}{\sqrt{9f^2 + f'^2}}[1 + O(r^2)] + \frac{1}{2}\frac{O([1 + [\alpha \pm \theta]^{-5}]r^2)}{[r^8 + 9f^2 + f'^2 + Q]^{\frac{3}{2}}} \\ &= \frac{-3f}{\sqrt{9f^2 + f'^2}} + O(r^2).\end{aligned}$$

To obtain the term $O(r^2)$ we use the continuity of f and f' in $(-\alpha + \varepsilon_0, \alpha - \varepsilon_0)$ and their asymptotic characteristics (3). In a similar[2] way we treat the next term and get

$$\left(\frac{r^{-1} w_\theta}{\sqrt{1 + |\nabla w|^2}}\right)_\theta = 9f\frac{ff'' - f'^2}{[9f^2 + f'^2]^{\frac{3}{2}}} + O(r^2).$$

So we are able to calculate $\operatorname{div} \mathbf{T}\, w$:

$$\operatorname{div} \mathbf{T}\, w = \frac{3f}{r} \cdot \frac{3ff'' - 9f^2 - 4f'^2}{[9f^2 + f'^2]^{3/2}} + O(r) = \frac{\kappa}{r} h + O(r). \tag{9}$$

To adopt the Comparison Principle and the corresponding corollaries respectively, we derive the following formula, by using formulae (4) and (9) and the definition of P

$$\operatorname{div} \mathbf{T}\, w - \kappa w - P = -\kappa A + O(r).$$

Then we can choose the constant A positive so that we have

$$\operatorname{div} \mathbf{T}\, w - \kappa w - P \leq 0. \tag{10}$$

Otherwise, by choosing A negative, we can achieve $\operatorname{div} \mathbf{T}\, w - \kappa w - P \geq 0$.

[2] Detailed calculations are given in [8].

Step 2: Here we look at $\operatorname{div}\mathbf{T}\,w-\kappa w-P$ in Ω. The aim is to show that $|\operatorname{div}\mathbf{T}\,w-\kappa w-P|<B<\infty$. From the first step one easily concludes this relation in Ω_ρ for $0<\rho<1$. So from now on we consider $r\geq\rho>0$. The "c" which occurs in the next calculations denotes some positive constant. When it appears repeatedly in a calculation, it does not always need to represent the same value.

It is sufficient to examine the case that θ tends to $\pm\alpha$ and r tends to infinity. So we can choose $\varepsilon_0>0$ small enough and r big enough that

$$r^8+9f^2+f'^2+(3fh+f'h')r^2+(h^2+h'^2)r^4\geq r^8+f'^2>f'^2>1 \tag{11}$$

is satisfied. Under this condition we can compute $|\operatorname{div}\mathbf{T}\,w|$, beginning with:

$$\begin{aligned}\left|\left(\frac{rw_r}{\sqrt{1+|\nabla w|^2}}\right)_r\right| &\leq \left|\frac{3f+3hr^2}{r^4\sqrt{1+|\nabla w|^2}}\right|+\left|\frac{1}{2}\frac{(3fr+hr^3)(8r^7+Q_r)}{r^{12}\left[1+|\nabla w|^2\right]^{\frac{3}{2}}}\right|\\ &\leq c\frac{|w_r|}{\sqrt{1+|\nabla w|^2}}\\ &\quad+\left|\frac{3fr^{-4}+hr^{-2}}{\sqrt{1+|\nabla w|^2}}\cdot\frac{4r^8+2(3fh+f'h')r^2+2(h^2+h'^2)r^4}{r^8\left[1+|\nabla w|^2\right]}\right|.\end{aligned}$$

Now, considering that fh, h, h' are bounded, we can further estimate

$$\begin{aligned}\left|\left(\frac{rw_r}{\sqrt{1+|\nabla w|^2}}\right)_r\right| &\leq c\frac{|\nabla w|}{\sqrt{1+|\nabla w|^2}}+\frac{|w_r|}{\sqrt{1+|\nabla w|^2}}\left|\frac{cr^8+f'h'r^2}{r^8\left[1+|\nabla w|^2\right]}\right|\\ &\leq c+\frac{c}{1+|\nabla w|^2}+c\left|\frac{f'h'r^2}{r^8\left[1+|\nabla w|^2\right]}\right|\\ &\overset{(11)}{\leq} c+c\left|\frac{f'h'r^2}{r^8+f'^2}\right|\leq c+c\left|\frac{f'h'r^2}{r^4f'}\right|\leq c.\end{aligned} \tag{12}$$

In the last line we used the inequality of arithmetic and geometric means. We get a similar result for the other term

$$\begin{aligned}\left|\left(\frac{r^{-1}w_\theta}{\sqrt{1+|\nabla w|^2}}\right)_\theta\right| &= \left|\frac{9f(ff''-f'^2)+3\left[3f^2h''-2f(2f'h'-f''h)-f'^2h\right]r^2}{\left[r^8+9f^2+f'^2+Q\right]^{3/2}}\right.\\ &\quad\left.+\frac{[3f(2hh''-h'^2)-4f'hh'+f''h^2]r^4+[h(hh''-h'^2)]r^6+f''r^8+h''r^{10}}{\left[r^8+9f^2+f'^2+Q\right]^{3/2}}\right|.\end{aligned}$$

Therefore by using the asymptotic behaviour of f, h and their derivatives, we can estimate the upper expression for some positive constant c, by three terms:

$$\begin{aligned}&\frac{c(\alpha\pm\theta)^{-11}}{\left[r^8+9f^2+f'^2+Q\right]^{3/2}}+\frac{cr^{10}}{\left[r^8+9f^2+f'^2+Q\right]^{3/2}}+\frac{c(\alpha\pm\theta)^{-5}r^8}{\left[r^8+9f^2+f'^2+Q\right]^{3/2}}\\ &\leq c\frac{(\alpha\pm\theta)^{-11}}{\left[f'^2\right]^{3/2}}+\frac{cr^{10}}{\left[r^8\right]^{3/2}}+cr\frac{(\alpha\pm\theta)^{-5}}{\left[r^8+f'^2\right]^{5/8}}\cdot\frac{r^7}{\left[r^8+f'^2\right]^{7/8}}\leq cr,\end{aligned} \tag{13}$$

where we have used (11). Therefore by combining (12), (13) and the considerations at the beginning of the step we get

$$|\operatorname{div} \mathbf{T}\, w| = r^{-1} \left| \left(\frac{r w_r}{\sqrt{1+|\nabla w|^2}} \right)_r + \left(\frac{r^{-1} w_\theta}{\sqrt{1+|\nabla w|^2}} \right)_\theta \right| \leq c,$$

in $\Omega \setminus \Omega_\rho$. Therefore by choosing A appropriate positive we can estimate in $\Omega \setminus \Omega_\rho$

$$\operatorname{div} \mathbf{T}\, w - \kappa w - P \leq c - \kappa\, h r^{-1} - \kappa A \leq 0, \tag{14}$$

since h is bounded and $r > \rho > 0$.
Otherwise, by choosing A negative, we can achieve $\operatorname{div} \mathbf{T}\, w - \kappa w - P \geq 0$.

3.3 Boundary behaviour

To prove the boundary condition we have to show

$$\nu \cdot \mathbf{T}\, w = \operatorname{sign}(\theta) \frac{r^{-1} w_\theta}{\sqrt{1+|\nabla w|^2}} = 1 \qquad \text{as } \theta \to \pm\alpha$$

for every $r > 0$. To show this we consider at first the case $0 < r < \rho < 1$ and $\theta \in (-\alpha, -\alpha+\varepsilon_0) \cup (\alpha-\varepsilon_0, \alpha)$, for a sufficiently small $\varepsilon_0 > 0$. Therefore we can use the asymptotic behaviour of the occurring functions. In detail, we have

$$\begin{aligned}
\frac{r^{-1} w_\theta}{\sqrt{1+|\nabla w|^2}} &= \frac{f' + h' r^2}{\sqrt{r^8 + 9f^2 + f'^2 + Q}} = \frac{f' + h' r^2}{\sqrt{9f^2 + f'^2}} \left[1 + \frac{r^8 + Q}{9f^2 + f'^2} \right]^{-1/2} \\
&= \frac{f'}{\sqrt{9f^2 + f'^2}} \left(1 + \frac{h'}{f'} r^2 \right) \left[1 + O\left(\frac{f' h'}{9f^2 + f'^2} r^2 \right) \right] \\
&= \frac{f'}{\sqrt{9f^2 + f'^2}} + O\left(\frac{f'^2 h'}{[9f^2 + f'^2]^{3/2}} r^2 \right).
\end{aligned}$$

Since $f \sim c_0(\alpha \pm \theta)^{-3}$ and $f' \sim c_1(\alpha \pm \theta)^{-4}$, with c_0, $c_1 \neq 0$, the last expression tends to ± 1 as θ tends to $\pm\alpha$. The other transformations are valid since h and h' are bounded (see Lemma 5).
Let $x \in \Omega$, $x = r(\cos\theta, \sin\theta)$ for $r > \rho > 0$, $\theta \in (0, \alpha)$. Hence the normal distance between x and Σ is $d = r \sin(\alpha - \theta)$, see Figure 2. To check the behaviour of $\nu \cdot \mathbf{T}\, w$ on Σ afar from the origin, the normal distance of x to Σ has to tend to 0. That is, we have in particular $d = r(\alpha - \theta) + O([r(\alpha - \theta)]^3)$. Thus we get

$$\begin{aligned}
r^2 h'/f' &\sim c_1(\alpha - \theta)[r(\alpha - \theta)]^2, \\
r^8/f'^2 &\sim c_2[r(\alpha - \theta)]^8, \\
9f^2/f'^2 &\sim c_3(\alpha - \theta)^2, \\
2(3fh + f'h')r^2/f'^2 &\sim c_4(\alpha - \theta)^4[r(\alpha - \theta)]^2, \\
(h^2 + h'^2) r^4/f'^2 &\sim c_5(\alpha - \theta)^8[r(\alpha - \theta)]^4,
\end{aligned} \tag{15}$$

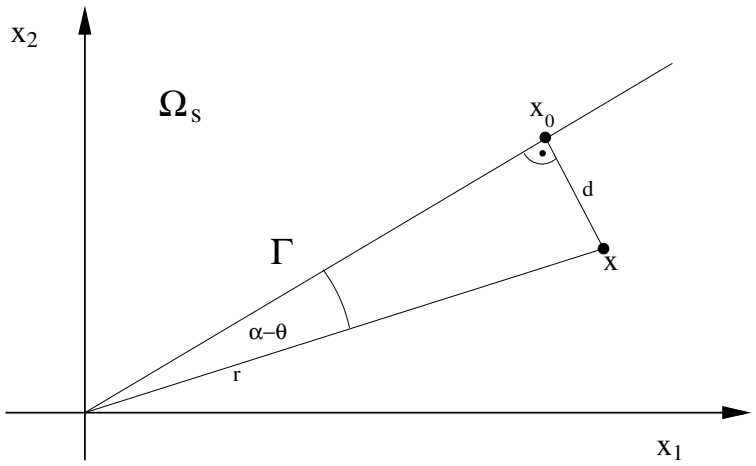

Fig. 2: Normal distance to Σ.

whereby $c_0, \ldots, c_6$ are some constants.
Combining the definition of $\nu \cdot \mathbf{T}\, w$ with (15) we get

$$\frac{r^{-1} w_\theta}{\sqrt{1+|\nabla w|^2}} = \frac{1 + \dfrac{h'}{f'} r^2}{\sqrt{1 + r^8/f'^2 + 9f^2/f'^2 + \dfrac{2(3fh + f'h')r^2 + (h^2 + h'^2)r^4}{f'^2}}},$$

which tends to 1 as x approaches Σ, since all expressions of (15) tend to 0 as $r\sin(\alpha - \theta)$ tends to 0. So the boundary condition is shown.

3.4 Applying the Comparison Principle

Let v be a solution of (1) and $w = fr^{-3} + hr^{-1} + A$. Firstly we apply the Comparison Principle to show that v satisfies $|v - fr^{-3} - hr^{-1}| \leq A$ in Ω for some positive constant A. And therewith we show $|v - fr^{-3} - hr^{-1} - B| < Ar$ in Ω_ρ, for a ρ with $0 < \rho < 1$.
After choosing $A > 0$ sufficiently big we combine (10) and (14) to get

$$\operatorname{div} \mathbf{T}\, w - \kappa w - P \leq 0 \text{ in } \Omega.$$

Therefore condition (i) of Corollary 3.a) is satisfied. By Section 3.3 condition (ii) of Corollary 3.a) is satisfied. So we get $v \leq w$ in Ω. Similarly by choosing A negative we get $v \geq w$. Therewith we have

$$\left| v(r,\theta) - \frac{f(\theta)}{r^3} - \frac{h(\theta)}{r} \right| \leq A \text{ in } \Omega,$$

for some constant $A > 0$.
Considering Ω_ρ instead of Ω and replacing w by $\tilde{w} = fr^{-3} + hr^{-1} + B + Ar$ results in just a slight modification of the upper calculations. Finally after applying Corollary 4 on $\tilde{w}$ we receive the second statement of Theorem 2, which proves the theorem.
The result for $\alpha \in (\pi/2, \pi)$ follows from the calculations above. ■

3.5 Properties of h

Lemma 5
Let h be the function defined by equation (5). Then we have:

a) $h \in \mathcal{C}^2([-\alpha, \alpha])$,

b) $h^{(k)} = O([\alpha \pm \theta]^{3-k})$ *as* θ *tends to* $\mp\alpha$ *and* $k = 0, 1, 2$,

c) h, h' *and* h'' *are bounded,*

d) $f'h' > 0$ *and* $3fh + f'h' > 0$ *as* θ *tends to* $\pm\alpha$.

Proof:
For the derivatives we have got

$$h' = h_1 \left[9f^2 + f'^2\right]^{-5/2} \qquad \text{and} \qquad h'' = h_2 \left[9f^2 + f'^2\right]^{-7/2},$$

where h_1 is a homogeneous polynomial of fifth degree in f, f', f'' and f''' and h_2 a homogeneous polynomial of seventh degree in f, f', f'', f''' and $f^{(4)}$.
Since f is a multiple of ψ, it is sufficient to examine ψ and its derivatives to verify the stated properties. Since $\psi, \ldots, \psi^{(4)}$ are continuous in $(-\alpha+\varepsilon_0, \alpha-\varepsilon_0)$ for a sufficiently small $\varepsilon_0 > 0$, and because of their asymptotic behaviour as θ tends to $\pm\alpha$ we obtain, that h, h' and h'' are continuous in $(-\alpha, \alpha)$.
More precisely we have got

$$\begin{gathered} \psi \sim 8(\alpha \pm \theta)^{-3}, \qquad \psi' \sim \mp 24(\alpha \pm \theta)^{-4}, \qquad \psi'' \sim 96(\alpha \pm \theta)^{-5}, \\ \psi''' \sim \mp 480(\alpha \pm \theta)^{-6}, \qquad \psi^{(4)} \sim 2880(\alpha \pm \theta)^{-7}, \end{gathered} \tag{16}$$

as θ tends to $\mp\alpha$. From (16) we obtain their behaviour near the boundary. So we see that in the definition of h the term $3ff'' - 4f'^2$ vanishes as θ tends to $\pm\alpha$. In the same way the highest singularities of h_1 and h_2 vanish, and we get

$$h^{(k)} \sim c_k(\alpha \pm \theta)^{3-k} \qquad \text{as } \theta \to \mp\alpha, \qquad k = 0, 1, 2$$

for some constants $c_k \neq 0$, depending only on k. Thus *a)* and *b)* are shown.
The property *c)* is an easy consequence of *a)* and *b)*. What remains to show is point *d)*.
Since the denominator of h' is positive, it is sufficient to show $f'h_1 > 0$ as θ tends to $\pm\alpha$. A close examination yields that we have, for a positive c:

$$\begin{aligned} h_1 &= c\psi^2(3\psi^2\psi''' - 8\psi\psi'\psi'' + 5\psi'^3) + O([\alpha \pm \theta]) \\ &= \mp c(\alpha \pm \theta)^{-18} + O([\alpha \pm \theta]). \end{aligned}$$

This proves property *d)*. □

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