Asymptotic Behavior of Capillary Problems governed by Disjoining Pressure Potentials: Circular Capillary Tube, horizontal Wall, parallel Walls

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Abstract. To determine the ascent of liquid in a cylinder is a well-known problem. But if the liquid has a very low dielectric constant scarcely nothing is known. In that case the arising boundary value problem holds a disturbance term. This disturbance causes a very different behaviour which is a common observed result in physical chemistry. The key to solve this boundary value problem is to modify a comparison principle of Concus and Finn. The amazing conclusion is, that this modified principle combined with the disturbance term yields an asymptotic result of the problem. Here we consider three special domains and achieve that the solution of the problem is always the negative of the disturbance term. This is a large difference to the classical case, where the liquid has a higher dielectric constant and strongly depends on the underlying domain.

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1. Introduction

A common chemical problem is to determine the absorbed amount of liquid in porous materials. For liquids with high dielectric constant, like water, general results are known. In this paper we examine the case of low[1] dielectric

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[1]That is, if the value of the dielectric constant of the liquid lies between the dielectric constants of the vapour and the solid phase.

constant, as one has at liquid helium, oxygen or rare gases. A result for arbitrary pores is hard to obtain, so we have a look on a type of pores that can be modeled as straight vertical cylinders over some domain Ω. After determining the arising surface, S, over Ω one can compute the absorbed liquid in this type of pores. Combining this result with statistics about the occurring types of pores leads to the total amount of absorbed liquid.

In detail we have a look on circular domains and extend the adopted methods to estimate the ascent of liquids on a vertical wall and between two parallel walls, respectively. The main tool to obtain results is to modify a comparison principle of Concus and Finn. This principle enables us to find the asymptotic behaviour of the ascent of the liquid.

Some energetic considerations, using variational methods, led to a boundary value problem. The classical procedure is performed in [1, Chapter 1]. In the present case we have to take the low dielectric constant of the liquid into consideration, which leads to the appearance of an additional term, namely the so called *disjoining pressure potential*, see [2, Chapter 11] for details. In other words, as described in [3] we are led to the boundary value problem

$$\begin{array}{rcll} \operatorname{div} \mathbf{T}\, u & = & \kappa u + P(x,u) + \lambda & \text{in } \Omega, \\ \nu \cdot \mathbf{T}\, u & = & 1 & \text{as } x \to \Sigma. \end{array} \tag{1.1}$$

Here u denotes the graph which describes the arising surface and Σ denotes the boundary of Ω. In (1.1) we have set $\mathbf{T}\, u = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$. P is the mentioned disjoining pressure potential, ν the outer normal on the cylinder wall, κ the (positive) capillary constant and λ the Lagrange multiplier.

The main task of this paper is to examine the behaviour of solutions of (1.1). Thereto it is necessary to study the term P. In general, the disjoining pressure potential of absorbed films can be computed by

$$P = P(x, u(x)) = c \int_{\Omega_s} \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2 \right]^{-p/2} \mathrm{d}y, \quad p > 3. \tag{1.2}$$

Thereby Ω_s denotes the solid outer domain, see Figure 1. The occurring constants in the upper equations are defined by the relations

$$\kappa = \rho g \sigma^{-1}, \qquad \lambda = \rho k T \ln(X) \sigma^{-1}, \qquad c = \mathcal{H} \pi^{-2} \sigma^{-1},$$

where

- ρ is the difference between the number densities of the liquid and the vapour phase,
- g is the (positive) gravitational constant,
- σ is the surface tension,
- k is the Boltzmann constant,
- T is the absolute temperature,
- X is the reduced (constant) vapour pressure, $0 < X < 1$,
- $\mathcal{H}$ is the (negative) Hamaker constant.

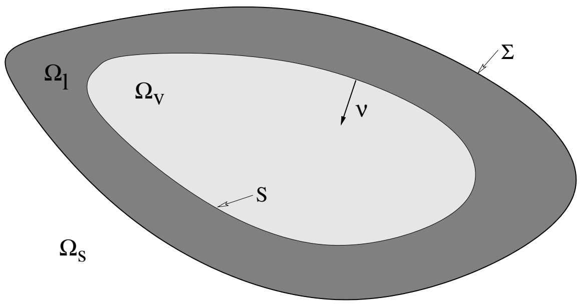

Figure 1. Solid, liquid and vapour domain.

In most practical cases the vapour density is negligibly small, so that ρ can be replaced by the density of the liquid phase. Since we consider the problem (1.1) for negative Hamaker constant $\mathcal{H}$ the constant c is also *negative*.

As mentioned above, formula (1.2) is given in a general form. In the phyiscal relevant cases we have $p = 6$. Cases with $3 < p < 6$ may arise in small systems without practical importance.

2. Asymptotic results

At first we state a comparison principle which enables us to find the desired asymptotic behaviour. The principle is a modification of a result first published by Concus and Finn, see [4]. After that we present a general result for arbitrary cross-sections, and then we give explicit results for some special domains. The reason for the existence of such a principle is the non-linearity of the underlying problem. Precisely this non-linearity ensures the boundedness of $\mathbf{T}\,u$, even if u becomes unbounded. Attempts to linearise the problem leads to wrong and non-physical “solutions“ of (1.1). Just like in the classical case the comparison principle yields the uniqueness of the solution of (1.1), see [1, Cor. 5.1] for details.

Responsible for the validity of the modified theorem is the monotonicity of P in u. In other words, the negativity of the constant c in the definition of P is an essential requirement.

Theorem 2.1. *Let $\kappa > 0$ and suppose that $\Sigma = \partial\Omega$ admits a decomposition $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$, where $\Sigma_\beta \in \mathcal{C}^1$ and Σ_0 has one-dimensional Hausdorff measure zero. Let $u, v \in \mathcal{C}^2(\Omega)$ with the properties*

$$\begin{array}{llcll} \text{(i)} & \operatorname{div}\mathbf{T}\,u - \kappa u - P(x,u) & \geq & \operatorname{div}\mathbf{T}\,v - \kappa v - P(x,v) & \textit{in } \Omega, \\ \text{(ii)} & u & \leq & v & \textit{as } x \to \Sigma_\alpha, \\ \text{(iii)} & \nu\cdot\mathbf{T}\,u & \leq & \nu\cdot\mathbf{T}\,v & \textit{as } x \to \Sigma_\beta. \end{array}$$

Then we have $u \leq v$ in Ω.

By writing

$$\nu\cdot\mathbf{T}\,u \leq \nu\cdot\mathbf{T}\,v \qquad \textit{as } x \to \Sigma_\beta$$

we mean

$$\lim_{\substack{x\in\Omega\\ x\to x_0\\ x_0\in\Sigma_\beta}} \nu\cdot\mathbf{T}\,u \le \lim_{\substack{x\in\Omega\\ x\to x_0\\ x_0\in\Sigma_\beta}} \nu\cdot\mathbf{T}\,v.$$

In property (ii) and in the following we use the corresponding abbreviations.

Now we will have a closer look on a special situation. If we assume some additional conditions on P and Ω we achieve a notable result for the solution of (1.1), namely:

Let v be a solution of the boundary problem (1.1), with $\Sigma_\alpha = \emptyset$ and $P(x,u)$ independent of the particular graph u. So let us denote it again by $P(x)$, and in addition let $\operatorname{div}\mathbf{T}\,P$ be uniformly bounded. With the notations of Theorem 2.1, the solution of (1.1), v, satisfies in particular

$$\operatorname{div}\mathbf{T}\,v - \kappa v = P(x) - \lambda \qquad \text{in } \Omega \tag{2.1}$$

and

$$\nu\cdot\mathbf{T}\,v \to 1 \qquad \text{as } x\to\Sigma_\beta.$$

Now define $w_0 = \kappa^{-1}(\lambda - P(x))$ and $w^+ = w_0 + A$, for a positive A, to get the estimate

$$\begin{aligned}\operatorname{div}\mathbf{T}\,w^+ - \kappa w^+ &= \operatorname{div}\mathbf{T}\,w_0 - \kappa w_0 - \kappa A\\ &= \operatorname{div}\mathbf{T}\,w_0 - \kappa A + P(x) - \lambda\\ &\overset{(2.1)}{=} \operatorname{div}\mathbf{T}\,w_0 - \kappa A + \operatorname{div}\mathbf{T}\,v - \kappa v\\ &\le \operatorname{div}\mathbf{T}\,v - \kappa v,\end{aligned}$$

for a constant A which satisfies $\operatorname{div}\mathbf{T}\,w_0 - \kappa A \le 0$. Since $\operatorname{div}\mathbf{T}\,P$ is bounded such a constant exists. That is, w^+ satisfies condition (i) of the comparison principle. We have by definition of P

$$|P(x)| \to \infty \qquad \text{as } x\to\Sigma_\beta$$

and so

$$\nu\cdot\mathbf{T}\,w^+ \to 1 \qquad \text{as } x\to\Sigma_\beta.$$

That is, in general w^+ satisfies the boundary condition (iii) of the comparison principle. Since condition (ii) is trivially satisfied, we can apply the principle to get $v \le w^+$.

An analogous result arises for $w^- = w_0 - A$, so that we get the following conclusion.

Theorem 2.2. *Let $\kappa > 0$ and suppose $\Sigma = \partial\Omega$ admits a decomposition $\Sigma = \Sigma_\beta \cup \Sigma_0$, where $\Sigma_\beta \in \mathcal{C}^1$ and Σ_0 has one-dimensional Hausdorff measure zero and let the following properties be satisfied*

(i) *$P(x,u)$ is independent of $u(x)$,*

(ii) *$\operatorname{div}\mathbf{T}\,P$ is uniformly bounded in Ω.*

Then there is a constant $A \geq 0$*, such that for the solution* $v(x)$ *of* (1.1) *follows*

$$\left|v(x) + \kappa^{-1}P(x)\right| \leq A \qquad \text{in } \Omega.$$

Now we consider some cases of special geometry. In the first case we observe a circular domain with constant radius $R > 0$. That is, we have $\Omega = B_R(0)$, where $B_R(0) = \left\{x_1^2 + x_2^2 < R^2\right\}$. Thus we are looking for a rotationally symmetric solution, depending only on the distance to the origin, which is $r = \sqrt{x_1^2 + x_2^2}$. The scaling $q = r/R$ transforms the domain into $B_1(0)$.

From now on we assume that the tube is an infinite cylinder, see Figure 2 and Figure 3, or that the bottom of the finite cylinder is covered by liquid. Then we get the following theorem.

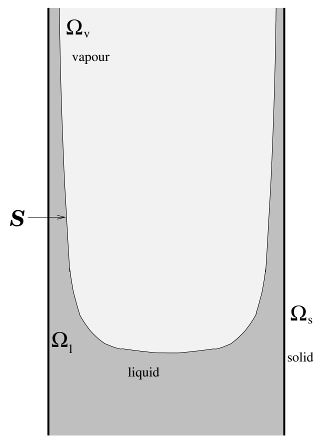

FIGURE 2. Open capillary tube.

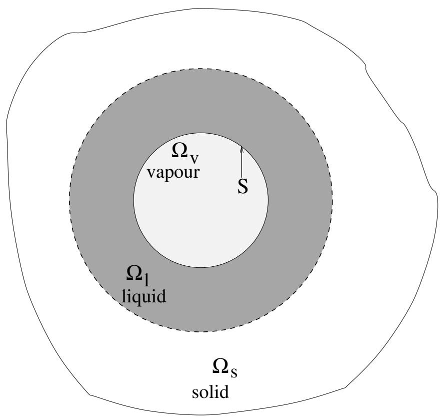

FIGURE 3. View from above.

Theorem 2.3. *Let* v *be the solution of* (1.1) *over the domain* $B_1(0)$*. Then there is a constant* $A \geq 0$*, with*

$$\left|v(q) - C \cdot \mathrm{F}(a, b; 1; q^2)\right| \leq A \qquad \text{in } B_1(0),$$

where $a = \frac{p-3}{2}$, $b = \frac{p-1}{2}$, $C = -\pi^{-\frac{1}{2}}\mathcal{H}\kappa^{-1}\sigma^{-1}R^{3-p}\Gamma(a)/\Gamma(p/2) > 0$ *and* $\mathrm{F}(a, b; 1; q^2)$ *is the hypergeometric function.*

In addition we present an example for the physically significant case $p = 6$. Again let v be the solution of (1.1), then we get with the function

$$\tilde{v}(q) = \tilde{C} \cdot \left\{\left(1 - q^2\right)^{-3} - \frac{3}{8}\left(1 - q^2\right)^{-2} - \frac{3}{32}\left(1 - q^2\right)^{-1}\right\}, \tag{2.2}$$

whereby $\tilde{C} = -\dfrac{4\mathcal{H}}{3\kappa\sigma\pi R^3} > 0$, the explicit result:

$$|v(q) - \tilde{v}(q)| \leq A \qquad \text{in } B_1(0), \tag{2.3}$$

with some positive constant A. From formulae (2.2) and (2.3) we see that v has got a leading singularity $\tilde{C}(1-q^2)^{-3}$ and some lower singularities. The leading term is the same ISRAELACHVILI conjectured in 1985, see [2, p. 194].

We will now start to examine the ascent of a fluid on an infinite vertical wall and between two parallel vertical walls with mutual distance $D > 0$. We choose the coordinate system in a way that we have $\Omega_a = \{x_1 \geq 0,\ -\infty < x_2 < \infty\}$ and $\Omega_b = \{-D \leq x_1 \leq D,\ -\infty < x_2 < \infty\}$, respectively.

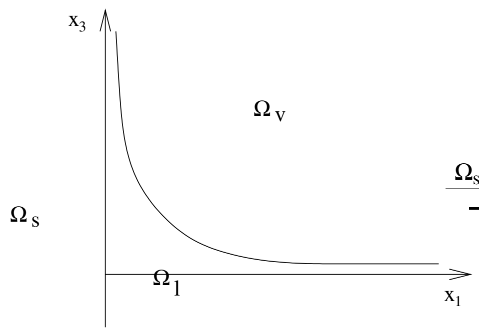

FIGURE 4. Ascent on a wall.

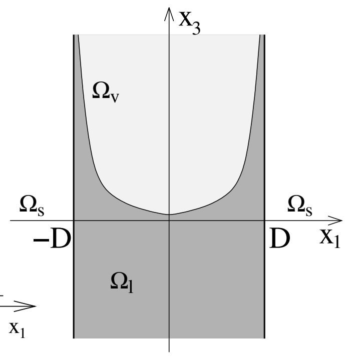

FIGURE 5. Ascent between two parallel plates.

In both cases the calculations lead to a similar result, namely to the following theorem.

Theorem 2.4. *Let v be the solution of the problem* (1.1) *over the domain Ω_a and Ω_b, respectively and define $C = \dfrac{-2\mathcal{H}}{\pi\kappa\sigma(p-2)(p-3)} > 0$. Then there is a constant $A \geq 0$ with*
a)

$$\left| v(x_1) - C \cdot x_1^{3-p} \right| \leq A \qquad \text{in } \Omega_a,$$

and
b)

$$\left| v(x_1) - C \cdot (D + x_1)^{3-p} - C \cdot (D - x_1)^{3-p} \right| \leq A \qquad \text{in } \Omega_b.$$

Let d denote the distance to the boundary of Ω. Then we conclude from Theorem 2.4 that we have

$$v \sim C \cdot d^{3-p} \qquad \text{as } d \to 0, \tag{2.4}$$

whereby v denotes the solution of (1.1). Inserting the most relevant case $p = 6$ into the upper equation, we obtain the same asymptotic behaviour as in the case of the circular tube.

3. Proofs

After proving the comparison principle we will use Theorem 2.2 to show Theorem 2.3 and Theorem 2.4, where we will verify the boundedness of $\operatorname{div} \mathbf{T} P$.

3.1. Proof of Theorem 2.1

This theorem is an extension of the comparison principle published by Finn and Hwang, see [5]. In their paper they proved the principle for positive κ and arbitrary Ω bounded or unbounded. After examining the mentioned proof it is sufficient to show, that the integral

$$G = \int_{\Omega_R} w \cdot (P(x,u) - P(x,v)) \mathrm{d}x$$

is non-negative, see [5] for the definition of w and Ω_R. To prove this, we show the monotonicity of $P(x,u)$ in u. That is we have

$$\begin{aligned}
\frac{\partial P}{\partial u} &= -pc \int_{\Omega_s} (u-y_3)\left[(x_1-y_1)^2+(x_2-y_2)^2+(u-y_3)^2\right]^{-p/2-1} \mathrm{d}y \\
&= -pc \int_{\Omega}\int_{-\infty}^{0} (u-y_3)\left[(x_1-y_1)^2+(x_2-y_2)^2+(u-y_3)^2\right]^{-p/2-1} \mathrm{d}y \\
&\quad - pc \int_{\mathbb{R}^2\setminus\Omega}\int_{-\infty}^{\infty} (u-y_3)\left[(x_1-y_1)^2+(x_2-y_2)^2+(u-y_3)^2\right]^{-p/2-1} \mathrm{d}y \\
&= -pc \int_{\Omega}\int_{-\infty}^{0} (u-y_3)\left[(x_1-y_1)^2+(x_2-y_2)^2+(u-y_3)^2\right]^{-p/2-1} \mathrm{d}y \\
&\geq 0,
\end{aligned}$$

since c is negative and u is positive. So we have $P(x,u) - P(x,v) \geq 0$ for $u \geq v$. Consequently we get

$$\begin{aligned}
G = &\int_{\Omega_{12}} (u - v - m_1)[P(x,u) - P(x,v)]\mathrm{d}x \\
&+ (m_2 - m_1)\int_{\Omega_2} [P(x,u) - P(x,v))]\mathrm{d}x \geq 0.
\end{aligned}$$

Thereby m_1, m_2 and Ω_{12} were introduced in [5] and $\Omega_2 \subset \Omega_R$ denotes the domain were $u - v \geq m_2$. This proves the theorem. ■

3.2. Proof of Theorem 2.3

The idea of the proof is to show, that $P(x,u)$ satisfies condition (ii) of Theorem 2.2. At first we examine the case of an infinite cylinder.

In the present symmetric case, it is helpful to introduce polar and cylindrical coordinates for (x_1, x_2) and (y_1, y_2, y_3), respectively and to transform the domain to the unit disk. Therefore we set $x_1 = Rq\cos\theta$, $x_2 = Rq\sin\theta$ and $y_1 = Rs\cos\varphi$, $y_2 = Rs\sin\varphi$, $y_3 = Rh$. Henceforward we denote $r = \sqrt{x_1^2 + x_2^2}$ and $q = r/R$.

By integrating over φ, the hypergeometric function $\mathrm{F}(a,b;c;z)$ appears. In detail we obtain

$$P(x,u(x)) = \tilde{C}\cdot \mathrm{F}\left(a,b;1;q^2\right), \quad \text{with } a=\frac{p-3}{2} \text{ and } b=\frac{p-1}{2},$$

where $\tilde{C} = \mathcal{H}\sigma^{-1}\pi^{-\frac{1}{2}}R^{3-p}\Gamma(\frac{p-3}{2})/\Gamma(\frac{p}{2}) < 0$, see [6, IV].
For the sake of simplicity, we set $\psi = \psi(q) = \mathrm{F}(a,b;1;q^2)$. To apply Theorem 2.2 it is sufficient to verify that $\operatorname{div}\mathbf{T}\,\psi$ is bounded. When using polar coordinates it is necessary to show that

$$|\operatorname{div}\mathbf{T}\,\psi| = \left|\frac{1}{q}\left(\frac{q\psi'}{\sqrt{1+\psi'^2}}\right)'\right| = \left|\frac{\psi'}{q\sqrt{1+\psi'^2}} + \frac{\psi''}{\left(1+\psi'^2\right)^{3/2}}\right| < \infty,$$

for all $q\in[0,1]$. At first we present the derivatives of $\psi(q)$. Some calculations to obtain them are given in the appendix[2]. The constants $A_1,\ldots,A_6$ occuring in equations (3.1) and (3.2) are explicitly known positive constants, depending only on p. In detail we have

$$\begin{aligned}\psi'(q) &= 2abq\mathrm{F}(a+1,b+1;2;q^2)\\ &= A_1 q\mathrm{F}(a+1,b+1;a+b;1-q^2)\\ &\qquad + A_2 q\left(1-q^2\right)^{-a-b}\mathrm{F}\left(1-a,1-b;1-a-b;1-q^2\right)\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\psi''(q) &= 2ab\mathrm{F}(a+1,b+1;2;q^2) + 2ab(a+1)(b+1)q^2\mathrm{F}\left(a+2,b+2;3;q^2\right)\\ &= A_3\mathrm{F}(a+1,b+1;a+b+1;1-q^2)\\ &\qquad + A_4 q^2\mathrm{F}\left(a+2,b+2;a+b+2;1-q^2\right)\\ &\qquad + A_5\left(1-q^2\right)^{-a-b}\mathrm{F}\left(1-a,1-b;1-a-b;1-q^2\right)\\ &\qquad + A_6 q^2\left(1-q^2\right)^{-1-a-b}\mathrm{F}\left(1-a,1-b;-a-b;1-q^2\right).\end{aligned} \tag{3.2}$$

In the first parts of formulae (3.1) and (3.2), respectively we can see that both ψ' and ψ'' are bounded for $q\in\left[0,\frac{1}{2}\right]$. In particular we have got

$$\left|\frac{\psi'}{q\sqrt{1+\psi'^2}}\right| \le \frac{|\psi'|}{q} = 2ab\left|\mathrm{F}\left(a+1,b+1;2;q^2\right)\right| < \infty$$

and

$$\begin{aligned}\left|\frac{\psi''}{(1+\psi'^2)^{3/2}}\right| \le |\psi''| \le 2ab&\left|\mathrm{F}\left(a+1,b+1;2;q^2\right)\right|\\ &+ 2ab(a+1)(b+1)q^2\left|\mathrm{F}\left(a+2,b+2;3;q^2\right)\right| < \infty.\end{aligned}$$

[2]Detailed calculations are given in [7].

To prove the estimate for $q \in \left(\frac{1}{2}, 1\right]$, we use the second parts of formulae (3.1) and (3.2), respectively. In this case we get

$$\left|\frac{\psi'}{q\sqrt{1+\psi'^2}}\right| \leq 2\left|\frac{\psi'}{\sqrt{1+\psi'^2}}\right| \leq 2,$$

and if $\mathrm{F}_1, \ldots, \mathrm{F}_6$ denote the according hypergeometric functions in formulae (3.1) and (3.2) we get

$$\left|\frac{\psi''}{(1+\psi'^2)^{3/2}}\right| \leq \tilde{\psi} \cdot \left(1-q^2\right)^{2a+2b-1}. \tag{3.3}$$

where

$$\tilde{\psi} \equiv \frac{(1-q^2)^{1+a+b}|A_3\mathrm{F}_3 + A_4q^2\mathrm{F}_4| + (1-q^2)A_5|\mathrm{F}_5| + A_6q^2|\mathrm{F}_6|}{\left[\left(1-q^2\right)^{2a+2b} + q^2\left\{A_1\left(1-q^2\right)^{a+b}\mathrm{F}_1 + A_2\mathrm{F}_2\right\}^2\right]^{3/2}}.$$

The the right-hand-side of (3.3) is bounded, since the function $\tilde{\psi}$, A_i and F_i for $i = 3, \ldots, 6$ are bounded for $q \in (\frac{1}{2}, 1]$ and since $2a+2b-1$ is positive. That is $\operatorname{div} \mathbf{T}\,\psi(q)$ is bounded for all $q \in [0,1]$. So we can apply Theorem 2.2 and Theorem 2.3 is proven for the infinite cylinder.

Let us look at the case of the closed tube. That is, we have

$$\Omega_s = \mathbb{R}^3 \setminus \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 \leq R^2,\ y_3 \geq 0\}.$$

Let in addition c be a positive constant with $v(x) \geq c$ for all $x \in \Omega$. Thereby v denotes the solution of (1.1). So we have

$$\begin{aligned}&\int_{\Omega_2} [(x_1-y_1)^2 + (x_2-y_2)^2 + (u-y_3)^2]^{-p/2} dy \\ &\qquad \leq K_1 \int_{-\infty}^{0} \frac{\mathrm{d}y_3}{|u-y_3|^p} \leq K_1 \int_c^\infty \frac{\mathrm{d}y_3}{y_3^p} \leq K_2 < \infty,\end{aligned}$$

where $\Omega_2 = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 < R^2,\, y_3 < 0\}$, K_1 and K_2 are some positive constants, depending only on R. So we can adopt the idea of the proof of Theorem 2.2 with a constant $A + K_2$ instead of A. This establishes the annex of Theorem 2.3. ∎

3.3. Proof of Theorem 2.4

Again we compute the disjoining pressure potential and verify the conditions of Theorem 2.2. In the first part of the theorem the solid domain is $\Omega_s = \{y \in \mathbb{R}^3 : y_1 < 0\}$, and so the computation of the disjoining pressure potential yields

$$P(x, u(x)) = \frac{2\mathcal{H}}{\pi\sigma(p-2)(p-3)} x_1^{3-p}.$$

Again we compute $\operatorname{div} \mathbf{T} P$, and is we get for some constants c_1 and c_2

$$\begin{aligned}
\operatorname{div} \mathbf{T} P &= \left(\frac{g'}{\sqrt{1+g'^2}} \right)' = g'' \cdot \left(1+g'^2\right)^{-\frac{3}{2}} \\
&= c_2 x_1^{1-p} \cdot \left\{ 1 + c_1^2 x_1^{4-2p} \right\}^{-\frac{3}{2}} \\
&= c_2 x_1^{2p-5} \cdot \left\{ x_1^{2p-4} + c_1^2 \right\}^{-\frac{3}{2}} .
\end{aligned} \tag{3.4}$$

Since $\operatorname{div} \mathbf{T} P$ is continuous for $x_1 \in (0, \infty)$ it is sufficient to show that $\operatorname{div} \mathbf{T} P$ is bounded as x_1 tends to 0 and ∞, respectively.
From the second line in equation (3.4) we conclude that $\operatorname{div} \mathbf{T} P$ tends to 0 as x_1 tends to infinity. We get the same result if x_1 tends to 0 from the last identity in formula (3.4), and so $\operatorname{div} \mathbf{T} P$ is bounded.

The calculations for the second part of the Theorem are similar to the preceding ones, and so the Theorem is proven. ■

4. Conclusions

The main observation is, that the ascent of the liquid always depends on the disjoining pressure potential and not on the domain Ω. So the striking result is that the solution always has the same asymptotic behaviour

$$v \sim c \cdot d^{3-p} \qquad \text{as } d \to 0, \tag{4.1}$$

where c is a positive constant and d denotes the distance to the boundary. This is a main difference to the classical setting, where the solution depends on the domain Ω.

The authors already obtained similar results for the case when the domain contains a corner. Hence a further research should confirm the conjecture, that the boundary behaviour is always as given as in (4.1). Other problems are the extension of the results to zero gravity, to other kind of pores, for example spherical ones and asymptotic expansion of the solutions. Last but not least the existence of a solution of problem (1.1) is not assured, neither for the general case nor for any special case.

Appendix

To compute the derivatives of ψ we use formula [8, 15.2.1]:

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathrm{F}(a, b; c; z) = \frac{ab}{c} \mathrm{F}(a+1, b+1; c+1; z).$$

To state the derivatives in an appropriate form we use formula [8, 15.3.6]

$$\mathrm{F}(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\mathrm{F}(a,b;a+b-c+1;1-z)$$
$$+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\mathrm{F}(c-a,c-b;c-a-b+1;1-z).$$

So we separate the highest poles in the according formulae.
In the case of $a+b \in \mathbb{Z}$ we need a slight modification of equations (3.1) and (3.2), respectively. Thereby we use formulae [8, 15.3.10-12] instead of [8, 15.3.6]. This modification yields the same result, which can be confirmed by an easy calculation. However, the result is that if q tends to 1, the leading singularities in both cases are of the same order.

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References

[1] R. Finn, *Equilibrium capillary surfaces*, vol. 284 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. New York: Springer-Verlag, 1986.

[2] J. N. Israelachvili, *Intermolecular and Surface Forces, Second Edition: With Applications to Colloidal and Biological Systems (Colloid Science)*. Academic Press, January 1992.

[3] P. Schiller, H.-J. Mögel, and E. Miersemann, "Stability of liquid layers on solid materials," *Journal of Mathematical Fluid Mechanics*, 2008.

[4] P. Concus and R. Finn, "On capillary free surfaces in the absence of gravity," *Acta Math.*, vol. 132, pp. 177–198, 1974.

[5] R. Finn and J.-F. Hwang, "On the comparison principle for capillary surfaces," *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, vol. 36, no. 1, pp. 131–134, 1989.

[6] J. R. Philip, "Adsorption and geometry: the boundary layer approximation," *The Journal of Chemical Physics*, vol. 67, pp. 1732–1741, 1977.

[7] O. Thomys, *Asymptotic Behaviour of Capillary Problems governed by Disjoining Pressure Potentials*. PhD thesis, 2009.

[8] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, vol. 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

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