Existence of front solutions for a nonlocal transport problem describing gas ionization

M. Günther [1] and G. Prokert [2]

[1] *Mathematisches Institut, Universität Leipzig, Johannisgasse 26, 04103 Leipzig, Germany*
email: guenther@mathematik.uni-leipzig.de
[2] *Department of Mathematics and Computer Science, Eindhoven, University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands*
email: g.prokert@tue.nl

Abstract

We discuss a moving boundary problem arising from a model of gas ionization in the case of negligible electron diffusion and suitable initial data. It describes the time evolution of an ionization front. Mathematically, it can be considered as a system of transport equations with different characteristics for positive and negative charge densities. We show that only advancing fronts are possible and prove short-time well-posedness of the problem in Hölder spaces of functions. Technically, the proof is based on a fixed point argument for a Volterra type system of integral equations involving potential operators. It crucially relies on estimates of such operators with respect to variable domains in weighted Hölder spaces and related calculus estimates.

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1. Introduction and problem formulation

Let $n \geq 2$, $\mathbb{T}^n := \mathbb{R}^n/(2\pi\mathbb{Z})^n$ be the n-dimensional torus and $\Pi := \mathbb{T}^n \times \mathbb{R}$. For $T > 0$, set $Q_T := \Pi \times [0,T]$.

We are concerned with the following system of PDEs for the nonlinear scalar functions $\phi, \rho, \sigma : Q_T \to \mathbb{R}$ and a vector valued function $E : Q_T \to \mathbb{R}^{n+1}$:

$$\left.\begin{aligned} \partial_t\sigma - \operatorname{div}(\sigma E) &= \sigma f(|E|) && \text{in } Q_T, \\ \partial_t\rho &= \sigma f(|E|) && \text{in } Q_T, \\ E &= -\nabla\phi && \text{in } Q_T, \\ \operatorname{div} E &= \rho - \sigma && \text{in } Q_T. \end{aligned}\right\} \tag{1.1}$$

Here $t \in [0,T]$ is the time variable, and the operators ∇ and div refer only to the $n+1$ spatial variables of Π.

This system occurs as a (dimensionless) minimal model for ionization processes in certain gases. In particular, it is used as a mathematical model for so-called electric streamers, i.e. discharge phenomena travelling in space, see e.g. [1, 2, 3, 6] and further references given there. In this model, $\sigma \geq 0$ and $\rho \geq 0$ are the electron and ion density, respectively, E is the electric field, and ϕ is its potential. The first two equations of (1.1) describe the creation of free electrons and ions by impact ionization. The rate of this process depends linearly on σ and nonlinearly on $|E|$. The function $f : [0, \infty) \to \mathbb{R}$ is given and in all further considerations assumed to be strictly increasing, to satisfy $f(0) = 0$ and to be such that the mapping $\mathbb{R}^{n+1} \ni E \mapsto f(|E|)$ is smooth. A usual choice is given by the so-called Townsend approximation

$$f(|E|) = |E|e^{-1/|E|}.$$

Due to their larger mass, the ions are considered to be immobile. On the relevant timescale, recombination of ions and electrons to noncharged atoms plays no role. Moreover, as our interest is in ionization fronts, electron diffusion is neglected. Consequently, the electron transport is purely convective, driven by the local electric field. Finally, $(1.1)_3$ and $(1.1)_4$ are standard equations of electrostatics prescribing the net charge as source of the electric field which is conservative as no magnetic effects are included.

As in [2], we demand the following conditions for E at infinity that constitute the external forcing:

$$\left.\begin{aligned} &E \to 0 && \text{as } z \to -\infty, \\ &E \to E_\infty e_{n+1} && \text{as } z \to +\infty, \end{aligned}\right\} \tag{1.2}$$

where $z \in \mathbb{R}$ is the ("nonperiodic") last coordinate of Π, and e_{n+1} the corresponding unit vector. The system has to be completed by prescribing suitable initial conditions σ_0 and ρ_0 for the electron and ion densities.

We are interested in classical solutions representing propagating ionization fronts, i.e. solutions where σ and ρ vanish on some part of Q_T and are differentiable on its complement. In view of $(1.1)_2$ it is reasonable to assume that in the complement of this part both σ and ρ are positive. Accordingly, we define the ionized phase Ω_i and the nonionized phase Ω_n by

$$\begin{aligned} \Omega_i(t) &:= \big\{x \in \Pi \mid \rho(x,t) > 0,\ \sigma(x,t) > 0\big\}, \\ \Omega_n(t) &:= \operatorname{int}\big\{x \in \Pi \mid \rho(x,t) = \sigma(x,t) = 0\big\}. \end{aligned} \tag{1.3}$$

Additionally we set $Q_{i,T} := \bigcup_{t\in[0,T]} \Omega_i(t) \times \{t\}$ and demand:

(F1) $\Omega_i(t)$ and $\Omega_n(t)$ are domains such that $\Omega_i(t) \supset \mathbb{T}^n \times (-\infty, -M(t))$, $\Omega_n(t) \supset \mathbb{T}^n \times (M(t), \infty)$ for some sufficiently large $M(t)$, $t \in [0,T]$,

(F2) $\Pi = \overline{\Omega_i(t)} \cup \overline{\Omega_n(t)}$, $t \in [0,T]$,

(F3) $\Gamma(t) := \partial\Omega_i(t) = \partial\Omega_n(t)$ for $t \in [0,T]$, and $\Sigma := \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}$ is a connected C^1 - hypersurface in Q_T.

(F4) ρ and σ are differentiable with respect to all variables in $\bar{Q}_{i,T}$. Moreover, $\rho(\cdot,t) - \sigma(\cdot,t)$ is integrable on $\Omega_i(t)$.

By the divergence theorem, this implies

$$\int_\Pi \big(\rho(x,t) - \sigma(x,t)\big)\,dx = \int_{\Omega_i(t)} \big(\rho(x,t) - \sigma(x,t)\big)\,dx = E_\infty, \tag{1.4}$$

provided the convergence in $(1.2)_1$ is uniform with respect to the first n spatial variables.

The following lemma states Rankine-Hugoniot type conditions across Σ. We will denote extensions of ρ and σ from $Q_{i,T}$ to Σ by $\bar{\rho}$ and $\bar{\sigma}$.

Lemma 1.1. *(Weak solutions)*
Let (F1)-(F4) *be valid and assume that* (ρ, σ, E, ϕ) *satisfy* $(1.1)_1$, $(1.1)_2$ *in* $Q_{i,T}$. *Then for* (ρ, σ, E, ϕ) *to satisfy* $(1.1)_1$, $(1.1)_2$ *in the sense of distributions in* Q_T *it is necessary and sufficient that*

$$\bar{\sigma}(V_n + E \cdot \nu) = 0, \quad \bar{\rho} V_n = 0 \qquad \text{on } \Gamma(t), \tag{1.5}$$

where ν *is the outer unit normal vector to* $\Omega_i(t)$ *and* V_n *is the normal velocity of* $\Gamma(t)$ *in this direction. In this case,* $V_n \geq 0$, *i.e. the mapping* $t \mapsto \Omega_i(t)$ *is increasing for any front solution.*

Proof. Observe that our smoothness assumptions in (F4) are sufficient to apply integration by parts. Thus, for any test function $\psi \in Q_T$ we find from $(1.1)_1$

$$0 = \int_{Q_T} \sigma\big(-\psi_t + E \cdot \nabla_x \psi - \psi f(|E|)\big)\, dx\, dt = \int_\Sigma \bar{\sigma}\psi \,\big((-1, E) \cdot N\big)\, d\Sigma,$$

where $N = (1 + |V_n|^2)^{-1/2}(-V_n, \nu)$ is the outer unit normal to $Q_{i,T}$. As ψ is arbitrary, this is equivalent to $\bar{\sigma}(V_n + E \cdot \nu) = 0$. The second equation in (1.5) is related to $(1.1)_2$ in an analogous way.

Assume $V_n < 0$ in some point of Σ. Then, by continuity, $V_n < 0$ and consequently $\bar{\rho} = 0$ in an Σ-neighborhood of some point $(x_0, t_0) \in \Sigma$ with $t_0 \in (0, T)$. Hence there exists a point $(x_1, t_1) \in \Sigma$, $t_1 > t_0$ with $x_1 \in \Omega_i(t)$ for $t \in [t_0, t_1)$ and $\bar{\rho}(x_1, t_1) = 0$. This leads to a contradiction as $\rho_t \geq 0$ and $\rho(x_1, t_0) > 0$. Thus $V_n \geq 0$ on Σ. □

Clearly, under the nondegeneracy assumption $\bar{\sigma} > 0$ on $\Gamma(t)$, the necessary conditions (1.5) provided in Lemma 1.1 imply the surface motion law $V_n = -E \cdot \nu$ on $\Gamma(t)$ and analogously, if $E \cdot \nu < 0$ on $\Gamma(t)$, then $\rho = 0$ on $\Gamma(t)$. Hence, motivated by these considerations, we are led to the following moving boundary problem:

Throughout this paper let $\Omega_0 \subset \Pi$ be a fixed $C^{1+\alpha}$-domain, $0 < \alpha < 1$, such that Ω_0 and $\Pi \setminus \bar{\Omega}_0$ are domains satisfying (F1), i.e.

$$\Omega_0 \supset \mathbb{T}^n \times (-\infty, -M), \quad \Pi \setminus \bar{\Omega}_0 \supset \mathbb{T}^n \times (M, \infty)$$

with some $M > 0$. We are looking for a family $t \mapsto \Omega(t)$, $t \in [0, T]$, of $C^{1+\alpha}$-domains and functions $E(\cdot, t) : \Pi \to \mathbb{R}^{n+1}$, $\sigma(\cdot, t), \rho(\cdot, t) : \bar{\Omega}(t) \to \mathbb{R}$ such that

$$\Omega(0) = \Omega_0, \quad \sigma(\cdot, 0) = \sigma_0,\ \rho(\cdot, 0) = \rho_0 \ \text{ on } \ \Omega_0 \tag{1.6}$$

with given initial data σ_0, ρ_0 and, using notation as above,

$$\left.\begin{aligned} \partial_t \sigma - \operatorname{div}(\sigma E) &= \sigma f(|E|) && \text{in } \Omega(t), \\ \partial_t \rho &= \sigma f(|E|) && \text{in } \Omega(t), \\ V_n &= -E \cdot \nu(t) && \text{on } \Gamma(t), \\ \rho &= 0 && \text{on } \Gamma(t), \end{aligned}\right\} \tag{1.7}$$

where V_n is the normal velocity of the moving boundary $t \mapsto \Gamma(t) := \partial\Omega(t)$ and $\nu(t)$ is its outer unit normal and the electric field E is determined by

$$\left.\begin{aligned} E &= -\nabla\phi && \text{in } Q_T, \\ \operatorname{div} E &= \rho - \sigma && \text{in } \Omega(t), \\ \operatorname{div} E &= 0 && \text{in } \Pi \setminus \Omega(t), \\ E &\to 0 && \text{as } z \to -\infty, \\ E &\to E_\infty e_{n+1} && \text{as } z \to +\infty, \end{aligned}\right\} \tag{1.8}$$

Note that for classical solutions, E_∞ is defined by (1.4) and independent of t due to conservation of total charge.

Previous research on this moving boundary problem has been concentrated on special types of solutions, motivated by the aim to replace it by simpler approximations (see e.g [2, 6]). In this context, planar travelling waves are most prominent, for similar investigations concerning cylindrical and spherical geometries see [1].

Our interest here is in constructing solutions (for short times and under suitable initial conditions) in a fairly more general situation. The main result of this paper, stated slightly informally, is the following:

Theorem 1.2. *Let $\sigma_0, \rho_0 \in C^{1+\alpha}(\bar{\Omega}_0)$ and such that*

(i) $\rho_0 = 0$ *on* Γ_0,

(ii) $\sigma_0 - \rho_0$ *decays exponentially as* $z \to -\infty$,

(iii) $\partial_{\nu_0}\rho_0 E_0 \cdot \nu_0 = \sigma_0 f(|E_0|)$ *on* Γ_0,

(iv) $E_0 \cdot \nu_0 > 0$ *on* Γ_0

where $\nu_0 := \nu(0)$, $E_0 := E(\cdot, 0)$.

Then the Cauchy problem (1.6)-(1.8) *has precisely one solution on some short time interval* $[0, T]$ *depending on the data such that* $\bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$ *is a* $C^{1+\alpha}$*-manifold and* σ *and* ρ *are* $C^{1+\alpha}$*-functions (in space and time) on* $\bigcup_{t\in(0,T)} \Omega(t) \times \{t\}$.

This theorem will follow from Theorem 3.1 and the remark after Lemma 3.6.

All the assumptions made here are satisfied in a special, essentially one dimensional situation of travelling planar fronts as discussed in [2, 3]. Theorem 1.2 provides sufficient conditions on the initial data (including the initial domain) that guarantee the existence of solutions to (1.1), (1.2) that qualitatively resemble these planar fronts in a certain sense: there is a sharp, forward moving front, the electron density jumps across it while the ion density (but not its spatial derivative) are globally continuous. Moreover, it will also be shown that the total charge density $\sigma - \rho$ decays exponentially far behind the front.

The contents of this paper is as follows: We will treat the moving boundary problem (1.7) by transformation to the fixed reference domain Ω_0; due to its character as a transport problem, this leads to a system of Volterra type integral equations (2.11). Preliminary to this, we have to discuss the determination of E from $\rho - \sigma$ on the varying domain. This will be done essentially by potential operators and corresponding estimates. Finally, the system (2.11) will be solved essentially by a usual Banach fixed point argument. This necessitates estimates for compositions of Hölder functions and interpolation inequalities. Some technical aspects are discussed in the Appendix.

2. The transformed problem

We will represent the family of domains $\{\Omega(t)\,|\,t\in[0,T]\}$ as images of Ω_0 under a family of diffeomorphisms $X=\{X(\cdot,t)\,|\,t\in[0,T]\}$ arising from the transport equation $(1.1)_1$ for σ. As a preparation for this, we introduce a nonlocal solution operator for (1.8) which, loosely speaking, determines the electric field from the charge distribution. This will be done first on a fixed domain, and in a second step we consider the dependence of this operator on perturbations of the domain.

Whenever necessary, we will write $x=(x',z)$ for $x\in\Pi$, where $x'\in\mathbb{T}^n$, $z\in\mathbb{R}$. As no confusion seems likely, we will write $|x_1-x_2|$ for the distance between two points x_1,x_2 in Π. Fix $\alpha,\lambda\in(0,1)$. Define the exponentially weighted Hölder space

$$C^\alpha_\lambda(\bar\Omega_0):=\{g\in C^\alpha(\bar\Omega_0)\,|\,[(x',z)\mapsto e^{-\lambda z}g(x',z)]\in C^\alpha(\bar\Omega_0)\}$$

with norm

$$\|g\|_{\alpha,\lambda}:=\|[(x',z)\mapsto e^{-\lambda z}g(x',z)]\|_{C^\alpha(\bar\Omega_0)}.$$

Spaces $C^{k+\alpha}_\lambda(\bar\Omega_0)$ with $k\in\mathbb{N}$ and spaces of vector valued functions $C^{k+\alpha}_\lambda(\bar\Omega_0,\mathbb{R}^{n+1})$ are defined in a analogous way. Throughout the paper, we are going to use the properties of Hölder spaces concerning products and compositions as discussed e.g. in the appendix of [4] without explicit mentioning.

For $g\in L^1(\Pi)$ we consider the problem

$$\left.\begin{array}{rcll}\operatorname{div}E &=& g & \text{in } \Pi,\\ E &=& -\nabla\phi & \text{in } \Pi,\\ E &\to& 0 & \text{as } z\to-\infty,\\ E &\to& \int_\Pi g\,dx\,e_{n+1} & \text{as } z\to+\infty.\end{array}\right\}\tag{2.1}$$

Essentially, of course, ϕ is a volume potential with density g, however, some issues concerning the conditions at infinity and the convergence of the convolution integral have to be addressed, as g may have noncompact support.

In particular, we will be interested in the case where g is C^α_λ on a domain near Ω_0 and zero outside this domain. We will discuss (2.1) first under the weaker assumption that g is in a weighted L^2-type space on Π. As a preparation, we will discuss a one-dimensional version first.

Let $L^2_\lambda(\mathbb{R})$ be the space of all functions $u\in L^2(\mathbb{R})$ such that

$$\|u\|^2_{L^2_\lambda}:=\int_{\mathbb{R}}e^{2\lambda|t|}|u|^2(t)\,dt<\infty.$$

This space is a Banach space under the norm $\|\cdot\|_{L^2_\lambda}$, and $C_0(\mathbb{R})$ is a dense subspace. We have $u\in L^2_\lambda(\mathbb{R})$ if and only if

$$[t\mapsto e^{\pm\lambda t}u(t)]\in L^2(\mathbb{R}).$$

Note that the moments of order zero and one

$$f\mapsto M_0(f):=\int_{\mathbb{R}}f(\tau)\,d\tau,\qquad f\mapsto M_1(f):=\int_{\mathbb{R}}\tau f(\tau)\,d\tau$$

are continuous linear functionals on $L^2_\lambda(\mathbb{R})$.

Fix $\psi_0\in C^\infty_0(\mathbb{R})$ such that $M_0(\psi_0)=1$, $M_1(\psi_0)=0$.

Lemma 2.1. (i) *For any $f \in L^2_\lambda(\mathbb{R})$ there is precisely one $u \in L^2_\lambda(\mathbb{R})$ such that*

$$u' = f - M_0(f)\psi_0 \quad \text{on } \mathbb{R}.$$

It satisfies an estimate

$$\|u\|_{L^2_\lambda} \leq C\|f\|_{L^2_\lambda}$$

with C independent of f.

(ii) *For any $f \in L^2_\lambda(\mathbb{R})$ there is precisely one $w \in L^2_\lambda(\mathbb{R})$ such that*

$$w'' = f - M_0(f)\psi_0 + M_1(f)\psi_0' \quad \text{on } \mathbb{R}.$$

It satisfies an estimate

$$\|w\|_{L^2_\lambda} + \|w'\|_{L^2_\lambda} \leq C\|f\|_{L^2_\lambda}$$

with C independent of f.

Proof. (i) It is sufficient to show the result in the case $M_0(f) = 0$ and also, by density arguments, for $f \in C_0(\mathbb{R})$. Let

$$u(t) := \int_{-\infty}^{t} f(\tau)\, d\tau, \qquad v(t) := e^{\lambda t} u'(t).$$

Then v vanishes for $|t|$ sufficiently large, hence $v \in L^2(\mathbb{R})$,

$$v'(t) = \lambda v(t) + e^{\lambda t} f(t),$$

and

$$0 = \int_{\mathbb{R}} v'(t)v(t)\, dt = \lambda \int_{\mathbb{R}} v^2(t)\, dt + \int_{\mathbb{R}} e^{\lambda t} f(t)v(t)\, dt.$$

Therefore

$$\|v\|_{L^2} \leq C\|f\|_{L^2_\lambda}.$$

Replacing λ by $-\lambda$ and repeating the argument yields the estimate. The uniqueness result is straightforward.

(ii) Applying (i) to the equations

$$u' = f - M_0(f)\psi_0,$$
$$w' = u - M_0(u)\psi_0$$

and using that due to our choice of ψ_0

$$M_0(u) = -\int_{\mathbb{R}} \tau u'(\tau)\, d\tau = -M_1(f)$$

yields the results. □

Lemma 2.2. *Assume $f \in L^2_\lambda(\mathbb{R})$, $k \geq 1$. The unique solution $u \in L^2(\mathbb{R})$ of the equation*

$$-u'' + k^2 u = f \quad \text{on } \mathbb{R} \tag{2.2}$$

is in $L^2_\lambda(\mathbb{R})$ and satisfies an estimate

$$k^2\|u\|_{L^2_\lambda} + k\|u'\|_{L^2_\lambda} \leq C\|f\|_{L^2_\lambda}$$

where C is independent of f and k.

Proof. Again, we can restrict ourselves to the case $f \in C_0(\mathbb{R})$. Assume $\operatorname{supp} f \subset [t_1, t_2]$. Then $u(t) = c_1 e^{kt}$ for $t < t_1$ and $u(t) = c_2 e^{-kt}$ for $t > t_2$. Multiply (2.2) by $e^{\lambda t}$ and substitute $v(t) := e^{\lambda t} u(t)$. Then $v \in L^2(\mathbb{R})$ and

$$-v'' + 2\lambda v' + (k^2 - \lambda^2) v = e^{\lambda t} f \quad \text{on } \mathbb{R}$$

and as $k^2 - \lambda^2$ is (uniformly) positive we find by standard arguments that

$$k^2 \|v\|_{L^2} + k\|v'\|_{L^2} \le C\|f\|_{L^2_\lambda}.$$

Repeating the arguments with λ replaced by $-\lambda$ yields the estimate. □

To treat a parallel problem in Π we introduce the space $L^2_\lambda(\Pi)$ consisting of the functions in $L^2(\Pi)$ for which

$$\|u\|^2_{L^2_\lambda} := \int_{\mathbb{T}^n} \int_{\mathbb{R}} e^{2\lambda|z|} |u|^2(x', z)\, dz dx' < \infty.$$

Analogous remarks as in the one-dimensional case apply. We introduce the modified moments

$$f \mapsto M_0^\Pi(f) := \int_\Pi f\, dz dx', \qquad f \mapsto M_1^\Pi(f) := \int_\Pi z f\, dz dx'$$

and find the following result:

Lemma 2.3. *For any $g \in L^2_\lambda(\Pi)$ there is precisely one $\phi \in L^2_\lambda(\Pi)$ such that*

$$-\Delta\phi = g - M_0^\Pi(g)\psi_0(z) + M_1^\Pi \psi_0'(z). \tag{2.3}$$

It satisfies an estimate

$$\|\phi\|_{L^2_\lambda} + \|\nabla\phi\|_{L^2_\lambda} + \|\nabla^2\phi\|_{L^2_\lambda} \le C\|g\|_{L^2_\lambda}.$$

Proof. Representing both g and ϕ in terms of Fourier series

$$g(x', z) = \sum_{k \in \mathbb{Z}^n} g_k(z) e^{ik\cdot x'}, \qquad \phi(x', z) = \sum_{k \in \mathbb{Z}^n} \phi_k(z) e^{ik\cdot x'},$$

yields $g_k \in L^2_\lambda(\mathbb{R})$, $\sum_k \|g_k\|^2_{L^2_\lambda} \le C\|g\|^2_{L^2_\lambda}$,

$$-\phi_k'' + |k|^2 \phi_k = g_k \quad \text{on } \mathbb{R}, \quad k \ne 0,$$

and

$$-\phi_0'' = g_0 - M_0^\Pi(g)\psi_0 + M_1^\Pi(g)\psi_0' = g_0 - M_0(g_0)\psi_0 + M_1(g_0)\psi_0'.$$

The lemma is obtained now by applying Lemmas 2.1 and 2.2 and using that $\phi \in L^2_\lambda(\Pi)$ if and only if $\phi_k \in L^2_\lambda(\mathbb{R})$ for all k and

$$\|\phi\|^2_{L^2_\lambda} \sim \sum_k \|\phi_k\|^2_{L^2_\lambda},$$

as well as corresponding representations for the derivatives. □

Observe, moreover, that for any $a, b \in \mathbb{R}$

$$G(x', z) := \frac{1}{2(2\pi)^n} \left(-|z| + \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|k|} e^{-|k||z|} e^{ik\cdot x'} \right) + az + b$$

and in particular, for $a = \pm 1/(2(2\pi)^n)$, $b = 0$,

$$G_\pm(x', z) := \frac{1}{(2\pi)^n}\left(-z_\mp + \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{2|k|} e^{-|k||z|} e^{ik\cdot x'}\right)$$

are fundamental solutions for the Laplacian on Π with

$$\big[(x', z) \mapsto e^{\pm\lambda z} G_\pm(x', z)\big] \in L^1(\Pi). \tag{2.4}$$

For functions $g \in L^2_\lambda(\Pi)$ that satisfy $M_0^\Pi(g) = M_1^\Pi(g) = 0$, the convolution $u^* := G \star g$ is well defined and independent of a and b. In particular, $u^* = G_\pm \star g$, and consequently

$$e^{\pm\lambda z} u^*(x', z) = \int_\Pi e^{\pm\lambda(z-\zeta)} G_\pm(x' - \xi', z - \zeta) e^{\pm\lambda\zeta} g(\xi', \zeta)\, d\xi' d\zeta \tag{2.5}$$

and therefore $u^* \in L^2_\lambda(\Pi)$ by (2.4) and Young's inequality. Thus, the solution ϕ to (2.3) can be represented as

$$\phi = G \star (g - M_0^\Pi(g)\psi_0 + M_1^\Pi(g)\psi_0')$$

and the solution $E = E[g]$ to (2.1) is found to be

$$E[g] = -\nabla\big(G \star (g - M_0^\Pi(g)\psi_0 + M_1^\Pi(g)\psi_0')\big) + \big(M_0^\Pi(g)\psi_1 - M_1^\Pi(g)\psi_0\big)e_{n+1},$$

where

$$\psi_1(z) := \int_{-\infty}^z \psi_0(\zeta)\, d\zeta.$$

Let $Z : \Omega_0 \to Z[\Omega_0] \subset \Pi$ be a $C^{1,\alpha}$-diffeomorphism. To consider the dependence of our nonlocal solution operator on such diffeomorphisms we introduce the operator $\mathscr{E}[Z]$ by

$$\mathscr{E}[Z]g := E[g \circ Z^{-1}]|_{Z[\bar\Omega_0]} \circ Z, \quad g \in C^\alpha_\lambda(\bar\Omega_0), \tag{2.6}$$

where $g \circ Z^{-1}$ is understood to be extended to Π by 0.

We will need Lipschitz dependence of $\mathscr{E}$ on Z. The proof is mainly based on potential estimates that go back to Lichtenstein [5], §3. For convenience, we quote his original result in modern notation, generalized to $\mathbb{R}^m$, $m \ge 2$: For a compactly supported, bounded function ϕ let $V(\phi)$ be the volume potential with density ϕ, given by

$$V(\phi)(x) := \int_{\mathbb{R}^m} P(x - y)\phi(y)\, dy,$$

where P denotes the standard fundamental solution for the Laplacian on $\mathbb{R}^m$. Let $\Sigma \subset \mathbb{R}^m$ be a bounded $C^{1+\alpha}$-domain. For a $C^{1+\alpha}$-diffeomorphism Z on Σ, define $\mathcal{V}[Z] \in \mathcal{L}(C^\alpha(\bar\Sigma), C^{2+\alpha}(\bar\Sigma))$ by (cf. (2.6))

$$\mathcal{V}[Z]\phi := V(\phi \circ Z^{-1})|_{Z(\bar\Sigma)} \circ Z$$

where $\phi \circ Z^{-1}$ has to be extended to $\mathbb{R}^m$ by zero.

Lemma 2.4. *(Dependence of the volume potential on domain variations)* *There is a neighborhood O of 0 in $C^{1+\alpha}(\Sigma, \mathbb{R}^m)$ such that*

$$\big[z \mapsto \mathcal{V}[Z]\big] \in \mathrm{Lip}\big(\mathrm{id} + O, \mathcal{L}\big(C^\alpha(\bar\Sigma), C^{2+\alpha}(\bar\Sigma)\big)\big).$$

The situation we have to discuss is slightly different in three aspects: We work on Π instead of $\mathbb{R}^m$, and we have to consider unbounded domains and (consequently) weighted Hölder spaces.

Lemma 2.5. *(Dependence of $\mathscr{E}$ on domain perturbations)*
For a sufficiently small open neighborhood O of 0 in $C^{1+\alpha}(\bar\Omega_0,\mathbb{R}^{n+1})$ we have

(i) $\big[Z \mapsto \mathscr{E}[Z]\big] \in \mathrm{Lip}\big(\mathrm{id}+O, \mathcal{L}\big(C^\alpha_\lambda(\bar\Omega_0), C^{1+\alpha}_\lambda(\bar\Omega_0,\mathbb{R}^{n+1})\big)\big)$,

(ii) $\big[Z \mapsto \mathscr{E}[Z]\big] \in \mathrm{Lip}\big(\mathrm{id}+O, \mathcal{L}\big(C_\lambda(\bar\Omega_0), C_\lambda(\bar\Omega_0,\mathbb{R}^{n+1})\big)\big)$.

Proof. We are going to show (i). Define the convolution operator G by

$$\mathsf{G}u := G_- \star u.$$

(This operator is clearly well-defined on $C^\alpha_\lambda(\Omega_0)$.) Then

$$\begin{aligned}
\mathscr{E}_i[Z]g &= Z^* E_i Z_* g \\
&= -Z^*\partial_i \mathsf{G} Z_* g \\
&\quad + M_0^\Pi(Z_*g)\, Z^*(\psi_1\delta_{i,n+1} + \partial_i\mathsf{G}\psi_0) - M_1^\Pi(Z_*g)\, Z^*(\psi_0\delta_{i,n+1} + \partial_i \mathsf{G}\psi_0'),
\end{aligned}$$

where Z^* and Z_* denote the pull-back and push-forward by Z. Here and in what follows, restrictions and extensions by zero are suppressed in the notation for the sake of brevity.

Using

$$\begin{aligned}
M_0^\Pi(Z_*g) &= \int_{\Omega_0} g|\det DZ|\, dx'dz \\
M_1^\Pi(Z_*g) &= \int_{\Omega_0} Z_{n+1} g|\det DZ|\, dx'dz
\end{aligned}$$

and the facts that ψ_0, ψ_1, and $\mathsf{G}\psi_0$ are smooth functions in $C^\alpha_\lambda(\bar\Omega_0)$ we easily get

$$\begin{aligned}
[Z \mapsto [g \mapsto M_0^\Pi(Z_*g)\, Z^*(\psi_1\delta_{i,n+1} + \partial_i\mathsf{G}\psi_0)]] &\in \mathrm{Lip}(\mathrm{id}+O, \mathcal{L}(C^\alpha_\lambda(\bar\Omega_0), C^{1+\alpha}_\lambda(\bar\Omega_0))), \\
[Z \mapsto [g \mapsto M_1^\Pi(Z_*g)\, Z^*(\psi_0\delta_{i,n+1} + \partial_i\mathsf{G}\psi_0')]] &\in \mathrm{Lip}(\mathrm{id}+O, \mathcal{L}(C^\alpha_\lambda(\bar\Omega_0), C^{1+\alpha}_\lambda(\bar\Omega_0))).
\end{aligned}$$

It remains to consider the term $Z^*\partial_i\mathsf{G}Z_*g$. We will show

$$[Z \mapsto [g \mapsto \partial_j Z^*\partial_i \mathsf{G} Z_* g]] \in \mathrm{Lip}(\mathrm{id}+O, \mathcal{L}(C^\alpha_\lambda(\bar\Omega_0), C^\alpha_\lambda(\bar\Omega_0))),$$

the remaining statement

$$[Z \mapsto [g \mapsto Z^*\partial_i \mathsf{G} Z_* g]] \in \mathrm{Lip}(\mathrm{id}+O, \mathcal{L}(C^\alpha_\lambda(\bar\Omega_0), C^0_\lambda(\bar\Omega_0)))$$

is simpler and can be proved along the same lines. By the chain rule,

$$\partial_j Z^*\partial_i\mathsf{G}Z_*g = \sum_l Z^*\partial_{il}\mathsf{G}Z_*g\; \partial_j Z_l$$

and $\partial_j Z_l \in C^\alpha(\bar\Omega_0)$, hence it will be sufficient to show

$$[Z \mapsto [g \mapsto Z^*\partial_{il}\mathsf{G}Z_*g]] \in \mathrm{Lip}(\mathrm{id}+O, \mathcal{L}(C^\alpha_\lambda(\bar\Omega_0), C^\alpha_\lambda(\bar\Omega_0)). \tag{2.7}$$

In the sequel, we will fix i and l and write

$$\mathcal{G}(Z) := Z^*\partial_{il}\mathsf{G}Z_*.$$

Assume without loss of generality

$$\{z \,|\, (x', z) \in \partial\Omega_0\} \subset (0,1)$$

and define for $k \in \mathbb{N}$

$$\Xi_k := \Omega_0 \cap \big(\mathbb{T}^n \times (-k, -k+2)\big).$$

For $v \in C^\alpha_\lambda(\bar\Omega_0)$ and $v_k := v|_{\Xi_k}$ we have

$$\|v\|_{C^\alpha_\lambda(\bar\Omega_0)} \sim \sup_{k\in\mathbb{N}} e^{\lambda k}\|v_k\|_{C^\alpha(\bar\Xi_k)}$$

in the sense of norm equivalence.

Consequently, to show (2.7) it will be sufficient to prove

$$[Z \mapsto [g \mapsto e^{\lambda k}(\mathcal{G}(Z)g)|_{\Xi_k}]] \in \mathrm{Lip}(\mathrm{id}+O, \mathcal{L}(C^\alpha_\lambda(\bar\Omega_0), C^\alpha(\bar\Xi_k)). \tag{2.8}$$

with a Lipschitz constant independent of k.

For this purpose, let $\chi \in C^\infty_0(\mathbb{R})$ be a cutoff function such that $\operatorname{supp}\chi \subset [-1,3]$, $\chi \equiv 1$ on $(-1/2, 5/2)$, set

$$\tilde g_k(x', z) := g(x', z)\chi(z-k), \quad \hat g_k := g - \tilde g_k,$$

and decompose

$$e^{\lambda k}(\mathcal{G}(Z)g)|_{\Xi_k} = e^{\lambda k}(\mathcal{G}(Z)\tilde g_k)|_{\Xi_k} + e^{\lambda k}(\mathcal{G}(Z)\hat g_k)|_{\Xi_k}. \tag{2.9}$$

For the first term we get parallel to Lemma 2.4 for $Z_1, Z_2 \in \mathrm{id}+O$, O sufficiently small

$$\begin{aligned} \|(\mathcal{G}(Z_1) - \mathcal{G}(Z_2))\tilde g_k\|_{C^\alpha(\bar\Xi_k)} &\le C\|Z_1 - Z_2\|_{C^{1+\alpha}(\bar\Xi'_k)}\|\tilde g_k\|_{C^\alpha(\bar\Xi'_k)} \\ &\le Ce^{-\lambda k}\|Z_1 - Z_2\|_{C^{1+\alpha}(\bar\Omega_0)}\|\tilde g_k\|_{C^\alpha_\lambda(\bar\Omega_0)}, \end{aligned} \tag{2.10}$$

where both O and C are independent of k, and

$$\Xi'_k := \bigcup_{|j-k|\le 1} \Xi_j.$$

(As mentioned above, we need a slight modification of the result in Lemma 2.4 as we work with the fundamental solution for the Laplacian on Π rather than on $\mathbb{R}^{n+1}$, however, the necessary changes are straightforward and unessential, as G and P have the same behavior near the singularity.)

To investigate the second term in (2.9), we use that for $x = (x', z) \in \Xi_k$, $Z_1, Z_2 \in \mathrm{id}+O$, we have

$$\begin{aligned} e^{-\lambda z}\big((\mathcal{G}(Z_1) - \mathcal{G}(Z_2))\hat g_k\big)(x) &= \int_{\Omega_0} e^{-\lambda(z-\zeta)}(L_1(x,\xi) - L_2(x,\xi))e^{-\lambda\zeta}\hat g_k(\xi)\,d\xi, \\ L_i(x,\xi) &:= K(Z_i(x) - Z_i(\xi))\det DZ_i(\xi) \end{aligned}$$

$\xi = (\xi', \zeta)$, $i = 1, 2$. Here, $K : \Pi \longrightarrow \mathbb{R}$ can be chosen to be a smooth function such that

$$(x', z) \mapsto e^{-\lambda z}K(x', z)$$

decays exponentially as $z \to \pm\infty$. Therefore, by Lemma A.2, we have

$$\|(x,\xi) \mapsto e^{-\lambda(z-\zeta)}(L_1(x,\xi) - L_2(x,\xi))\|_{C^\alpha(\Xi_k\times\Omega_0)} \le C\|Z_1 - Z_2\|_{C^\alpha(\bar\Omega_0)}.$$

Thus,

$$\begin{aligned} e^{\lambda k}\|(\mathcal{G}(Z_1)-\mathcal{G}(Z_2))\hat{g}_k|_{\Xi_k}\|_{C^\alpha(\Xi_k)} &\le C\|Z_1-Z_2\|_{C^\alpha(\bar\Omega_0)}\|[x\mapsto e^{-\lambda z}\hat{g}_k(x)]\|_{C^\alpha(\bar\Omega_0)} \\ &\le C\|Z_1-Z_2\|_{C^\alpha(\bar\Omega_0)}\|g\|_{C^\alpha_\lambda(\bar\Omega_0)}. \end{aligned}$$

Together with (2.9) and (2.10), this proves (2.8) and hence the proof of (i) is complete.

The proof of (ii) along the same lines is easier, as no regularization is involved and the singularity of the kernel is integrable (cf. (2.4)). □

Using the nonlocal operator $\mathscr{E}$, we can rewrite (1.1), (1.2) as a system of Volterra integral equations for $t\mapsto X(\cdot,t)$. For $t\in[0,T]$ define

$$\hat\sigma(\cdot,t):=\sigma\big(X(\cdot,t),t\big),\quad \hat\rho(\cdot,t):=\rho\big(X(\cdot,t),t\big),\quad \hat E(\cdot,t):=\mathscr{E}[X(\cdot,t)](\hat\rho-\hat\sigma)(\cdot,t).$$

Then we get for $x\in\Omega_0$, $t\in[0,T]$

$$\left.\begin{aligned} X(x,t) &= x-\int_0^t \hat E(x,\tau)\,d\tau,\\ \hat\sigma(x,t) &= \sigma_0(x)+\int_0^t \hat\sigma(x,\tau)\big(f(|\hat E(x,\tau)|)+(\hat\rho-\hat\sigma)(x,\tau)\big)\,d\tau,\\ \hat\rho(x,t) &= \rho_0\big(X(x,t)\big)\\ &\quad+\int_{\Theta(X,x,t)}^t \hat\sigma\big(X^{-1}(X(x,t),\tau),\tau\big)f\big(|\hat E(X^{-1}(X(x,t),\tau),\tau)|\big)\,d\tau. \end{aligned}\right\}\qquad(2.11)$$

In the last equation, $X^{-1}(\cdot,\tau)$ denotes the inverse of $X(\cdot,\tau)$. Moreover, $\Theta(X,x,t)$ is the uniquely defined smallest time such that $X(x,t)\in X[\Omega_0,\tau]$ for $\tau>\Theta(X,x,t)$ and the pull back in the integrand makes sense. When $\Theta(X,x,t)$ is positive then the first summand has to be neglected (or, equivalently, ρ_0 has to be extended by zero outside Ω_0.) See Fig. 1.

In the sequel, we will abuse notation and omit all hats, still working with the functions defined on the fixed domain Ω_0.

3. Existence of solutions

We are going to prove the solvability of (2.11) by a contraction argument. This will be done under the assumptions (cf. Theorem 1.2)

$$\sigma_0,\rho_0\in C^{1+\alpha}(\bar\Omega_0)\quad\text{such that}\quad \sigma_0-\rho_0\in C^{1+\alpha}_\lambda(\bar\Omega_0)\tag{3.1}$$

together with the compatibility conditions

$$\rho_0=0,\quad \partial_{\nu_0}\rho_0\,E_0\cdot\nu_0=\sigma_0 f(|E_0|)\ \text{ on }\ \partial\Omega_0,\tag{3.2}$$

where $E_0:=\mathscr{E}[\mathrm{id}](\rho_0-\sigma_0)$ and

$$E_0\cdot\nu_0>\gamma>0\ \text{ on }\ \partial\Omega_0.\tag{3.3}$$

Further, for given $\varepsilon>0$, $K=(K_1,K_2)$, $K_i>0$ let $M(\varepsilon,K)$ be the set of functions

$$M(\varepsilon,K):=\big\{u=(X,\sigma,\rho)\in\mathcal{X}\mid X,\sigma,\rho \text{ satisfy (M1), (M2), (M3)}\big\},$$

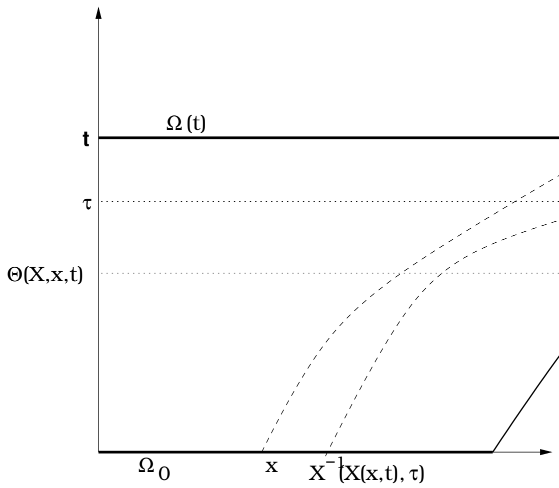

Figure 1: Schematic sketch of the transformations involved in (2.11). (For simplicity, the moving domain is represented as a half line here.) Due to the immobility of the ions in the model, the transport equations for σ and ρ have different characteristics.

where

$$\mathcal{X} := C^\alpha\big(I, C^{1+\alpha}(\bar\Omega_0, \mathbb{R}^{n+1})\big) \times B\big(I, C^{1+\alpha}(\bar\Omega_0)\big) \times B\big(I, C^{1+\alpha}(\bar\Omega_0)\big)$$

and the conditions (M1)-(M3) are given by

(M1) $X - \mathrm{id} \in C^\alpha(I, C^{1+\alpha}_\lambda)$ with

$$\|X - \mathrm{id}\|_{C^\alpha(I, C^{1+\alpha}_\lambda)} \le \varepsilon, \tag{3.4}$$

(M2) $\sigma - \sigma_0 \in C^\alpha(I, C^\alpha_\lambda)$ with

$$\|\sigma\|_{B(I,C^{1+\alpha})}, \|\sigma - \sigma_0\|_{C^\alpha(I,C^\alpha_\lambda)} \le K_1, \tag{3.5}$$

(M3) $\rho - \rho_0 \in B(I, C^\alpha_\lambda) \cap C^\alpha(I, C_\lambda)$ with

$$\|\rho - \rho_0\|_{C^\alpha(I,C_\lambda)}, \|\rho - \rho_0\|_{B(I,C^\alpha_\lambda)} \le K_1, \quad \|\rho\|_{B(I,C^{1+\alpha})} \le K_2. \tag{3.6}$$

Our main result is the following:

Theorem 3.1. *Let Ω_0, ρ_0, σ_0 be given and satisfy* (3.1)-(3.3). *For sufficiently large $K_1 > 0$ (depending on the data and on α, λ, γ), sufficiently large $K_2 > 0$, and sufficiently small $\varepsilon, T > 0$ (all depending on the data and on α, λ, γ, and K_1),* (2.11) *has precisely one solution $(X\sigma, \rho)$ in $M(\varepsilon, K)$.*

This theorem will be proved by applying the Banach Fixed Point theorem, i.e. it will follow directly from Lemmas 3.5 and 3.6 below.

As a preparation, we investigate the properties of the map Θ. Let ν_0 denote the outer unit normal vector on $\Gamma_0 := \partial\Omega_0$ and let $\mathrm{dist}\,(\cdot, \Gamma_0)$ denote the signed distance function to Γ_0, taken positive outside Ω_0. For $\delta > 0$ define the "one-sided neighborhood"

$$U_\delta := \big\{x \in \Pi \,|\, \mathrm{dist}\,(x, \Gamma_0) \in [0, \delta)\big\}.$$

Furthermore, to shorten notation, let $I := [0,T]$ and $\Pi := \bar{\Omega}_0 \times I$. For spaces of functions defined on $\bar{\Omega}_0$ we will simply write C, $C^{k+\alpha}$ instead of $C(\bar{\Omega}_0)$, $C^{k+\alpha}(\bar{\Omega}_0, \Pi)$ etc. Moreover, for functions X defined on Π we will not distinguish notationally between X and the function $t \mapsto \mathsf{X}(\cdot, t)$ valued in appropriate function spaces on $\bar{\Omega}_0$. Finally, let id denote both the identity on $\bar{\Omega}_0$ and the canonical projection of Π onto $\bar{\Omega}_0$.

Lemma 3.2. *Let $K, \gamma > 0$ be given and assume $X \in C^{1+\alpha}(\bar{\Omega}_0 \times I, \Pi)$ with*

$$X(\cdot, 0) = \mathrm{id}, \quad \|X\|_{C^{1+\alpha}} \leq K, \tag{3.7}$$

$$\partial_t X(\cdot, 0) \cdot \nu_0 \geq \gamma > 0 \ \text{ on } \ \Gamma_0. \tag{3.8}$$

There exist $\delta, M, \tau > 0$ depending only on K and γ and functions

$$(\xi_X, \theta_X) \in C^{1+\alpha}\big(U_\delta, \Gamma_0 \times [0, \tau)\big) \quad \text{with} \quad \|(\xi_X, \theta_X)\|_{1+\alpha} \leq M \tag{3.9}$$

such that for all $z \in U_\delta$, $(\xi, \theta) = (\xi_X(z), \theta_X(z))$ is the only solution to

$$X(\xi, \theta) = z, \quad \xi \in \Gamma_0, \theta \in [0, \tau). \tag{3.10}$$

Moreover, if $X_1, X_2 \in C^{1+\alpha}(\bar{\Omega}_0 \times I, \Pi)$ both satisfy (3.7), (3.8), *then*

$$\|\xi_{X_1} - \xi_{X_2}\|_{C^\alpha(U_\delta)}, \|\theta_{X_1} - \theta_{X_2}\|_{C^\alpha(U_\delta)} \leq M \|X_1 - X_2\|_{C^\alpha(\bar{\Omega}_0 \times I, \Pi)}. \tag{3.11}$$

For the proof of this lemma we use the following quantitative version of the Inverse Function Theorem. It basically asserts that "locally, inversion of a function is Lipschitz with respect to C^α-norms", provided the functions to be inverted are $C^{1+\alpha}$.

Lemma 3.3. *Let $K, r > 0$ be given and*

$$g \in C^{1+\alpha}(W, \mathbb{R}^m), \quad W := \big\{x \in \mathbb{R}^m \;\big|\; |x| \leq r\big\}$$

with $g(0) = 0$ and a non-singular derivative $Dg(0)$ such that

$$\|g\|_{C^{1+\alpha}(W)} \leq K, \quad \|Dg(0)^{-1}\| \leq K. \tag{3.12}$$

Then there exist constants $M, N, r_0, r_1, r_2 > 0$, depending only on K, α, m and functions

$$g^{-1} \in C^{1+\alpha}(V, \mathbb{R}^m), \ V := \big\{y \in \mathbb{R}^m \;\big|\; |y| \leq r_0\big\} \quad \text{with} \quad \|g^{-1}\|_{C^{1+\alpha}(V)} \leq M$$

such that $x = g^{-1}(y)$ is the uniquely determined solution to

$$g(x) = y, \quad |x| \leq r_1$$

for all $y \in V$. Further, if $g_1, g_2 \in C^{1+\alpha}(W, \mathbb{R}^m)$ with $g_1(0) = g_2(0) = 0$ both satisfy (3.12), *then*

$$\|g_1^{-1} - g_2^{-1}\|_{C^\alpha(V')} \leq N \|g_1 - g_2\|_{C^\alpha(W)}, \quad V' := \big\{y \in \mathbb{R}^m \;\big|\; |y| \leq r_2\big\}. \tag{3.13}$$

Proof. We are going to prove (3.13) only. Choose r_2 small enough to ensure

$$V' + B(0, \|g_1 - g_2\|_{C^0(V')}) \subset V,$$

this is possible due to (3.12) and $g_1(0) = g_2(0)$. Now, by (3.12) and Lemma A.2,

$$\begin{aligned}
\|g_1^{-1} - g_2^{-1}\|_{C^\alpha(V')} &\leq C(K) \|g_1^{-1} \circ g_2 - g_2^{-1} \circ g_2\|_{C^\alpha(g_2^{-1}(V'))} \\
&= C(K) \|g_1^{-1} \circ g_2 - g_1^{-1} \circ g_1\|_{C^\alpha(g_2^{-1}(V'))} \\
&\leq C(K) \|g_1^{-1}\|_{C^{1+\alpha}(V)} \|g_1 - g_2\|_{C^\alpha(W)}.
\end{aligned}$$

This implies (3.13). □

Proof of Lemma 3.2: Applying usual extension theorems, w.l.o.g. we can assume that $X(\cdot,t)$ is defined for all $t \in I' = [-T,T]$. Denote by G the restriction of X to $\Gamma_0 \times (-T,T)$. Fix $x_0 \in \Gamma_0$ and observe that $G(x_0,0) = x_0$ and the derivative DG of G in this point is surjective due to (3.8) with

$$\|DG(x_0,0)^{-1}\| \le C,$$

where C is independent of X, but depends on K, γ and Γ_0. Therefore, by the Inverse Function theorem, we find $\tau_0, \delta_0 > 0$ and functions

$$(\xi,\theta) \in C^1(V, \Gamma_0 \times (-\tau,\tau)), \quad V := B(x_0,\delta_0)$$

such that $(\xi,\theta) = (\xi(z),\theta(z))$ is the only solution to

$$X'(\xi,\theta) = z, \quad \xi \in \Gamma_0,\ \theta \in (-\tau_0,\tau_0).$$

Differentiation of this equation with respect to z at (x_0) yields (in matrix notation)

$$D_x X(x_0,0) D_z \xi(x_0) + \partial_t X(x_0,0) \nabla_z \theta(x_0)^\top = I,$$

and after multiplication by $\nu_0(x_0)$ from the right and by its transpose from the left we get from (3.8)

$$\partial_{\nu_0}\theta(x_0) = (\partial_t X(x_0,0)\cdot \nu_0(x_0))^{-1} > 0, \tag{3.14}$$

so $\theta(z)$ is positive whenever $z \in V \setminus \Omega_0$ and δ_0 sufficiently small. Hence for such z, (ξ,θ) also solves the original equation (3.10). All further statements of the lemma follow now from Lemma 3.3 by combining the local results near sufficiently many points of Γ_0. □

Observe that under the assumptions of the lemma, we have

$$\Theta(X,x,t) = \theta_X(X(x,t)), \tag{3.15}$$

where θ has to be extended by zero inside Ω_0.

The assumptions (3.7), (3.8) ensure that for small T, the mappings $X(\cdot,t)$ are diffeomorphisms satisfying $\Omega_0 \subset X(\Omega_0,t)$. For technical reasons, we have to extend them to a slightly larger set

$$\Omega_1 = \Omega_1(\delta) := \Omega_0 + B(0,\delta), \qquad \delta > 0 \text{ small},$$

with preservation of these properties.

Note first that $\partial\Omega_1 = \{\xi + \delta\nu_0(\xi) \,|\, \xi \in \Gamma_0\}$ and

$$\nu_1(\tilde\xi) = \nu_0(\xi), \quad \tilde\xi := \xi + \delta\nu_0(\xi), \quad \xi \in \Gamma_0, \tag{3.16}$$

and ν_1 is the outer unit normal on Ω_1.

Lemma 3.4. *Under the assumptions of Lemma 3.2, there are constants $\delta, T > 0$ depending only on γ, K such that for any $t \in I$, $X(\cdot,t)$ has an extension $\tilde X(\cdot,t) \in \mathrm{Diff}^{1+\alpha}(\Omega_1, \tilde X(\Omega_1,t))$ such that*

(i) $\tilde X(\cdot,t)|_{\Omega_0} = X(\cdot,t)$,

(ii) $X(\Omega_0,t) \subset \Omega_1 \subset \tilde X(\Omega_1,t)$,

(iii) $t \mapsto \tilde X(\cdot,t) \in C^1(I, C^{1+\alpha}(\bar\Omega_1,\Pi))$.

Proof. Let $\mathsf{E} = \mathsf{E}(\delta) \in \mathcal{L}(C^s(\Omega_0), C^s(\Omega_1))$, $s \in [0, 1+\alpha]$ be a usual extension operator where δ is small enough to satisfy

$$\gamma - \|\mathsf{E}(\delta)\|_{\mathcal{L}(C^\alpha(\Omega_0), C^\alpha(\Omega_1))} K\delta^\alpha > 0. \tag{3.17}$$

Define

$$\tilde{X}(\cdot, t) := \mathsf{E}(X(\cdot, t) - \mathrm{id}_{\Omega_0}) + \mathrm{id}_{\Omega_1}.$$

Then (i) and (iii) are clear. Furthermore,

$$\|\tilde{X}(\cdot, t) - \mathrm{id}\|_{C^1} \le CKT\delta^\alpha$$

and hence $\tilde{X}(\cdot, t) \in \mathrm{Diff}^{1+\alpha}(\Omega_1, \tilde{X}(\Omega_1, t))$ if T is small. The first inclusion in (ii) is also clear for T small. Finally,

$$\partial_t \tilde{X}(\cdot, t) = \mathsf{E}\partial_t X(\cdot, t)$$

and therefore by (3.16), (3.17)

$$\begin{aligned} \partial_t \tilde{X}(\tilde{\xi}, 0) \cdot \nu_1(\tilde{\xi}) \;&\ge\; \partial_t X(\xi, 0) \cdot \nu_0(\xi) - |\partial_t \tilde{X}(\tilde{\xi}, 0) - \partial_t \tilde{X}(\xi, 0)| \\ &\ge\; \gamma - \|\mathsf{E}\|_{\mathcal{L}(C^\alpha(\Omega_0), C^\alpha(\Omega_1))} \|\partial_t X(\cdot, 0)\|_{C^\alpha} |\tilde{\xi} - \xi|^\alpha > 0. \end{aligned}$$

This implies the second inclusion in (ii). □

On $M(\varepsilon, K)$ we define the mapping $F = (F_1, F_2, F_3)$ given by (cf. (2.11))

$$\begin{aligned} F_1(u)(x,t) &:= Y(x,t) := x - \int_0^t E(x,\tau)\, d\tau, \\ F_2(u)(x,t) &:= \sigma_0(x) + \int_0^t \sigma(x,\tau)\big(f(|E(x,\tau)|) + (\rho - \sigma)(x,\tau)\big)\, d\tau, \\ F_3(u)(x,t) &:= \rho_0\big(Y(x,t)\big) + \int_{\Theta(Y,x,t)}^t \tilde{\sigma}\big(Z_t(x,\tau),\tau\big) f\big(|\tilde{E}(Z_t(x,\tau),\tau)|\big)\, d\tau \end{aligned} \tag{3.18}$$

where ρ is extended by zero outside Ω_0, $\tilde{\sigma} := \mathsf{E}\sigma$, $\tilde{E} := \mathsf{E}E$, and the abbreviations Z_t and E are given by

$$Z_t(x,\tau) := \tilde{Y}^{-1}\big(Y(x,t),\tau\big), \tag{3.19}$$

$$E(\cdot,t) := \mathscr{E}[X(\cdot,t)]((\rho - \sigma)(\cdot,t)). \tag{3.20}$$

Note that Lemma 3.4 ensures that $Z_t(x,\tau)$ is well defined for all $t, \tau \in I$ and takes values in Ω_1, provided ε and T are small. Moreover, $Z_t(x,\tau) \in \Omega_0$ if and only if $\tau > \Theta(Y,x,t)$ so that F is independent of the extension operator E.

It easily follows from this observation that (under suitable regularity assumptions) the fixed point problem

$$u = F(u), \quad u \in M(\varepsilon, K)$$

is equivalent to the solution of (2.11). Note that differing from (2.11) we have used $Y \equiv F_1$ instead of X in the definition of F_3. This is mainly to make use of better regularity properties of Y with respect to the time variable t.

Lemma 3.5. *Let $K_1 > \|\sigma_0\|_{C^{1+\alpha}}$, then for K_2 sufficiently large and sufficiently small $\varepsilon, T > 0$, F maps $M(\varepsilon, K)$ into itself.*

Proof. Let $(X,\sigma,\rho)\in M(\varepsilon,K)$, the conditions to be satisfied by ε, T, and K_2 will be gathered during the proof. Unless otherwise indicated, constants denoted by C in this proof are allowed to depend on Ω_0,α,λ, and the ρ_0, σ_0 as well as γ but not on K.

Step 1: Estimate of $\|F_1(u)-\mathrm{id}\|_{C^\alpha(I,C^{1+\alpha}_\lambda)}$.

From (M2), (M3) and the assumption (3.1) we see

$$\big[t\mapsto(\rho-\sigma)(\cdot,t)\big]\in B(I,C^{1+\alpha})\cap B(I,C^\alpha_\lambda)\cap C^\alpha(I,C_\lambda), \tag{3.21}$$

thus, remembering (3.20) and using Lemma 2.5, we find

$$\big[t\mapsto E(\cdot,t)\big]\in B(I,C^{1+\alpha}_\lambda)\cap C^\alpha(I,C_\lambda) \tag{3.22}$$

together with estimates

$$\begin{aligned}\|E\|_{B(I,C^{1+\alpha}_\lambda)}&\le C\|\rho-\sigma\|_{B(I,C^\alpha_\lambda)}\le C(K_1+\|\rho_0-\sigma_0\|_{C^\alpha_\lambda})\le C(K_1+1),\\ \|E\|_{C^\alpha(I,C_\lambda)}&\le C\big\{\|X\|_{C^\alpha(I,C^{1+\alpha})}\|\rho-\sigma\|_{B(I,C_\lambda)}+\|\rho-\sigma\|_{C^\alpha(I,C_\lambda)}\big\}\le CK_1.\end{aligned}$$

Therefore, by Lemma A.1 (ii), we have

$$[t\mapsto(F_1(u)-\mathrm{id})(\cdot,t)]\in\mathrm{Lip}(I,C^{1+\alpha}_\lambda)\cap C^{1+\alpha}(I,C_\lambda)$$

with

$$\|F_1(u)-\mathrm{id}\|_{B(I,C^{1+\alpha}_\lambda)}\le C(K_1+1)T,\quad \|F_1(u)-\mathrm{id}\|_{\mathrm{Lip}(I,C^{1+\alpha}_\lambda)}\le CK_1(T+1)$$

Thus choosing $T>0$ sufficiently small, this implies

$$\|F_1(u)-\mathrm{id}\|_{C^\alpha(I,C^{1+\alpha}_\lambda)}\le\varepsilon,$$

hence $F_1(u)$ satisfies condition (M1). Moreover we find from (3.22)

$$\|F_1(u)-\mathrm{id}\|_{C^{1+\alpha}(I,C_\lambda)}\le CK_1(T+1),$$

and consequently

$$\|F_1(u)\|_{C^{1+\alpha}(\bar\Omega_0\times I)}\le C(K_1+1). \tag{3.23}$$

Step 2: Estimate of $\|F_2(u)\|_{B(I,C^{1+\alpha})}$ *and* $\|F_2(u)-\sigma_0\|_{C^\alpha(I,C^{1+\alpha}_\lambda)}$.

In view of (3.22), the smoothness of $y\mapsto f(|y|)$, and $f(0)=0$ we have

$$\big[t\mapsto f(|E(\cdot,t)|)\big]\in B(I,C^{1+\alpha}_\lambda)\cap C^\alpha(I,C_\lambda), \tag{3.24}$$

and by this and (3.21), the integrand in the definition of F_2 is in $B(I,C^{1+\alpha})\cap B(I,C^\alpha_\lambda)\cap C^\alpha(I,C_\lambda)$, and its norm in this space is bounded by a constant depending (for given f) only on K. Consequently, due to Lemma A.1 (ii) we have

$$F_2(u)-\sigma_0\in\mathrm{Lip}(I,C^{1+\alpha})\cap\mathrm{Lip}(I,C^\alpha_\lambda)$$

with corresponding estimates

$$\begin{aligned}\|F_2(u)-\sigma_0\|_{B(I,C^{1+\alpha})},\ \|F_2(u)-\sigma_0\|_{B(I,C^\alpha_\lambda)}&\le C(K)T,\\ \|F_2(u)-\sigma_0\|_{\mathrm{Lip}(I,C^{1+\alpha})},\ \|F_2(u)-\sigma_0\|_{\mathrm{Lip}(I,C^\alpha_\lambda)}&\le C(K)(T+1).\end{aligned}$$

This implies via interpolation

$$\|F_2(u)\|_{B(I,C^{1+\alpha})},\ \|F_2(u)-\sigma_0\|_{C^\alpha(I,C^\alpha_\lambda)} \le K_1,$$

if $T>0$ is chosen sufficiently small and $K_1 > \|\sigma_0\|_{C^{1+\alpha}}$.

Step 3: Estimate of $\|F_3(u)(\cdot,t)\|_{C^{1+\alpha}(\bar\Omega_0)}$.

Fix $t\in I$ and define $D_t := Y^{-1}(\Omega_0,t)$,

$$\begin{aligned}\psi(x,\tau) &:= \tilde\sigma\big(Z_t(x,\tau),\tau\big) f\big(|\tilde E(Z_t(x,\tau),\tau)|\big), \quad \tau\in I\\ \Theta(x) &:= \Theta(Y,x,t).\end{aligned}\tag{3.25}$$

Note that

$$[\tau\mapsto Z_t(\cdot,\tau)] \in B\big(I,C^{1+\alpha}(\bar\Omega_0)\big)\cap C\big(I,C(\bar\Omega_0)\big).$$

and consequently

$$\tau\mapsto\psi(\cdot,\tau)\in B(I,C^{1+\alpha}(\bar\Omega_0)\cap C(I,C(\bar\Omega_0).$$

with

$$\|\tau\mapsto\psi(\cdot,\tau)\|_{B(I,C^{1+\alpha})} \le C(K_1).\tag{3.26}$$

The estimate will be given by showing

$$F_3(u)(\cdot,t)|_{\overline{D}_t}\in C^{1+\alpha}(\overline{D}_t),\quad F_3(u)(\cdot,t)|_{\bar\Omega_0\setminus D_t}\in C^{1+\alpha}(\bar\Omega_0\setminus D_t)$$

and continuity of F_3 and its first spatial derivatives across $\partial D_t = Y^{-1}(\Gamma_0,t)$. Then

$$\|F_3(u)(\cdot,t)\|_{C^{1+\alpha}(\bar\Omega_0)} \le C(\|F_3(u)(\cdot,t)|_{\bar D_t}\|_{C^{1+\alpha}(\bar D_t)} + \|F_3(u)(\cdot,t)|_{\bar\Omega_0\setminus D_t}\|_{C^{1+\alpha}(\bar\Omega_0\setminus D_t)})\tag{3.27}$$

with a constant C that can be chosen independently of t as the boundaries ∂D_t are "uniformly $C^{1+\alpha}$"-manifolds as they are images of Γ_0 under $C^{1+\alpha}$-diffeomorphisms that are uniformly bounded in this norm.

To estimate the first term on the right, observe that $\Theta(x)=0$ for $x\in\bar D_t$, $\rho_0\circ Y(\cdot,t)\in C^{1+\alpha}(\overline{D}_t)$ and apply (3.26) to get

$$\|F_3(u)(\cdot,t)|_{\bar D_t}\|_{C^{1+\alpha}(\bar D_t)} \le C(K_1).\tag{3.28}$$

For $x\in\bar\Omega_0\setminus D_t$ we have

$$\begin{aligned}F_3(u)(x,t) &= \int_{\Theta(x)}^t \psi(x,\tau)\,d\tau,\\ \partial_i F_3(u)(x,t) &= -\partial_i\Theta(x)\psi(x,\Theta(x)) + \int_{\Theta(x)}^t \partial_i\psi(x,\tau)\,d\tau.\end{aligned}\tag{3.29}$$

Observe that $\partial_t Y(\cdot,0)\cdot\nu_0 = -E_0\cdot\nu_0 > 0$ on $\partial\Omega_0$ due to (3.2). Therefore, by Lemma 3.2 and (3.15) we find $\Theta\in C^{1+\alpha}(\bar\Omega_0\setminus D_t)$ for Y fixed and T sufficiently small. This implies C^α-smoothness for the first term in (3.29). To get this for the second term, pick $x_1,x_2\in\bar\Omega_0\setminus D_t$ such that without loss of generality $\Theta(x_1)\le\Theta(x_2)$. Then, using (3.26) again,

$$\left|\int_{\Theta(x_1)}^t \partial_i\psi(x_1,\tau)\,d\tau - \int_{\Theta(x_2)}^t \partial_i\psi(x_2,\tau)\,d\tau\right|$$
$$\le \int_{\Theta(x_1)}^{\Theta(x_2)} |\partial_i\psi(x_1,\tau)|\,d\tau + \int_{\Theta(x_2)}^t |\partial_i\psi(x_1,\tau) - \partial_i\psi(x_2,\tau)|\,d\tau$$
$$\le C(K_1)|x_1 - x_2|^\alpha.$$

Consequently, also

$$\|F_3(u)(\cdot,t)|_{\bar\Omega_0\setminus D_t}\|_{C^{1+\alpha}} \le C(K_1). \tag{3.30}$$

Let $\xi \in \partial D_t$. By (3.2) and continuity of Θ we have for the one-sided limits

$$\lim_{D_t\ni x\to\xi} F_3(u)(x,t) = \lim_{D_t\not\ni x\to\xi} F_3(u)(x,t) = \int_0^t \psi(\xi,\tau)\,d\tau,$$

hence both $F_3(u)(\cdot,t)$ and its tangential derivatives are continuous across ∂D_t. To show continuity of the complete gradient it is sufficient now to consider the directional derivative in the nontangential direction $\nu := (DY(\cdot,t)^\top)^{-1}\nu_0$. We will write

$$\partial_\nu^\pm u(\xi) := \lim_{h\to\pm 0} h^{-1}\big(u(\xi+h\nu) - u(\xi)\big)$$

for functions u defined either in $\bar D_t$ or $\bar\Omega_0 \setminus D_t$. From the inside, we get

$$\partial_\nu^- F_3(u)(\xi,t) = \partial_{\nu_0}\rho_0\big(Y(\xi,t)\big) + \int_0^t \partial_\nu^-\psi(\xi,\tau)\,d\tau.$$

From the outside, using $Z_t(\xi,0) = Y(\xi,t)$, $\Theta(x) = \theta(Y(x,t))$, and (cf. (3.14))

$$\partial_{\nu_0}\theta = -(E_0\cdot\nu_0)^{-1} \quad \text{on } \partial\Omega_0,$$

we get with $y = Y(x,t)$

$$\partial_\nu^+ F_3(\xi,t) = \big(E_0(y)\cdot\nu_0(y)\big)^{-1}\sigma_0(y)f\big(|E_0(y)|\big) + \int_0^t \partial_\nu^+\psi(\xi,\tau)\,d\tau,$$

and the equality of both limits follows from (3.2).

Thus $F_3(u)(\cdot,t) \in C^{1+\alpha}(\bar\Omega_0)$, and from (3.27), (3.28), and (3.30)

$$\|F_3(u)(\cdot,t)\|_{1+\alpha}^{\bar\Omega_0} \le C(K_1) \le K_2,$$

if K_2 is chosen sufficiently large.

Step 4: Estimate of $\|F_3(u) - \rho_0\|_{B(I,C_\lambda^\alpha)}$ *and* $\|F_3(u) - \rho_0\|_{C^\alpha(I,C_\lambda)}$.

We first estimate

$$F_3(u) - \rho\circ Y = [(x,t)\mapsto \int_{\Theta(Y,x,t)}^t \psi(x,\tau)\,d\tau].$$

Observe (cf. (3.15)) that the mappings $t \mapsto \Theta(Y,x,t)$ and $x \mapsto \Theta(Y,x,t)$ are Lipschitz continuous with uniform bounds. Moreover, the integrand of $F_3(u)$ is C^α with respect to all arguments. Using the estimate

$$\left|\int_{\Theta(x_1)}^{\Theta(x_2)} \psi(x_1,\tau)\,d\tau\right| \le C(K_1)|\Theta(x_2)-\Theta(x_1)|^{\alpha}|\Theta(x_2)-\Theta(x_1)|^{1-\alpha}$$
$$\le C(K_1)|x_2-x_1|^{\alpha}T^{1-\alpha}$$

and estimates as given in Step 3, one shows

$$\|F_3(u)-\rho_0\circ Y\|_{B(I,C^\alpha)}, \|F_3(u)-\rho_0\circ Y\|_{C^\alpha(I,C)} \le C(K_1)T^{1-\alpha}.$$

More precisely, using (3.24) we analogously get

$$\|F_3(u)-\rho_0\circ Y\|_{B(I,C^\alpha_\lambda)}, \|F_3(u)-\rho_0\circ Y\|_{C^\alpha(I,C_\lambda)} \le C(K_1)T^{1-\alpha}. \tag{3.31}$$

Furthermore, one straightforwardly gets

$$\|\rho_0\circ Y-\rho_0\|_{C^\alpha(I,C_\lambda)} \le C(K_1)\|Y-\mathrm{id}\|_{C^\alpha(I,C_\lambda)} \le C(K_1)T^{1-\alpha} \tag{3.32}$$

and by Lemma A.3

$$\|\rho_0\circ Y-\rho_0\|_{B(I,C^\alpha_\lambda)} \le C(K_1)\|Y-\mathrm{id}\|_{B(I,C^\alpha_\lambda)} \le C(K_1)T. \tag{3.33}$$

Choosing T small, we get from (3.31)–(3.33)

$$\|F_3(u)-\rho_0\|_{B(I,C^\alpha_\lambda)}, \|F_3(u)-\rho_0\|_{C^\alpha(I,C_\lambda)} \le K_1$$

as demanded in (M3).

□

On $M(\varepsilon,K)$ we define the metric d by

$$\mathsf{d}(u_1,u_2) := \|X_1-X_2\|_{B(I,C^{1+\alpha}_\lambda)} + \|\sigma_1-\sigma_2\|_{B(I,C^\alpha_\lambda)} + \|\rho_1-\rho_2\|_{B(I,C^\alpha_\lambda)},$$

$u_i := (X_i,\sigma_i,\rho_i)$, $i=1,2$. It follows from Lemma A.1 that $M(\varepsilon,K)$ is complete with respect to d.

Lemma 3.6. *Assume $\varepsilon, T>0$ and K such that $F: M(\varepsilon,K)\to M(\varepsilon,K)$ according to Lemma 3.5. Then F is contractive with respect to the metric d, provided $T>0$ is sufficiently small.*

Proof. Fix $u_1=(X_1,\sigma_1,\rho_1), u_2=(X_2,\sigma_2,\rho_2)\in M(\varepsilon,K)$ and denote the corresponding quantities by Y_i, E_i, ψ_i, $i=1,2$ (see (3.25)). As

$$\|\rho_i-\sigma_i\|_{B(I,C^\alpha_\lambda)} \le C(K_1), \quad i=1,2,$$
$$\|(\rho_1-\sigma_1)-(\rho_2-\sigma_2)\|_{B(I,C^\alpha_\lambda)} \le \|\rho_1-\rho_2\|_{B(I,C^\alpha_\lambda)} + \|\sigma_1-\sigma_2\|_{B(I,C^\alpha_\lambda)},$$

we obtain from Lemma 2.5 immediately

$$\|E_1-E_2\|_{B(I,C^{1+\alpha}_\lambda)} \le C(K_1)\mathsf{d}(u_1,u_2),$$

hence

$$\|F_1(u_1)-F_1(u_2)\|_{B(I,C^{1+\alpha}_\lambda)} \le CT\mathsf{d}(u_1,u_2). \tag{3.34}$$

In the same manner (using the smoothness assumptions on f) we find

$$\|\sigma_1 f(|E_1|)-\sigma_2 f(|E_2|)\|_{B(I,C^\alpha_\lambda)} \le C(K_1)\mathsf{d}(u_1,u_2)$$

as well as

$$\|(\sigma_1-\rho_1)\sigma_1-(\sigma_2-\rho_2)\sigma_2\|_{B(I,C^\alpha_\lambda)}\le C(K_1)\mathsf{d}(u_1,u_2),$$

thus

$$\|F_2(u_1)-F_2(u_2)\|_{B(I,C^\alpha_\lambda)}\le CT\mathsf{d}(u_1,u_2). \tag{3.35}$$

It remains to consider the third component. We write $F_3(u)$ in the form

$$F_3(u)(x,t)=H(u)\big(Y(x,t),t\big),\quad x\in\Omega_0,t\in I$$

with H given by

$$H(u)(x,t):=\rho_0(x)+\int_{\theta_Y(x)}^t\eta(u)(x,\tau)\,d\tau,$$
$$\eta(u)(x,\tau):=\tilde\sigma\big(\tilde Y^{-1}(x,\tau),\tau\big)f\big(|E(\tilde Y^{-1}(x,\tau),\tau)|\big)$$

for $t,\tau\in I$ and $x\in\Omega_1$. (Cf. Lemma 3.4 and (3.18). If t is fixed and ψ is defined by (3.25) then $\psi(\cdot,\tau)=\eta(\cdot,\tau)\circ Y(x,t)$.) Then we have

$$\|\eta(u)(\cdot,\tau)\|_{C^{1+\alpha}_\lambda(\Omega_1)}\le C(K_1).$$

Further, by Lemma 3.3

$$\|\tilde Y_1^{-1}(\cdot,t)-\tilde Y_2^{-1}(\cdot,t)\|_{C^\alpha_\lambda}\le C(K_1)\|Y_1(\cdot,t)-Y_2(\cdot,t)\|_{C^\alpha_\lambda}\le C(K_1)T\mathsf{d}(u_1,u_2),$$

and by Lemma A.3

$$\|\eta(u_1)(\cdot,\tau)-\eta(u_2)(\cdot,\tau)\|_{C^\alpha_\lambda(\Omega_1)}\le C(K_1)\mathsf{d}(u_1,u_2). \tag{3.36}$$

Moreover, as in the proof of Lemma 3.5, step 3 we find $H(u)(\cdot,t)\in C^{1+\alpha}(\Omega_1)$ with

$$\|H(u)(\cdot,t)\|_{C^{1+\alpha}(\Omega_1)}\le C(K_1),$$

and consequently, again by Lemma A.3,

$$\begin{aligned}\big\|H(u_1)\big(Y_1(\cdot,t),t\big)-H(u_1)\big(Y_2(\cdot,t),t\big)\big\|_{C^\alpha_\lambda}&\le C(K_1)\|Y_1(\cdot,t)-Y_2(\cdot,t)\|_{C^\alpha_\lambda}\\&\le C(K_1)T\mathsf{d}(u_1,u_2).\end{aligned}$$

Splitting

$$\begin{aligned}H(u_1)(\cdot,t)-H(u_2)(\cdot,t):=&\int_{\theta_{Y_1}(\cdot)}^t\eta(u_1)(\cdot,\tau)-\eta(u_2)(\cdot,\tau)\,d\tau\\&+\int_{\theta_{Y_1}(\cdot)}^{\theta_{Y_2}(\cdot)}\eta(u_2)(\cdot,\tau)\,d\tau=:I_1(\cdot,t)+I_2,\end{aligned}$$

we obtain using (3.36)

$$\|I_1(\cdot,t)\|_{C^\alpha_\lambda}\le C(K_1)\big(T+T^{1-\alpha}\|\theta_{Y_1}(\cdot)\|_{C^\alpha}\big)\mathsf{d}(u_1,u_2).$$

On the other hand, by Lemma 3.2 we have

$$\|\theta_{Y_1}-\theta_{Y_2}\|_{C^\alpha}\le\|Y_1-Y_2\|_{C^\alpha(\bar\Omega_0\times I,\Pi)}\le C(K_1)T\mathsf{d}(u_1,u_2),$$

and consequently

$$\|I_2\|_{C^\alpha_\lambda}\le C(K_1)T\mathsf{d}(u_1,u_2).$$

Summarizing the above estimates we have finally

$$\|F_3(u_1) - F_3(u_2)\|_{B(I,C^\alpha_\lambda)} \le C(K_1)T\mathsf{d}(u_1,u_2). \tag{3.37}$$

The estimates (3.34), (3.35), (3.37) imply the assertion for $T > 0$ sufficiently small and the proof is complete. $\square$

Remark: Using our previous results it is not hard to see that if $(X,\sigma,\rho) \in M(\varepsilon, K)$ is the fixed point of F then $t \mapsto X(x,t)$, $t \mapsto \sigma(x,t)$, and $t \mapsto \rho(x,t)$ are $C^{1+\alpha}$. This implies the additional smoothness statements in Theorem 1.2.

A. Some auxiliary results

Let $((\mathcal{X}_\theta, \|\cdot\|_\theta) \,|\, \theta \in [0,1])$ be a scale of Banach spaces such that $\mathcal{X}_\theta \hookrightarrow \mathcal{X}_0$ and

(A1) for any $x \in \mathcal{X}_1$, the mapping $\theta \mapsto \|x\|_\theta$ is nondecreasing,

(A2) for any $x \in \mathcal{X}_1$, the interpolation inequality

$$\|x\|_\theta \le C\|x\|_1^\theta \|x\|_0^{1-\theta}$$

holds,

(A3) for any $x \in \mathcal{X}_0$, we have $x \in \mathcal{X}_1$ iff $x \in \mathcal{X}_\theta$ for all $\theta \in [0,1)$ and $\sup_\theta \|x\|_\theta < \infty$, and in this case

$$\|x\|_1 = \sup_\theta \|x\|_\theta.$$

Note that for fixed $k \in \mathbb{N}$, $\alpha, \lambda \in (0,1)$ the scales of spaces given by

$$\mathcal{X}_\theta := C^{\theta(k+\alpha)}(\bar{\Omega}_0), \quad \mathcal{X}_\theta := C^{\theta(k+\alpha)}([0,T],Z), \quad \text{or} \quad \mathcal{X}_\theta := C^\alpha_{\theta\lambda}(\bar{\Omega}_0)$$

(with appropriate norms) satisfy these assumptions, where Z is any Banach space.

Lemma A.1. *Let a scale of Banach spaces $((\mathcal{X}_\theta, \|\cdot\|_\theta) \,|\, \theta \in [0,1])$ satisfy* (A1)–(A3). *Then, for $\theta \in (0,1)$,*

(i) *For any sequence (u_n) in $\mathcal{X}_1$ satisfying $\|u_n\|_1 \le K$ and $u_n \to u^*$ in $\mathcal{X}_0$ we have $u^* \in \mathcal{X}_1$, $\|u^*\|_1 \le K$, and $u_n \to u^*$ in $\mathcal{X}_\theta$.*

(ii) *Let $T > 0$ and $u \in C([0,T], \mathcal{X}_0)$ with $u(t) \in \mathcal{X}_1$ and $\|u(t)\|_1 \le K$ for all $t \in [0,T]$. Then*

$$\int_0^T u(t)\,dt \in \mathcal{X}_1, \qquad \left\| \int_0^T u(t)\,dt \right\|_1 \le KT$$

and consequently, if

$$(Iu)(t) := \int_0^t u(\tau)\,d\tau$$

then

$$Iu \in \mathrm{Lip}([0,T], \mathcal{X}_1), \quad \|Iu\|_{\mathrm{Lip}([0,T],\mathcal{X}_1)} \le (T+1)K.$$

(iii)

$$\begin{aligned}
B([0,T],\mathcal{X}_1)\cap C([0,T],\mathcal{X}_0) &\hookrightarrow C([0,T],\mathcal{X}_\theta),\\
B([0,T],\mathcal{X}_1)\cap C^\alpha([0,T],\mathcal{X}_0) &\hookrightarrow C^{\alpha(1-\theta)}([0,T],\mathcal{X}_\theta),\\
B([0,T],\mathcal{X}_1)\cap \mathrm{Lip}([0,T],\mathcal{X}_0) &\hookrightarrow C^{1-\theta}([0,T],\mathcal{X}_\theta).
\end{aligned}$$

(iv) *Let* (u_n) *be a sequence in* $B([0,T],\mathcal{X}_1)\cap C([0,T],\mathcal{X}_0)$ *with* $\|u_n(t)\|_1\le K$. *Assume* (u_n) *converges in* $B([0,T],\mathcal{X}_0)$. *Then the limit* u^* *is in* $B([0,T],\mathcal{X}_1)\cap C([0,T],\mathcal{X}_\theta)$ *and satisfies* $\|u^*(t)\|_1\le K$.

(v) *Let* (u_n) *be a sequence in* $\mathrm{Lip}([0,T],\mathcal{X}_1)\cap C([0,T],\mathcal{X}_0)$ *with* $\|u_n\|_{\mathrm{Lip}([0,T],\mathcal{X}_1)}\le K$. *Assume* (u_n) *converges in* $B([0,T],\mathcal{X}_0)$. *Then the limit* u^* *is in* $\mathrm{Lip}([0,T],\mathcal{X}_1)\cap C([0,T],\mathcal{X}_\theta)$ *and satisfies* $\|u^*\|_{\mathrm{Lip}([0,T],\mathcal{X}_1)}\le K$.

Proof: (i) As (u_n) is a Cauchy sequence in $\mathcal{X}_0$ we have because of (A2)

$$\|u_n-u_m\|_\theta\le C\|u_n-u_m\|_0^{1-\theta}\|u_n-u_m\|_1^\theta\to 0\quad\text{as } m,n\to\infty$$

for any $\theta\in[0,1)$. Thus, (u_n) is a Cauchy sequence in $\mathcal{X}_\theta$, and $u_n\to u^*$ in $\mathcal{X}_\theta$ as well. Moreover, $\|u^*\|_\theta=\lim_{n\to\infty}\|u_n\|_\theta\le K$, and the remaining assertions follow from (A3).

(ii) Note that

$$\int_0^T u(t)\,dt=\lim_{n\to\infty} I_n\qquad\text{in } \mathcal{X}_0,$$

where

$$I_n:=\frac{T}{n}\sum_{k=0}^{n-1}u(kT/n),$$

and hence $I_n\in\mathcal{X}_1$, $\|I_n\|_1\le KT$ for all $n\in\mathbb{N}$. The assertions follow now from (i), applied to the sequence (I_n).

(iii) The first embedding is an immediate consequence of (i). The second and third follow easily from (A2).

(iv) For $t\in[0,T]$ we have $u_n(t)\to u^*(t)$ in $\mathcal{X}_0$ and $\|u_n(t)\|_1$ is bounded uniformly in n and t. Therefore by (i) $\|u^*(t)\|\le K$. Furthermore $u^*\in C([0,T],\mathcal{X}_0)$ by uniform convergence, and therefore by (iii) $u^*\in C([0,T],\mathcal{X}_\theta)$.

(v) Fix $s,t\in[0,T]$. By assumption, the sequence $u_n(t)-u_n(s)$ is convergent in $\mathcal{X}_0$ and $\|u_n(t)-u_n(s)\|_1\le K|t-s|$. Thus, by (i), $u^*(t)-u^*(s)\in\mathcal{X}_1$ and $\|u^*(t)-u^*(s)\|_1\le K|t-s|$. This proves the result. □

We provide a proof of the following result on superposition operators in Hölder spaces.

Lemma A.2. *Let* $\Omega\subset\mathbb{R}^m$ *be a domain,* $g_1,g_2\in C^\alpha(\Omega,\mathbb{R}^k)$,

$$\Xi:=\{y\in\mathbb{R}^k\,|\,\mathrm{dist}\,(y,g_1(\Omega))\le\|g_1-g_2\|_{C^0}\},$$

and $F\in C^{1+\alpha}(\Xi)$. *Then*

$$\|F\circ g_1-F\circ g_2\|_{C^\alpha}\le\|F\|_{C^{1+\alpha}}\|g_1-g_2\|_{C^\alpha}.$$

Proof: Let $x\in\Omega$. Then

$$\begin{aligned}
|F(g_1(x))-F(g_2(x))| &\le \int_0^1|\nabla F(g_2(x)+s(g_1(x)-g_2(x)))||g_1(x)-g_2(x)|\,ds\\
&\le \|F\|_{C^1}\|g_1-g_2\|_{C^0}.
\end{aligned}$$

Let $x_1, x_2 \in \Omega$ and define $\Delta_i := g_1(x_i) - g_2(x_i)$, $i = 1, 2$. Then

$$|\Delta_i| \le \|g_1 - g_2\|_{C^0}, \quad |\Delta_1 - \Delta_2| \le \|g_1 - g_2\|_{C^\alpha}|x_1 - x_2|^\alpha.$$

Now

$$\begin{aligned} & |F(g_1(x_1)) - F(g_2(x_1)) - F(g_1(x_2)) + F(g_2(x_2))| \\ \le\ & |F(g_1(x_1)) - F(g_1(x_2)) - F(g_1(x_1) - \Delta_1) + F(g_1(x_2) - \Delta_1)| \\ & + |F(g_1(x_2) - \Delta_1) - F(g_1(x_2) - \Delta_2)| =: I_1 + I_2, \end{aligned}$$

and the terms on the right can be estimated separately by

$$\begin{aligned} I_1 & \le \int_0^1 |\nabla F(g_1(x_1) - s\Delta_1) - \nabla F(g_1(x_2) - s\Delta_1)||\Delta_1|\, ds \\ & \le \|F\|_{C^{1+\alpha}}\|g_1 - g_2\|_{C^0}|x_1 - x_2|^\alpha, \\ I_2 & \le \|F\|_{C^1}|\Delta_1 - \Delta_2| \le \|F\|_{C^1}\|g_1 - g_2\|_{C^\alpha}|x_1 - x_2|^\alpha. \end{aligned}$$

This proves the result. □

Let now Ω_0 be as above and recall the definition of the weighted spaces $C^\alpha_\lambda(\Omega_0)$. We provide a version of Lemma A.2 for these spaces.

Lemma A.3. *Let $g_1, g_2 \in C^\alpha_\lambda(\Omega_0, \mathbb{R}^k)$, let Ξ be defined as in Lemma* A.2 *and $F \in C^{1+\alpha}(\Xi))$. Then*

$$\|F \circ g_1 - F \circ g_2\|_{C^\alpha_\lambda} \le C\|F\|_{C^{1+\alpha}}\|g_1 - g_2\|_{C^\alpha_\lambda}$$

with C depending on λ and Ω_0 only.

Proof: For $\zeta \in \mathbb{R}$, denote

$$\Omega^{(\zeta)} := \{x = (x', z) \in \Omega_0 \,|\, z < \zeta\}$$

and observe that $C^\alpha_\lambda(\Omega_0)$ can be equipped with the equivalent norm $|||\cdot|||_{C^\alpha_\lambda}$ given by

$$|||u|||_{C^\alpha_\lambda} := \sup_{\zeta\in\mathbb{R}} e^{-\lambda\zeta}\|u|_{\Omega^{(\zeta)}}\|_{C^\alpha(\bar\Omega^{(\zeta)})}.$$

For any $\zeta \in \mathbb{R}$ we have by Lemma A.2

$$e^{-\lambda\zeta}\|F \circ g_1 - F \circ g_2\|_{C^\alpha(\bar\Omega^{(\zeta)})} \le e^{-\lambda\zeta}\|F\|_{C^{1+\alpha}}\|g_1 - g_2\|_{C^\alpha(\bar\Omega^{(\zeta)})} \le \|F\|_{C^{1+\alpha}}|||g_1 - g_2|||_{C^\alpha_\lambda},$$

and the result follows. □

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