On the Uniqueness of the Solution of the Capillary Problem in Porous Materials

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Abstract

The behaviour of capillary effects in porous materials is often considered in physical chemistry. Here the the qualitative ascent of fluids on the walls of the pores is examined. The principal point is the extension of a comparison principle of Concus and Finn on the current circumstances. So for strait cylindrical cavities with arbitrary cross-section a comparison principle is obtained. For the special case of a circular cross-section explicit results are stated. Finally an assumption of Israelachvili on the order of the ascent in cavities is verified.

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1 Introduction

This paper deals with cavities in porous materials, were we regard the pores as straight cylinders with the basement Ω. The goal is to find an equation for the interface S between the fluid and the vapour area. Mainly the order of the observed singularities on the cavity walls are of particular interest. The wide difference to the classical capillary problem is, that there is no fixed barrier between the fluid and the solid domain. That means the fluid may diffuse between the solid and the fluid region. This characteristic leads to an additional term in the typical capillary equation, which is

$$2H = \kappa u - \mu. \tag{1}$$

According to certain energy considerations, which for example are performed in [1], on obtains the equation

$$2H = F(u) + \kappa u - \mu \quad \text{on } S. \tag{2}$$

A solution $u(x_1, x_2)$ of equation (2) describes the graph of the capillary surface S, which establishes in the regarded pore. A different formulation is the following. If Ω is the two-dimensional cross-section of the straight cylinder, than for $(x_1, x_2) \in \Omega$, $u(x_1, x_2)$ indicates the rise in hight of the liquid in the cylinder.

In equation (2) H is the mean curvature, κ the positive capillary constant, μ a negative constant. More precisely $\kappa = \frac{g\rho}{\sigma}$ depends on the gravity g, the surface tension σ, and ρ the difference between the number densities of the liquid and the vapour phase.

The constant μ is determined via $\mu = \sigma^{-1}\rho kit \ln(X)$, were k is the Boltzmann constant, T the absolute temperature and $X \in (0, 1)$ is the reduced vapour pressure. Also in equation (2), the disjoining pressure potential appears, which is given by

$$F\left(x, u(x)\right) = c \int_{\Omega_s} \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2\right]^{-p/2} \mathrm{d}y, \tag{3}$$

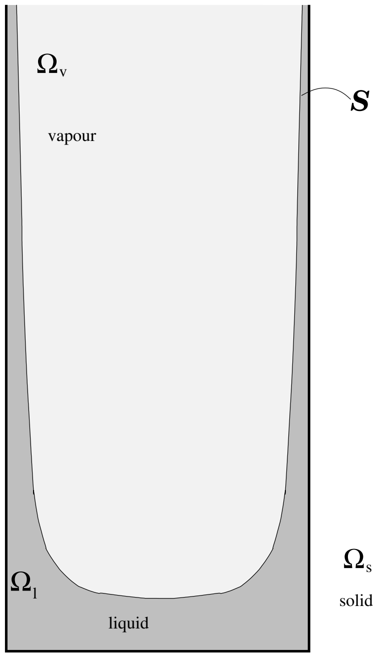

Figure 1: Closed capillary tube

for some constant $p > 3$, depending on the materials, and $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$. Here $c = \mathcal{H}/(\sigma\pi^2)$ is *negative*, because $\mathcal{H}$ is the Hamaker constant, see [2]. Ω_s is the (infinite) outer domain of the pore, cf. Fig. 1.

Even the disjoining pressure potential is the reason for the difference to the classical capillary equation (1). Here u appears just as a linear term, which simplyfies the calculations. As Finn in [3, p. 11] shows, on can write the mean curvature as a non-liner expression of ∇u, namely

$$2H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right). \tag{4}$$

Here $\operatorname{div}(f)$ denotes the sum of the first partial derivatives of the components of f. Because of this non-linearity of u in equation 4, on can formulate a comparison principle, which enables us to proof the uniqueness of the solution of equation (2). And in addition the comparison principle implicates in Section 3 the order of the singularity of $u(x_1, x_2)$, when (x_1, x_2) tends to $\partial\Omega$.

In the past, the capillary problems often was treated with linearised problems. But this procedure leads to different (and non-physically) results and would not bear something similar like the comparison principle.

So, when we replace $2H$, equation (2) turns into

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = F(u) + \kappa u - \mu.$$

To simplify the formula, on introduces the abbreviations

$$G(u) = F(u) + \kappa u - \mu \text{ and } \mathbf{T}\, u := \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}.$$

With this notation we get

$$\operatorname{div} \mathbf{T} u = G(u). \tag{5}$$

2 Comparison Principle

In this section a modification of the comparison principle of Concus and Finn is formulated and proven. Actually the principle also counts for unbounded domains Ω in $\mathbb{R}^n$. But here just the proof for the case $n = 2$ is given, because this is physically reasonable. The case $n > 2$ is treated in a similar way.
Following the proof, on recognises that the principle is in particular applicable, when ∇u becomes unbounded. This is a consequence of the nonlinearity of the problem.
Before we formulate the theorem, we still set

$$\mathbb{M}u := \operatorname{div} \mathbf{T} u - G(u).$$

The basic idea is, to show that $G(u)$ is monotonously increasing in u.

Theorem 1 *(Comparison principle) Let $\kappa > 0$ and suppose $\Sigma = \partial\Omega$ admits a decomposition $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$, where $\Sigma_\beta \in C^1$ and Σ_0 has a one-dimensional Hausdorff measure zero. Let $u, v \in C^2(\Omega)$ with the properties*

(i) $\mathbb{M}u \geq \mathbb{M}v$ *in* Ω,

(ii) $\operatorname{div} \boldsymbol{T} u - \operatorname{div} \boldsymbol{T} v \geq G(u) - G(v)$,

(iii) $(\boldsymbol{T} u - \boldsymbol{T} v)(x) \cdot \nu(x) \leq 0$ *almost everywhere on* Σ_β *as a limit from points of* Ω.

Then $v \geq u$ in Ω; equality holds at any point if and only if $v \equiv u$.

Proof: We follow the proofs, given in [4], [3, p.111] or [5].
Suppose u and v satisfy (i), (ii), (iii) and for some $x_0 \in \Omega$ there holds $u(x_0) - v(x_0) > 0$. So we define the sets

$$\begin{aligned} \Omega_1 &= \{x \in \Omega : u - v < 0\}, \\ \Omega_2 &= \{x \in \Omega : 0 < u - v < M\}, \\ \Omega_3 &= \{x \in \Omega : u - v > M\}. \end{aligned}$$

Due to our assumption, we can choose M so that Ω_2 has positive measure.

For any $R > 0$, we set $B_R = \{x \in \mathbb{R}^2; |x| < R\}$, $\Omega_R = \Omega \cap B_R$, $\Gamma_R = \partial B_R \cap \Omega$ (see Figure 2). We consider only those R sufficiently large that $|\Omega_R \cap \Omega_2|$ has positive measure.
Let be

$$w(x) := \begin{cases} 0 & \text{, in } \Omega_1 \\ u - v & \text{, in } \Omega_2 \\ M & \text{, in } \Omega_3. \end{cases}$$

The basic idea of the proof is, to show the next integral vanishes.

$$\int_{\Omega_R} w \cdot (\mathbb{M}u - \mathbb{M}v) \mathrm{d}x$$

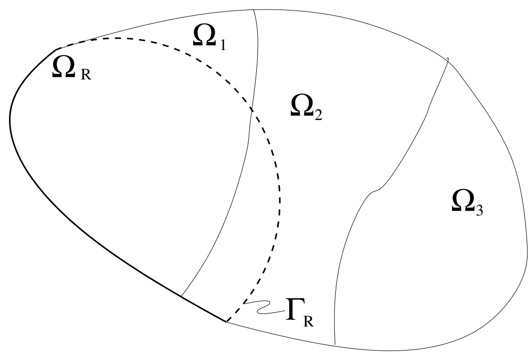

Figure 2: Partition of Ω

It is clear, that we have the next estimate.

$$
\begin{aligned}
0 &\leq \int_{\Omega_R} w \cdot (\mathbb{M}u - \mathbb{M}v)\mathrm{d}x \\
&= \int_{\Omega_R} w \cdot (\operatorname{div} \mathbf{T}\, u - \operatorname{div} \mathbf{T}\, v)\mathrm{d}x - \int_{\Omega_R} w \cdot (G(u) - G(v))\mathrm{d}x
\end{aligned}
\tag{6}
$$

To analyse the second integral we prove the monotonicity of $F(u)$, that is, we show $\frac{\partial F}{\partial u} \geq 0$.

$$
\begin{aligned}
\frac{\partial F}{\partial u} &= c \int_{\Omega_s} \frac{\partial}{\partial u} \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2\right]^{-p/2} \mathrm{d}y = \\
&= -pc \int_{\Omega_s} (u - y_3) \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2\right]^{-p/2-1} \mathrm{d}y \\
&= -pc \int_{\Omega} \left(\int_{-\infty}^{0} (u - y_3) \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2\right]^{-p/2-1} \mathrm{d}y_3 \right) \mathrm{d}y_1 \mathrm{d}y_2 \\
&\quad - pc \int_{\Omega^C} \left(\int_{-\infty}^{\infty} (u - y_3) \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2\right]^{-p/2-1} \mathrm{d}y_3 \right) \mathrm{d}y_1 \mathrm{d}y_2 \\
&= -pc \int_{\Omega} \left(\int_{-\infty}^{0} (u - y_3) \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2\right]^{-p/2-1} \mathrm{d}y_3 \right) \mathrm{d}y_1 \mathrm{d}y_2 \\
&\geq 0.
\end{aligned}
$$

The last inequality holds, because c is negative per assumption and the integrand is positive. Hence we obtain

$$
\begin{aligned}
\int_{\Omega_R} w \cdot (G(u) - G(v))\mathrm{d}x = &\int_{\Omega_R \cap \Omega_2} (u - v)^2 \mathrm{d}x + \int_{\Omega_R \cap \Omega_2} (u - v)(F(u) - F(v))\mathrm{d}x \\
&+ M \int_{\Omega_R \cap \Omega_3} (u - v)\mathrm{d}x + M \int_{\Omega_R \cap \Omega_3} (F(u) - F(v))\mathrm{d}x.
\end{aligned}
\tag{7}
$$

Since $u > v$ in $\Omega_2 \cup \Omega_3$ follows from the monotonicity that integral (7) is non negative.

How to handle the first integral on the right hand side in (6) we refer to [4, p. 193] and [5]. The operative point is, that this integral is non positive, which leads to a contradiction. That means it yields $v \geq u$ in Ω.

The other assertions of the theorem follows by the same reasoning such as Finn does in [3, p. 111 f]. ■

Remark 2 *By evaluating the integral*

$$\int_{\Omega_R} w(\operatorname{div} \boldsymbol{T}u - \operatorname{div} \boldsymbol{T}v)dx = \int_{\partial\Omega_R} w(\boldsymbol{T}u - \boldsymbol{T}v)\cdot\nu\, dx - \int_{\Omega_R} \nabla w\cdot(\boldsymbol{T}u - \boldsymbol{T}v)dx$$

we use a limiting procedure. In detail, we approximate the border of $\partial\Omega_R$ by an sequence $\{\Sigma_j\}$ of adequate surfaces, that converge to $\partial\Omega_R$ pointwise and in normal direction. For more details see [4, p. 193].

3 The circular Tube

From now on, we regard the special case, that the cylinder has got a circular cross section with constant radius $R > 0$. Thus we are looking for a rotationally symmetric solution, depending only on the distance from the origin.

At first we assume the case of an infinite tube, cf. Fig. 3. In this case we obtain an explicit result via integration. Here the solid domain is

$$D_s = \{(y_1, y_2, y_3) \in \mathbb{R}^3 :\ y_1^2 + y_2^2 > R^2,\ -\infty < y_3 < \infty\}.$$

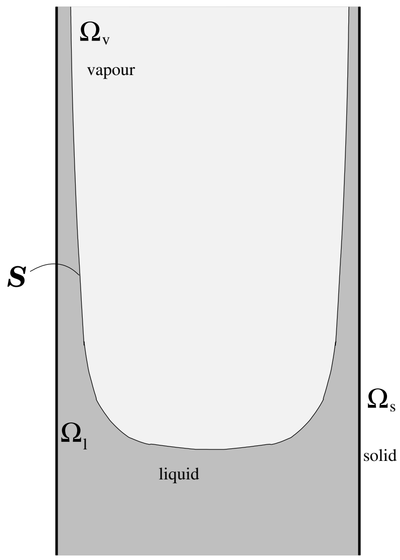

Figure 3: Open capillary tube

Now we convert Equation (3), by using polar and cylindrical coordinates for (x_1, x_2) and (y_1, y_2, y_3), respectively. We set $x_1 = Rq\cos\theta$, $x_2 = Rq\sin\theta$ and $y_1 = Rs\cos\varphi$, $y_2 = Rs\sin\varphi$, $y_3 = Rh$. Henceforward we denote $r = \sqrt{x_1^2 + x_2^2}$ and $q = r/R$. Then we set $U(q) = u(x_1, x_2)$ and $K(q, \theta, U(q)) = F(x, u(x_1, x_2))$ and thus the integral (3) changes to

$$\begin{aligned} K &= cR^{3-p}\int_1^\infty\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} s\left[(q\cos\theta - s\cos\varphi)^2 + (q\sin\theta - s\sin\varphi)^2 + (U-h)^2\right]^{-p/2} \mathrm{d}h\mathrm{d}\varphi\mathrm{d}s \\ &= cR^{3-p}\int_1^\infty\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} s\left[s^2 - 2qs\cos(\varphi-\theta) + q^2 + h^2\right]^{-p/2} \mathrm{d}h\mathrm{d}\varphi\mathrm{d}s \\ &= 2cR^{3-p}\int_1^\infty\int_{0}^{\pi}\int_{-\infty}^{\infty} s\left[s^2 - 2qs\cos\varphi + q^2 + h^2\right]^{-p/2} \mathrm{d}h\mathrm{d}\varphi\mathrm{d}s. \end{aligned}$$

The values θ and U can be omitted in the integral, because of the integration limits. So it turns out, that the disjoining pressure potential is a function that depends just on the one parameter q

$$F(x, u(x)) = cR^{3-p}\psi(q),\ q = r/R,$$

where $r = \sqrt{x_1^2 + x_2^2}$ is the distance to the origin and

$$\psi(q) = 2\int_{\Omega_1} s\left[s^2 - 2qs\cos\varphi + q^2 + h^2\right]^{-p/2} \mathrm{d}h\mathrm{d}\varphi\mathrm{d}s.$$

The domain of integration is $\Omega_1 = \{(s, \varphi, h) \in \mathbb{R}^3 :\ 1 < s < \infty,\ 0 < \varphi < \pi,\ -\infty < h < \infty\}$. Furthermore, we follow the achievements of Philip in [6, IV].

Integration over h leads to the Beta Function $\mathrm{B}(.,.)$, so that we obtain with the help of [7, Formula 3.621]

$$\begin{aligned}
\psi(q) &= 4\int_1^\infty \int_0^\pi s\left[s^2 - 2sq\cos\varphi + q^2\right]^{\frac{1-p}{2}} \mathrm{d}\varphi\mathrm{d}s \int_0^{\pi/2} (\cos t)^{p-2}\mathrm{d}t \\
&= 2^{p-1}\mathrm{B}\left(\frac{p-1}{2}, \frac{p-1}{2}\right) \int_1^\infty \int_0^\pi s\left[s^2 - 2sq\cos\varphi + q^2\right]^{\frac{1-p}{2}} \mathrm{d}\varphi\mathrm{d}s \\
&= 2^{p-1}\frac{\Gamma^2(\frac{p-1}{2})}{\Gamma(p-1)} \int_1^\infty \int_0^\pi s\left[s^2 - 2sq\cos\varphi + q^2\right]^{\frac{1-p}{2}} \mathrm{d}\varphi\mathrm{d}s.
\end{aligned}$$

By integrating over φ, the hypergeometric function $F(a, b; c; z)$ appears. For handling hypergeometric functions see for example [8, p. 555 ff]. In detail we obtain

$$\begin{aligned}
\psi(q) &= 2^{p-1}\frac{\Gamma^2(\frac{p-1}{2})}{\Gamma(p-1)} \int_1^\infty s^{2-p} \int_0^\pi \left[1 - 2\frac{q}{s}\cos\varphi + \frac{q^2}{s^2}\right]^{\frac{1-p}{2}} \mathrm{d}\varphi\mathrm{d}s \\
&= 2^{p-1}\frac{\Gamma^2(\frac{p-1}{2})}{\Gamma(p-1)}\mathrm{B}\left(\frac{1}{2}, \frac{1}{2}\right) \int_1^\infty s^{2-p}\mathrm{F}\left(\frac{p-1}{2}, \frac{p-1}{2}; 1; \left(\frac{q}{s}\right)^2\right) \mathrm{d}s \\
&= \pi^{\frac{3}{2}}\frac{\Gamma\left(\frac{p-3}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} q^{3-p} \int_0^{q^2} t^{\frac{p-5}{2}}\mathrm{F}\left(\frac{p-1}{2}, \frac{p-1}{2}; 1; t\right) \mathrm{d}t \\
&= \pi^{\frac{3}{2}}\frac{\Gamma\left(\frac{p-3}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\mathrm{F}\left(\frac{p-3}{2}, \frac{p-1}{2}; 1; q^2\right).
\end{aligned} \tag{8}$$

In formula (8) we have used the relations

$$\int_0^\pi \frac{\sin^{2\mu-1} x\mathrm{d}x}{(1 + 2a\cos x + a^2)^\nu} = \mathrm{B}\left(\mu, \frac{1}{2}\right)\mathrm{F}\left(\nu, \nu - \mu + \frac{1}{2}; \mu + \frac{1}{2}; a^2\right)$$

(cf. [7, 3.665.2]), $\mathrm{B}\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$, $\Gamma(z)/\Gamma(2z) = \pi^{1/2}2^{1-2z}/\Gamma\left(z + \frac{1}{2}\right)$, and [8, 15.2.3]. So we found for the case of the infinite tube, that the disjoining pressure potential becomes

$$F(x, u(x)) = C \cdot \mathrm{F}\left(a, b; 1; q^2\right),\ \text{ with } a = \frac{p-3}{2} \text{ and } b = \frac{p-1}{2}.$$

for $C = \frac{\mathcal{H}}{\sigma}\pi^{-\frac{1}{2}}R^{3-p}\frac{\Gamma(\frac{p-3}{2})}{\Gamma(\frac{p}{2})}$.
Consequently equation (5) turns into

$$\operatorname{div}\mathbf{T}\,U = cR^{3-p}\psi(q) + \kappa U - \mu.$$

Because of the present symmetry, the main problem - to solute the equation (2) - has changed into

$$\operatorname{div}\mathbf{T}\,U(q) \equiv \frac{1}{q}\left(\frac{U'(q)}{\sqrt{1+(U'(q))^2}}\right)' = cR^{3-p}\psi(q) + \kappa U(q) - \mu, \tag{9}$$

to a non-linear, second order ordinary differential equation for $q \in (0,1)$. At first view, this problem is much easier to handle, than the primary problem. But there are nevertheless some difficulties, the singularity at $q = 0$ for example.

Remark 3 *The Comparison Principle proven in the previous section is valid.*

This is a easy conclusion and follows from the proof of Theorem 1, since $\psi(q)$ is trivially monotone in U.

Lemma 4 *For $\psi(q) = F(a,b;1;q^2)$, with $a = \frac{p-3}{2}$ and $b = \frac{p-1}{2}$, yields*

$$\operatorname{div}\boldsymbol{T}\psi = O(1), \text{ uniformly in } [0,1].$$

Proof: It is sufficient to show that

$$|\operatorname{div}\mathbf{T}\,\psi(q)| = \left|\frac{1}{q}\left(\frac{q\psi'}{\sqrt{1+\psi'^2}}\right)'\right| = \left|\frac{\psi'}{q\sqrt{1+\psi'^2}} + \frac{\psi''}{(1+\psi'^2)^{3/2}}\right| \le K < \infty$$

for all $q \in [0,1]$.

At first we present the first derivations of $\psi(q)$. To bring the derivations in an appropriate form, we use formula [8, 15.3.6], so we separate the highest poles in the according formulas. In this cases $A_1, ..., A_6$ are explicit known positive constants, depending only on p, a concrete example is given following Theorem 6.

$$\begin{aligned}\psi'(q) &= 2abq\mathrm{F}(a+1,b+1;2;q^2)\\ &= A_1 q\mathrm{F}(a+1,b+1;a+b;1-q^2) + A_2\left(1-q^2\right)^{-a-b} q\mathrm{F}(1-a,1-b;1-a-b;1-q^2))\end{aligned} \tag{10}$$

$$\begin{aligned}\psi''(q) &= 2ab\mathrm{F}(a+1,b+1;2;q^2) + 2ab(a+1)(b+1)q^2\mathrm{F}\left(a+2,b+2;3;q^2\right)\\ &= A_3 F(a+1,b+1;3;1-q^2) + A_4 q^2\mathrm{F}\left(a+2,b+2;a+b+2;1-q^2\right)\\ &\quad + A_5\left(1-q^2\right)^{1-a-b}\mathrm{F}\left(1-a,1-b;2-a-b;1-q^2\right)\\ &\quad + A_6 q^2\left(1-q^2\right)^{-1-a-b}\mathrm{F}\left(1-a,1-b;a+b;1-q^2\right).\end{aligned} \tag{11}$$

In the case of $a+b \in \mathbb{Z}$ we need a slight modification of formulae (10) and (11), respectively, by using formulae [8, 15.3.10-12]. This modification yields the same result, which can be confirmed by an easy calculation. However, the result is, that the the leading singularities in both cases are from the same order.

From the first parts in formulae (10) and (11), respectively, we can see, that both ψ' and ψ'' are bounded for $q \in [0, \frac{1}{2}]$. In particular we have got

$$\begin{aligned}
\left| \frac{\psi'}{q\sqrt{1+\psi'^2}} \right| &\leq \frac{\psi'}{q} = 2ab\mathrm{F}(a+1, b+1; 2; q^2) < K \\
\left| \frac{\psi''}{(1+\psi'^2)^{3/2}} \right| &\leq \psi'' = 2ab\mathrm{F}(a+1, b+1; 2; q^2) \\
&\quad + 2ab(a+1)(b+1)q^2\mathrm{F}\left(a+2, b+2; 3; q^2\right) < K.
\end{aligned}$$

Remember that even the power series of the hypergeometric function $\mathrm{F}(a, b; c; z)$ converges absolutely for $|z| < 1$ and $\mathrm{F}(a, b; c; z) > 0$ if $a, b, c > 0$.

To prove the estimate for $q \in (\frac{1}{2}, 1]$, we use the second parts of formulae (10) and (11), respectively. In this case on gets

$$\left| \frac{\psi'}{q\sqrt{1+\psi'^2}} \right| \leq 2\frac{\psi'}{\sqrt{1+\psi'^2}} \leq 2,$$

and if $\mathrm{F}_1, ..., \mathrm{F}_6$ denote the according hypergeometric Functions from formula (10) and (11)

$$\left| \frac{\psi''}{(1+\psi'^2)^{3/2}} \right| = \left(1-q^2\right)^{2a+2b-1} \frac{\left(1-q^2\right)^{1+a+b}[A_3\mathrm{F}_3 + A_4q^2\mathrm{F}_4] + \left(1-q^2\right)^2\mathrm{F}_5 + A_5q^2\mathrm{F}_6}{\left[(1-q^2)^{2a+2b} + q^2\left\{A_1\left(1-q^2\right)^{a+b}\mathrm{F}_1 + A_2\mathrm{F}_2\right\}^2\right]^{3/2}}.$$

The right side of the upper equation is bounded. This is easy to see, because if we note that $A_1, ..., A_2$, $\mathrm{F}_1, ..., \mathrm{F}_2$ are bounded for $q \in (\frac{1}{2}, 1]$ or have no higher singularities than $\left(1-q^2\right)^{-1-a-b}$. Now, from $2a + 2b - 1 > 0$ even follows that the right side tends to zero as q tends to one.
So the lemma is proven. ■

Remark 5 *A more closely examination shows that $\psi'(q)$ tends to infinity as q tends to* 1 *and so it results in particular that* div $\boldsymbol{T}\psi$ *tends to* 1 *as q tends to* 1.

Now we are able to summarise our results in the next:

Theorem 6 *In the case of an open tube, the function $-\frac{cR^{3-p}}{\kappa}\psi(q)$ has as the same singularity, as the solution of equation (2).*

Proof: Define $\omega(q) = -\frac{cR^{3-p}}{\kappa}\psi(q)$ and remember, that c is negative. It is clear that for each constant A the equation $\operatorname{div}\mathbf{T}(\omega + A) = \operatorname{div}\mathbf{T}\,\omega$ is true and from Lemma 4 follows $\operatorname{div}\mathbf{T}\,\omega = \mathrm{O}(1)$, uniformly in $[0, 1]$.

If we note the solution of equation (9) with V and use the notation of Theorem 1, we get for a sufficient big constant A the estimate

$$\mathbb{M}(\omega + A) = \operatorname{div}\mathbf{T}(\omega + A) - cR^{3-p}\psi - \kappa(\omega + A) + \mu = \operatorname{div}\mathbf{T}\,\omega - \kappa A + \mu \leq 0.$$

And consequently, considering $\mathbb{M}(V) = 0$ and applying Theorem 1, we get the result $V \leq \omega + A$. The analogous considerations are also valid for $\omega - A$, so that we obtain

$$\omega - A \leq V \leq \omega + A,$$

for some positive constant A. And so the assertion of the theorem follows. ■

Now we present a concrete example for the common case $p = 6$. So we get from formula (8) with [8, 15.3.12] (and $m = 3$) an explicit expansion for $\psi(q)$:

$$\begin{aligned} \psi(q) &= \pi^{\frac{3}{2}} \frac{\Gamma(3/2)}{\Gamma(2)} \mathrm{F}(3/2, 5/2; 1; q^2) \\ &= \frac{4}{3}\pi \left(1 - q^2\right)^{-3} - \frac{\pi}{2} \left(1 - q^2\right)^{-2} - \frac{\pi}{16} \left(1 - q^2\right)^{-1} + R, \end{aligned} \tag{12}$$

where R is a convergent power series in $1 - q^2$. From formula (12) we see that there is a leading singularity $(1 - q^2)^{-3}$ and some lower singularities. The leading singularity is the same Israelachvili conjectured in 1992, cf. [2, p. 194].

This is also the generic case, that is, that the solution of the equation (2) is made up of a singular part and a bounded part. The singular part itself is the sum of some singularities, whereas the highest singularity bears like $(1 - q^2)^{3-p}$ when $q \to 1$.

Remark 7 *For the closed tube, that means, if we consider*

$$\Omega_s = \mathbb{R}^3 \setminus \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 \leq R^2,\ y_3 \geq 0\},$$

and if $c > 0$ is a positive constant with $u(x) \geq c$ for all $x \in \Omega$, where u is the solution of equation (2). So we can conclude that a solution of equation (2) has got the same singularity as in the case of the open tube.

This is easy to see if we notice that

$$\int_{\Omega_1} [(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2]^{-p/2} dy \leq C \int_{-\infty}^{0} \frac{\mathrm{d}y_3}{|u - y_3|^p} \leq C \int_c^{\infty} \frac{\mathrm{d}y_3}{y_3^p} \leq C < \infty,$$

where $\Omega_1 = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 < R^2,\ y_3 < 0\}$ and C is some positive constant, depending only on R. So we can adopt the idea of the proof of Theorem 6 with an constant $A + C$ instead of A.

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