THE FUNDAMENTAL GROUP OF PERIOD DOMAINS OVER FINITE FIELDS

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Abstract. We determine the fundamental group of period domains over finite fields. This answers a question of M. Rapoport raised in [R].

1. Introduction

Period domains over finite fields are open subvarieties of flag varieties defined by a semi-stability condition. They were introduced and discussed by M. Rapoport in [R]. In this paper we determine their fundamental groups which answers a question raised in loc.cit.

Let G be a reductive group over a finite field k. We fix an algebraic closure $\overline{k}$ of k and denote by $\Gamma = \Gamma_k$ the corresponding absolute Galois group of k. Let $\mathcal{N}$ be a conjugacy class of $\mathbb{Q}$-1-PS of $G_{\overline{k}}$. We denote by $E = E(G, \mathcal{N})$ the reflex field of the pair $(G, \mathcal{N})$. This is a finite extension of k which is characterized by its Galois group $\Gamma_E = \{\sigma \in \Gamma \mid \nu \in \mathcal{N} \Longrightarrow \nu^\sigma \in \mathcal{N}\}$. Every $\mathbb{Q}$-1-PS ν induces via Tannaka formalism a $\mathbb{Q}$-filtration $\mathcal{F}_\nu$ over $\bar{k}$ of the forgetful fibre functor $\omega^G : \mathrm{Rep_k(G)} \to \mathrm{Vec}_k$ from the category of algebraic G-representations over k into the category of k-vector spaces. Two $\mathbb{Q}$-1-PS are called par-equivalent if they define the same $\mathbb{Q}$-filtration. There exists a smooth projective variety $\boldsymbol{\mathcal{F}}(G, \mathcal{N})$ over E with

$$\boldsymbol{\mathcal{F}}(G, \mathcal{N})(\overline{k}) = \{\nu \in \mathcal{N} \ \text{ modulo par-equivalence } \} \ .$$

The variety is a generalized flag variety for G_E. More precisely, by a lemma of Kottwitz [K], there is a $\mathbb{Q}$-1-PS $\nu \in \mathcal{N}$ which is defined over $E = E(G, \mathcal{N})$. Thus we may write $\boldsymbol{\mathcal{F}}(G, \mathcal{N}) = G_E/P$, where $P = P(\nu)$ is the parabolic subgroup of G_E attached to ν. Further, after fixing a maximal torus and a Borel subgroup in G, we may suppose that ν is contained in the closure $\bar{C}_{\mathbb{Q}}$ of the corresponding rational Weyl chamber $C_{\mathbb{Q}}$.

A point $x \in \boldsymbol{\mathcal{F}}(G, \mathcal{N})(\bar{k})$ is called semi-stable if the induced filtration $\mathcal{F}_x(\mathrm{Lie}(G)_{\bar{k}})$ on the adjoint representation $\mathrm{Lie}(G)_{\bar{k}} = \mathrm{Lie}(G) \otimes_k \bar{k}$ of G is semi-stable. The latter means that for all k-subspaces U of $\mathrm{Lie}(G)$, the following inequality is satisfied

$$\frac{1}{\dim U}\Big(\sum\nolimits_y y \cdot \dim \mathrm{gr}^y_{\mathcal{F}|U_{\bar{k}}}(U_{\bar{k}})\Big) \leq \frac{1}{\dim \mathrm{Lie}(G)}\Big(\sum\nolimits_y y \cdot \dim \mathrm{gr}^y_{\mathcal{F}}(\mathrm{Lie}(G)_{\bar{k}})\Big).$$

In [DOR] it is shown that there is an open subvariety $\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}$ of $\boldsymbol{\mathcal{F}}(G,\mathcal{N})$ parametrizing all semi-stable points, i.e. $\boldsymbol{\mathcal{F}}(G,\mathcal{N})(\bar{k})^{ss} = \boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}(\bar{k})$. This open subvariety $\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}$ is called the *period domain* to $(G,\mathcal{N})$.

The most prominent example of a period domain is the Drinfeld upper half plane $\Omega_k^{(\ell+1)} = \mathbb{P}_k^\ell \setminus \cup \mathbb{P}(H)$ where H runs through all k-rational hyperplanes of $k^{\ell+1}$. This space corresponds to the pair $(G,\mathcal{N})$ where $G = \mathrm{PGL}_{\ell+1,k}$ and $\nu = (x_1, x_2, \ldots, x_2) \in \bar{C}_{\mathbb{Q}}$ with $x_1 > x_2$ and $x_1 + \ell \cdot x_2 = 0$. Here we identify $\bar{C}_{\mathbb{Q}}$ as usual with $(\mathbb{Q}^{\ell+1})^0_+ = \{(x_1, \ldots, x_{\ell+1}) \in \mathbb{Q}^{\ell+1} \mid \sum_i x_i = 0,\, x_1 \geq x_2 \geq \ldots \geq x_{\ell+1}\}$. The period domain $\Omega_k^{(\ell+1)}$ is isomorphic to a Deligne-Lusztig variety and admits therefore interesting étale coverings, cf. [DL]. In [OR] it is shown that $\Omega_k^{(\ell+1)}$ is essentially the only period domain which is at the same time a Deligne-Lusztig variety.

Period domains only depend on their adjoint data, cf. [OR], [DOR]. More precisely, let G_{ad} be the adjoint group of G, and let $\mathcal{N}_{\mathrm{ad}}$ be the induced conjugacy class of $\mathbb{Q}$-1-PS of G_{ad}. Then

$$\boldsymbol{\mathcal{F}}(G,\mathcal{N})(\bar{k})^{ss} \xrightarrow{\sim} \boldsymbol{\mathcal{F}}(G_{\mathrm{ad}},\mathcal{N}_{\mathrm{ad}})(\bar{k})^{ss} \ .$$

Also if G splits into a product $G = \prod_i G$, the corresponding period domain splits into products, as well. Thus for formulating our main result, we may assume that G is k-simple adjoint. Hence there is an absolutely simple adjoint group G' over a finite extension k' of k with $G = \mathrm{Res}_{k'/k} G'$. In this case $\mathcal{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_t)$ is given by a tuple of conjugacy classes $\mathcal{N}_j$ of $\mathbb{Q}$-1-PS of $G'_{\bar{k}}$, where $t = |k' : k|$. Thus ν is given by a tuple of $\mathbb{Q}$-1-PS $\nu = (\nu_1, \ldots, \nu_t)$.

Our main result is the following. Let ℓ be the (absolute) rank of G'. We denote by π_1 the functor which associates to a variety its geometric fundamental group.

Theorem 1. *Let G be absolutely simple adjoint over k. Then $\pi_1(\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}) = \{1\}$ unless $G = \mathrm{PGL}_{\ell+1,k}$ and $\nu = (x_1 \geq x_2 \geq \ldots \geq x_{\ell+1}) \in (\mathbb{Q}^{\ell+1})^0_+$ with $x_2 < 0$ or $x_\ell > 0$. In the latter case we have $\pi_1(\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}) = \pi_1(\Omega_k^{(\ell+1)})$.*

More generally, let $G = \mathrm{Res}_{k'/k} G'$ be k-simple adjoint. Then $\pi_1(\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}) = \{1\}$ unless $G' = \mathrm{PGL}_{\ell+1,k'}$ and such that the following two conditions are satisfied. Write $\nu_i = (x_1^{[i]} \geq x_2^{[i]} \geq \ldots \geq x_{\ell+1}^{[i]}) \in (\mathbb{Q}^{\ell+1})^0_+$, $i = 1, \ldots, t$. Then there is a unique $1 \leq j \leq t$, such that

(i) $x_2^{[j]} < 0$ or $x_\ell^{[j]} > 0$.

(ii) $\sum_{i \neq j} x_1^{[i]} < -x_2^{[j]}$ if $x_2^{[j]} < 0$ resp. $\sum_{i \neq j} x_{\ell+1}^{[i]} > -x_\ell^{[j]}$ if $x_\ell^{[j]} > 0$.

In the latter case we have $\pi_1(\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}) = \pi_1(\Omega_{k'}^{(\ell+1)})$.

Acknowledgements: I thank M. Rapoport for helpful remarks on this paper.

2. Some preparations

In this section we recall some results concerning the relation of period domains to Geometric Invariant Theory (GIT).

Let G be a reductive group over k and let $\mathcal{N} = \{\nu\}$ be a conjugacy class of $\mathbb{Q}$-1-PS of $G_{\bar{k}}$. We abbreviate $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}(G, \mathcal{N})$. We fix an *invariant inner product* $(\ ,\ )$ on G over k. Recall that this is a positive-definite bilinear form $(\ ,\ )$ on $X_*(T)_{\mathbb{Q}}$ for any maximal torus T of G defined over $\overline{k}$. The following conditions are required:

(i) For $g \in G(\overline{k})$, the inner automorphism $\mathrm{Int}(g)$ induces an isometry

$$\mathrm{Int}(g) : X_*(T)_{\mathbb{Q}} \longrightarrow X_*(T^g)\ ,\ T^g = g \cdot T \cdot g^{-1}\ .$$

(ii) Any $\sigma \in \Gamma$ induces an isometry

$$\sigma : X_*(T)_{\mathbb{Q}} \longrightarrow X_*(T^\sigma)_{\mathbb{Q}}\ .$$

The choice of such an inner invariant product induces together with the standard pairing $\langle\ ,\ \rangle : X_*(T)_{\mathbb{Q}} \times X^*(T)_{\mathbb{Q}} \to \mathbb{Q}$ an identification $X_*(T)_{\mathbb{Q}} \cong X^*(T)_{\mathbb{Q}}$ for all maximal tori T of G defined over $\overline{k}$. To the pair $(G, \mathcal{N})$ there is attached an ample homogeneous $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\boldsymbol{\mathcal{F}}$ given by

$$\mathcal{L} = G \times^P \mathbb{G}_{a,-\nu^*}\ .$$

Here ν^* denotes the rational character of T which corresponds to ν under the above identification (it extends to a character of P). The following theorem of Totaro [To] describes the semi-stable points $\boldsymbol{\mathcal{F}}^{ss}$ inside $\boldsymbol{\mathcal{F}}$ via GIT. Here we denote by $\mu^{\mathcal{L}}(x, \lambda)$ the slope of $x \in \boldsymbol{\mathcal{F}}(\bar{k})$ with respect to the 1-PS λ and the ample line bundle $\mathcal{L}$ in the sense of GIT, cf. [MFK].

Theorem 2.1. *Let $x \in \boldsymbol{\mathcal{F}}(\bar{k})$. Then $x \in \boldsymbol{\mathcal{F}}^{ss}(\bar{k})$ if and only if for all 1-PS λ of G_{der} defined over k the Hilbert-Mumford inequality holds, i.e.*

$$\mu^{\mathcal{L}}(x, \lambda) \geq 0\ .$$

Let $\Delta_k = \{\alpha_1, \ldots, \alpha_d\}$ be the set of relative simple roots with respect to a fixed maximal split torus $S \subset G$ and a Borel subgroup $B \subset G$ containing S. Note that G is quasi-split since k is a finite field. Let $T = Z(S)$ be the centralizer of S which is a maximal torus over k. We let Δ be the set of absolutely simple roots of G with respect to $T \subset B$. Then the relative simple roots are given by $\Delta_k = \{\alpha|S \mid \alpha \in \Delta, \alpha|S \neq 0\}$, cf. [Ti]. By conjugating ν with an element of the (absolute) Weyl group W, we may assume that ν is contained in the closure of the dominant Weyl chamber, i.e.,

$$\nu \in \bar{C}_{\mathbb{Q}} = \{\lambda \in X_*(T)_{\mathbb{Q}} \mid \langle \lambda, \alpha \rangle \geq 0\ \forall \alpha \in \Delta\}.$$

We denote by $(\omega_\alpha)_{\alpha\in\Delta} \subset X_*(T)_{\mathbb{Q}}$ the set of co-fundamental weights. Recall that they are defined by $(\omega_\alpha, \beta^\vee) = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$. For $1 \leq i \leq d$, let

$$\Psi(\alpha_i) = \{\beta \in \Delta \mid \beta|S = \alpha_i\}.$$

We set

$$\omega_i = \sum\nolimits_{\beta\in\Psi(\alpha_i)} \omega_\beta. \tag{2.1}$$

Up to multiplication by a positive scalar these are just the relative fundamental weights. In [O] we have shown[1] that in Theorem 2.1 it suffices to treat the vertices of the spherical Tits-complex [CLT] defined by Curtis, Lehrer and Tits. Thus

Proposition 2.2. *Let $x \in \boldsymbol{\mathcal{F}}(\bar{k})$. Then $x \in \boldsymbol{\mathcal{F}}^{ss}(\bar{k})$ iff for all $g \in G(k)$ and for all i the inequality $\mu^{\mathcal{L}}(x, \mathrm{Int}(g) \circ \omega_i) \geq 0$ is satisfied.*

We consider the closed complement $Y := \boldsymbol{\mathcal{F}} \setminus \boldsymbol{\mathcal{F}}^{ss}$ of $\boldsymbol{\mathcal{F}}^{ss}$. For any integer $1 \leq i \leq d$, we set

$$Y_i(\bar{k}) := \{x \in \boldsymbol{\mathcal{F}}(\bar{k}) \mid \mu^{\mathcal{L}}(x, \omega_i) < 0\}.$$

The sets $Y_i(\bar{k})$ are induced by closed subvarieties Y_i of Y which are defined over E. Let $P_i = P(\omega_i)$ be the parabolic subgroup corresponding to ω_i. If $n \in \mathbb{N}$ is some integer such that $n\omega_i \in X_*(T)$, then

$$P(\omega_i)(\bar{k}) = \{g \in G(\bar{k}) \mid \lim\nolimits_{t\to 0} \mathrm{Int}(n\omega_i(t)) \circ g \text{ exists in } G(\bar{k})\},$$

cf. [MFK]. This definition does not depend on n and P_i is defined over k since $\omega_i \in X_*(S)_{\mathbb{Q}}$. The natural action of G on $\boldsymbol{\mathcal{F}}$ restricts to an action of P_i on Y_i for every i. It is a consequence of Prop. 2.2 that we can write Y as the union

$$Y = \bigcup\nolimits_{i=1,\ldots,d} \bigcup\nolimits_{g\in G(k)} gY_i. \tag{2.2}$$

In [O] we proved that the varieties Y_i are unions of Schubert cells. More precisely, denote by $W_P \subset W$ the parabolic subgroup induced by P. We identify the elements of $W^P := W/W_P$ with representatives of shortest length in W.

Proposition 2.3. *We have*

$$\begin{aligned} Y_i &= \bigcup_{\substack{w\in W^P \\ (\omega_i, w\nu)>0}} P_i wP/P \\ &= \bigcup_{\substack{w\in W^P \\ (\omega_i, w\nu)>0}} BwP/P. \end{aligned}$$

[1]Actually, in loc.cit. we considered the dual basis of Δ_k which consists of certain positive multiples of $(\omega_i)_i$. This does not affect the statement.

The proof follows from the identity

$$\mu^{\mathcal{L}}(pw[\nu], \omega_i) = -(\omega_i, w\nu),$$

for all $p \in P_i(\bar{k})$, $w \in W$. Here $[\nu]$ denotes the point of $\boldsymbol{\mathcal{F}}(E)$ induced by ν.

We conclude by (2.2) that

$$\dim Y = \max_{i=1,\ldots,d} \dim Y_i.$$

On the other hand, each subvariety Y_i is a union of the Schubert cells BwP/P, $w \in W^P$, with $(\omega_i, w\nu) > 0$. The dimension of BwP/P is $\ell(w)$, cf. [Bo]. Thus we deduce that

$$\dim Y_i = \max\,\{\ell(w) \mid w \in W^P,\ (\omega_i, w\nu) > 0\}. \tag{2.3}$$

Let w_0 resp. w_0^P be the longest element of the Weyl group W resp. of W^P. Then $w_0 = w_0^P \cdot w_P$ where w_P is the longest element in W_P. In particular

$$w_0\nu = w_0^P\nu \tag{2.4}$$

and

$$\dim \boldsymbol{\mathcal{F}} = \ell(w_0^P). \tag{2.5}$$

We shall examine in the next section when it happens that $\dim Y = \dim \boldsymbol{\mathcal{F}} - 1$, i.e., $\operatorname{codim} Y = 1$.

3. The proof of Theorem 1

From now on we assume that G is k-simple adjoint, i.e., $G = \mathrm{Res}_{k'/k}G'$ for some finite extension k'/k of degree t, cf. [Ti]. Let ℓ be the (absolute) rank of G'. We start with the case where G is absolutely simple adjoint i.e., $k' = k$.

Proposition 3.1. *Let G be absolutely simple adjoint over k. Then* $\operatorname{codim} Y \geq 2$ *unless* $G = \mathrm{PGL}_{\ell+1}$ *and* $\nu = (x_1 \geq x_2 \geq \ldots \geq x_\ell \geq x_{\ell+1}) \in (\mathbb{Q}^{\ell+1})^0_+$ *with* $x_2 < 0$ *or* $x_\ell > 0$.

Proof. The elements of length $\ell(w_0) - 1$ in W are given by the expressions sw_0, where $s \in W$ is a simple reflection. We deduce from (2.3) - (2.5) that there is some integer $1 \leq i \leq d$ with $\operatorname{codim} Y_i = 1$, if and only if there is a simple reflection $s_\beta \in W$, $\beta \in \Delta$, with

$$(\omega_i, s_\beta w_0 \nu) > 0. \tag{3.1}$$

By the equivariance of $(\ ,\ )$ we get

$$(\omega_i, s_\beta w_0 \nu) = (s_\beta \omega_i, w_0 \nu). \tag{3.2}$$

1st *case:* G is split.

Thus we have $\Delta_k = \Delta$. Further, by [Bou] ch. VI, 1.10, we have[2]

$$s_\beta \omega_i = \begin{cases} \omega_i & \text{if } \beta \neq \alpha_i \\ \omega_i - \alpha_i & \text{if } \beta = \alpha_i \end{cases}.$$

Since $w_0\nu \in -\bar{C}_{\mathbb{Q}}$ we get $(\omega_i, w_0\nu) < 0$. Thus we conclude that $\beta = \alpha_i$ is a necessary condition in order that (3.1) holds. Further, in this situation we get by (3.2) $(\omega_i, s_\beta w_0 \nu) > 0$ if and only if

$$(\omega_i, w_0\nu) > (\alpha_i, w_0\nu). \tag{3.3}$$

We start to investigate inequality (3.3) for the root system of type $A_\ell (\ell \geq 1)$. In this case the data is given as follows:

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1},\ i = 1, \ldots, \ell, \\ \omega_i &= \frac{1}{\ell+1}\big((\ell+1-i)^{(i)}, -i^{(\ell+1-i)}\big),\ i = 1, \ldots, \ell, \\ \bar{C}_{\mathbb{Q}} &= (\mathbb{Q}^{\ell+1})^0_+. \end{aligned}$$

Here in the definition of ω_i the exponent (j) means that we repeat the corresponding entry j times. Further, w_0 acts on $\mathbb{Q}^{\ell+1}$ via

$$w_0(x_1, x_2, \ldots, x_{\ell+1}) = (x_{\ell+1}, x_\ell, \ldots, x_1).$$

Let $\nu = (x_1 \geq x_2 \geq \ldots \geq x_{\ell+1}) \in (\mathbb{Q}^{\ell+1})^0_+$. Then

$$(\omega_i, w_0\nu) = x_{\ell+1} + \ldots + x_{\ell-i+2}$$

and

$$(\alpha_i, w_0\nu) = x_{\ell-i+2} - x_{\ell-i+1}.$$

Thus inequality (3.3) is satisfied if and only if

$$x_{\ell+1} + \ldots + x_{\ell-i+3} > -x_{\ell-i+1} \ \text{ if } \ 1 < i < \ell \tag{3.4}$$

resp.

$$x_\ell > 0 \ \text{ if } \ i = 1$$

resp.

$$x_2 < 0 \ \text{ if } \ i = \ell.$$

Let $1 < i < \ell$. Then

$$x_1 + \ldots + x_{\ell-i} + x_{\ell-i+2} \geq x_{\ell+1} + \ldots + x_{\ell-i+3} + x_{\ell-i+1}$$

[2]Here we make use of the identification $X_*(T)_{\mathbb{Q}} = X^*(T)_{\mathbb{Q}}$

as $x_{\ell-i+2} \geq x_{\ell-i+3}$, $x_{\ell-i} \geq x_{\ell-i+1}$ and $\sum_{j=1}^{\ell-i-1} x_j \geq 0$ resp. $\sum_{j=0}^{i-3} x_{\ell+1-j} \leq 0$. Thus (3.4) cannot be satisfied if $1 < i < \ell$ since the sum over all entries in ν vanishes. Hence the proof follows in the case of the root system $A_\ell (\ell \geq 1)$.

For the other split root systems, i.e., of type $B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4, G_2$, we proceed as follows. We write down $\nu = \sum_{i=1}^{\ell} n_i \omega_i$ as linear combination of the co-fundamental weights with non-negative coefficients $n_i \geq 0$. Note that $n_i = (\nu, \alpha_i^\vee)$, $i = 1, \ldots, \ell$. We get

$$w_0 \nu = -\sum_{j=1}^{\ell} n_j \omega_{\tau(j)}.$$

where τ is the opposition involution of $\{1, \ldots, \ell\}$, cf. [Ti]. In the case of B_ℓ, C_ℓ, $D_\ell(\ell \text{ even}), E_7, E_8, F_4, G_2$ we have $\tau = \text{id}$. For $D_\ell(\ell \text{ odd})$, we have $\tau = (\ell - 1, \ell)$. Finally in the case E_6 we have $\tau = (1,6)(2,5)(3,4)$. In all cases

$$(\omega_i, w_0 \nu) = -\sum_{j=1}^{\ell} n_j (\omega_i, \omega_{\tau(j)}).$$

and

$$(\alpha_i, w_0 \nu) = -n_{\tau^{-1}(i)} \cdot \frac{1}{2} \cdot (\alpha_i, \alpha_i)$$

as $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$. Since $(\omega_i, \omega_j) \geq 0$ for all i, j, cf. [Bou], ch. VI, 1.10, we get

$$(\omega_i, w_0 \nu) \leq -n_{\tau^{-1}(i)} \cdot (\omega_i, \omega_i). \tag{3.5}$$

Further one checks case by case by the explicit representation of the co-fundamental weights in loc.cit. p. 265-290, that

$$(\omega_i, \omega_i) \geq \frac{1}{2} \cdot (\alpha_i, \alpha_i) \text{ for } i = 1, \ldots, \ell.$$

Hence we get by using (3.5)

$$(\omega_i, w_0 \nu) \leq (\alpha_i, w_0 \nu).$$

Thus we deduce that the inequality (3.3) cannot be satisfied for root systems different from A_ℓ. Let us illustrate this argument for the root system of type G_2. Here the data is given by

$$\begin{aligned} \alpha_1 &= \epsilon_1 - \epsilon_2, \ \alpha_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3, \\ \omega_1 &= \epsilon_3 - \epsilon_2, \ \omega_2 = -\epsilon_1 - \epsilon_2 + 2\epsilon_3. \end{aligned}$$

Let $\nu = n_1\omega_1 + n_2\omega_2$ with $n_1, n_2 \geq 0$. We get $w_0\nu = -n_1\omega_1 - n_2\omega_2$. Then

$$(\omega_1, w_0\nu) = -n_1(\omega_1, \omega_1) - n_2(\omega_1, \omega_2) = -2n_1 - 3n_2$$

and

$$(\omega_2, w_0\nu) = -n_1(\omega_2, \omega_1) - n_2(\omega_2, \omega_2) = -3n_1 - 6n_2.$$

Further, we compute

$$(\alpha_1, w_0\nu) = -n_1 \cdot \frac{1}{2} \cdot (\alpha_1, \alpha_1) = -n_1$$

and

$$(\alpha_2, w_0\nu) = -n_2 \cdot \frac{1}{2} \cdot (\alpha_2, \alpha_2) = -3n_2.$$

Hence

$$(\omega_1, w_0\nu) \leq -n_1(\omega_1, \omega_1) = -2n_1 \leq (\alpha_1, w_0\nu) = -n_1$$

and

$$(\omega_2, w_0\nu) \leq -n_2(\omega_2, \omega_2) = -6n_2 \leq (\alpha_2, w_0\nu) = -3n_2.$$

2nd *case:* G is not split.

Recall that $\omega_i = \sum_{\beta \in \Psi(\alpha_i)} \omega_\beta$, cf. (2.1). We get

$$s_\beta \omega_i = \begin{cases} \omega_i & \text{if } \beta \notin \Psi(\alpha_i) \\ \omega_i - \beta & \text{if } \beta \in \Psi(\alpha_i) \end{cases}.$$

Again we conclude that $\beta \in \Psi(\alpha_i)$ is a necessary condition in order that (3.1) holds. Further $(\omega_i, s_\beta w_0\nu) > 0$, if and only if

$$(\omega_i, w_0\nu) > (\beta, w_0\nu). \tag{3.6}$$

Now we have

$$(\omega_i, w_0\nu) = \sum\nolimits_{\beta \in \Psi(\alpha_i)} (\omega_\beta, w_0\nu) \leq (\omega_\beta, w_0\nu) \text{ for all } \beta \in \Psi(\alpha_i).$$

Thus by the computation in the 1st case, we conclude that a necessary condition in order that (3.6) holds is that the root system of $G_{\bar{k}}$ is of type $A_\ell (\ell \geq 1)$.

In this case the group $G = \mathrm{PU}_{\ell+1}$ is the projective unitary group of (absolute) rank ℓ and $d = [\frac{\ell+1}{2}]$, cf. [Ti]. The co-fundamental weights $(\omega_i)_i$ of $\mathrm{PU}_{\ell+1}$ are given as follows. Let $\Delta = \{\beta_1 = \epsilon_1 - \epsilon_2, \ldots, \beta_\ell = \epsilon_\ell - \epsilon_{\ell+1}\}$ be the set of standard simple roots of type A_ℓ. Then

$$\omega_i = \omega_{\beta_i} + \omega_{\beta_{\ell+1-i}},\ i = 1, \ldots, d-1$$

and

$$\omega_d = \begin{cases} \omega_{\beta_d} & \text{if } \frac{\ell+1}{2} \in \mathbb{Z} \\ \omega_{\beta_d} + \omega_{\beta_{d+1}} & \text{if } \frac{\ell+1}{2} \notin \mathbb{Z} \end{cases}.$$

Thus by the explicit computation in the $\mathrm{PGL}_{\ell+1}$-case, we see that if inequality (3.6) is satisfied, then we necessarily have $i = 1$ and $\beta = \beta_1$ or $\beta = \beta_\ell$. But we compute

$$(\omega_1, w_0\nu) = x_{\ell+1} - x_1$$

and

$$(\beta_1, w_0\nu) = x_{\ell+1} - x_\ell$$

resp.

$$(\beta_\ell, w_0\nu) = x_2 - x_1.$$

Hence we see that inequality (3.6) cannot be satisfied for $G = \mathrm{PU}_{\ell+1}$ either. □

Next we determine explicitely the period domains for which the codimension of the closed complement is 1. So by Prop. 3.1 we may assume that $G = \mathrm{PGL}_{\ell+1,k}$ and $\nu = (x_1, x_2, \ldots, x_{\ell+1}) \in (\mathbb{Q}^{\ell+1})^0_+$. We rewrite ν in the shape $\nu = (y_1^{(n_1)}, \ldots, y_r^{(n_r)})$ with $y_1 > y_2 > \cdots > y_r$ and $n_i \geq 1$, $i = 1, \ldots, r$. Let $V = k^{\ell+1}$. Then $\boldsymbol{\mathcal{F}}(G, \mathcal{N})(\bar{k})$ is given by the set of filtrations

$$(0) \subset \mathcal{F}^{y_1} \subset \mathcal{F}^{y_2} \subset \ldots \subset \mathcal{F}^{y_r} = V_{\bar{k}}$$

with

$$\dim \mathcal{F}^{y_i} = n_1 + \cdots + n_i.$$

If $x_2 < 0$ then $n_1 = 1$ resp. if $x_\ell > 0$ then $n_r = 1$. In order to determine the period domain, one can replace in the definition of a semi-stable filtration the Lie Algebra $\mathrm{Lie}(G)$ by V, cf. [DOR]. Thus a point $\mathcal{F}^\bullet$ is semi-stable if for all k-subspaces U of V the following inequality is satisfied

$$\frac{1}{\dim U}\Big(\sum\nolimits_y y \cdot \dim \mathrm{gr}^y_{\mathcal{F}|U_{\bar{k}}}(U_{\bar{k}})\Big) \leq \frac{1}{\dim V}\Big(\sum\nolimits_y y \cdot \dim \mathrm{gr}^y_{\mathcal{F}}(V_{\bar{k}})\Big).$$

Then one computes easily that

$$\boldsymbol{\mathcal{F}}^{ss}(\bar{k}) = \{\mathcal{F}^\bullet \in \boldsymbol{\mathcal{F}}(\bar{k}) \mid \mathcal{F}^{y_1} \text{ is not contained in any } k\text{-rational hyperplane}\}$$

resp.

$$\boldsymbol{\mathcal{F}}^{ss}(\bar{k}) = \{\mathcal{F}^\bullet \in \boldsymbol{\mathcal{F}}(\bar{k}) \mid \mathcal{F}^{y_r} \text{ does not contain any } k\text{-rational line}\}.$$

Thus the projections

$$\begin{array}{ccccccc} \boldsymbol{\mathcal{F}} & \to & \mathbb{P}^\ell_k & \text{resp.} & \boldsymbol{\mathcal{F}} & \to & \check{\mathbb{P}}^\ell_k \\ \mathcal{F}^\bullet & \mapsto & \mathcal{F}^{y_1} & & \mathcal{F}^\bullet & \mapsto & \mathcal{F}^{y_r} \end{array}$$

induce surjective proper maps

$$\boldsymbol{\mathcal{F}}^{ss} \to \Omega_k^{(\ell+1)} \quad \text{resp.} \quad \boldsymbol{\mathcal{F}}^{ss} \to \check{\Omega}_k^{(\ell+1)} \tag{3.7}$$

in which the fibres are generalized flag varieties.

Proof of Theorem 1 in the absolute simple case: The proof follows from Proposition 3.1 and the following facts on fundamental groups of algebraic varieties. If $\operatorname{codim} Y \geq 2$, then we get $\pi_1(\boldsymbol{\mathcal{F}}^{ss}) = \pi_1(\boldsymbol{\mathcal{F}}) = \{1\}$, since $\boldsymbol{\mathcal{F}}$ is simply connected, cf. [SGA1], ch.

XI, Cor. 1.2. If $\operatorname{codim} Y = 1$ we are in the situation (3.7). Then the statement follows from [SGA1] Cor. 6.11 since the fibres of the maps (3.7) are simply connected. Note that the fundamental groups of $\Omega_k^{(\ell+1)}$ and $\check{\Omega}_k^{(\ell+1)}$ are the same since both varieties are isomorphic. □

Now we consider the general case of an k-simple adjoint group G.

Proposition 3.2. *Let $G = \mathrm{Res}_{k'/k}G'$ be k-simple adjoint. Then* $\operatorname{codim} Y \geq 2$ *unless $G' = \mathrm{PGL}_{\ell+1}$ and there is a unique $1 \leq j \leq t$, such that the following two conditions are satisfied. Let $\nu_j = (x_1^{[j]} \geq x_2^{[j]} \geq \ldots \geq x_{\ell+1}^{[j]}) \in (\mathbb{Q}^{\ell+1})^0_+$, $j = 1, \ldots, t$. Then*

(i) ν_j as in the absolutely simple case, i.e., with $x_2^{[j]} < 0$ or $x_\ell^{[j]} > 0$.

(ii) $\sum_{i \neq j} x_1^{[i]} < -x_2^{[j]}$ if $x_2^{[j]} < 0$ resp. $\sum_{i \neq j} x_{\ell+1}^{[i]} > -x_\ell^{[j]}$ if $x_\ell^{[j]} > 0$.

Proof. We conclude by the same argument as in the proof of Proposition 3.1, 2nd case, that $\operatorname{codim} Y_i = 1$ if and only if there is a simple root $\beta \in \Psi(\alpha_i)$ such that

$$(\omega_i, w_0\nu) > (\beta, w_0\nu). \tag{3.8}$$

Let $\mathrm{Gal}(k'/k) = \{\sigma^j \mid 0 \leq j \leq t-1\}$ and denote by W' the Weyl group of G'. Since $G = \mathrm{Res}_{k'/k}G'$ we have $W = \prod_{j=1}^t W'$ and $w_0 = (w_0', \ldots, w_0') \in W$. Further, the natural restriction map $\Delta'_{k'} \to \Delta_k$ is bijective where $\Delta'_{k'} = \{\alpha_1', \ldots, \alpha_d'\}$ is the set of relative simple roots of G' with respect to a maximal k'-split torus S' such that $S(k) \subset S'(k')$. It follows that $\omega_i = \sum_{j=0}^{t-1} \sigma^j \omega_i'$. Here $(\omega_i')_i \in X_*(S')_{\mathbb{Q}}$ is defined with respect to $(\alpha_i')_i \in X^*(S')_{\mathbb{Q}}$. Furthermore, Δ is formed by t copies of the set Δ' of absolute simple roots to G'. We conclude that for each $\beta \in \Psi(\alpha_i)$ there is an index $j(\beta) = j$, $1 \leq j \leq t$, with

$$(\beta, w_0\nu) = (\beta, w_0'\nu_j).$$

For all other indices $h \neq j$, we have $(\beta, w_0'\nu_h) = 0$. We compute

$$(\omega_i, w_0\nu) = \sum\nolimits_{j=0}^{t-1} (\sigma^j\omega_i', w_0\nu) \leq (\sigma^j\omega_i', w_0\nu) = (\omega_i', w_0'\nu_j). \tag{3.9}$$

Thus by the computation in the proof of Proposition 3.1 we conclude that a necessary condition in order that (3.8) holds is that G' is split and that the root system of G' is of type $A_\ell (\ell \geq 1)$.

So let $G' = \mathrm{PGL}_{\ell+1,k'}$. Then Δ is given by the set $\{\alpha_i^{[j]} \mid 1 \leq i \leq \ell, 1 \leq j \leq t\}$, where

$$\alpha_i^{[j]} = \epsilon_i^{[j]} - \epsilon_{i+1}^{[j]}.$$

Here $\epsilon_i^{[j]}$ is the appropriate coordinate function on $T_{\bar{k}} \cong \prod_{j=1}^t S_{\bar{k}}$, where S is the diagonal torus in $\mathrm{PGL}_{\ell+1,k'}$. Furthermore, the sets $\Psi(\alpha_i)$ are given by

$$\Psi(\alpha_i) = \{\alpha_i^{[j]} \mid 1 \leq j \leq t\}.$$

Let $\nu = (\nu_1, \dots, \nu_t) \in \bar{C}_{\mathbb{Q}}$. We get $w_0\nu = (w_0'\nu_1, \dots, w_0'\nu_t)$, where the entries are given by $w_0'\nu_j = (x_{\ell+1}^{[j]}, x_\ell^{[j]}, \dots, x_1^{[j]})$, $j = 1, \dots, t$. In the proof of Proposition 3.1 we have seen that if the inequalities (3.8) and (3.9) are satisfied then $\beta = \alpha_1^{[j]}$ and $x_\ell^{[j]} > 0$ resp. $\beta = \alpha_\ell^{[j]}$ and $x_2^{[j]} < 0$ for some integer j with $1 \le j \le t$.

Let $\beta = \alpha_1^{[j]}$ and $x_\ell^{[j]} > 0$. Then

$$(\omega_1, w_0\nu) = \sum_{i=1}^{t} x_{\ell+1}^{[i]}$$

and

$$(\beta, w_0\nu) = x_{\ell+1}^{[j]} - x_\ell^{[j]}.$$

Thus the inequality (3.8) is satisfied if and only if

$$\sum\nolimits_{i \neq j} x_{\ell+1}^{[i]} > -x_\ell^{[j]}.$$

Furthermore, we claim that the integer j is uniquely determined. In fact, suppose first that h is another integer with $1 \le h \le t$ and

$$\sum\nolimits_{i \neq h} x_{\ell+1}^{[i]} > -x_\ell^{[h]}.$$

Without loss of generality we may assume that $-x_\ell^{[j]} \le -x_\ell^{[h]}$. Then

$$-x_\ell^{[j]} \le -x_\ell^{[h]} < \sum\nolimits_{i \neq h} x_{\ell+1}^{[i]} \le x_{\ell+1}^{[j]} \le -x_\ell^{[j]},$$

which is a contradiction. Here the latter inequality follows from the fact that $x_{\ell+1}^{[j]} + x_\ell^{[j]} \le 0$, since $\nu_j \in (\mathbb{Q}^{\ell+1})_+^0$.

If in the opposite direction h is another integer with $1 \le h \le t$ and

$$\sum\nolimits_{i \neq h} x_1^{[i]} < -x_2^{[h]}$$

then

$$x_1^{[j]} \le \sum\nolimits_{i \neq h} x_1^{[i]} < -x_2^{[h]} \le -x_{\ell+1}^{[h]} \le -\sum\nolimits_{i \neq j} x_{\ell+1}^{[i]} < x_\ell^{[j]},$$

which is a contradiction, as well.

The case $\beta = \alpha_\ell^{(j)}$ and $x_2^{[j]} < 0$ behaves dually and yields $\sum_{i \neq j} x_1^{[i]} < -x_2^{[j]}$. □

Again we determine explicitly the period domains where the codimension of the closed complement is 1. So let $\nu = (\nu_1, \dots, \nu_t) \in \bar{C}_{\mathbb{Q}}$ such that $\operatorname{codim} Y = 1$. After reindexing we may suppose that $\nu_1 \in (\mathbb{Q}^{\ell+1})_+^0$ is the vector with $\sum_{i \neq 1} x_1^{[i]} < -x_2^{[1]}$ or $\sum_{i \neq 1} x_{\ell+1}^{[i]} > -x_\ell^{[1]}$. Over the algebraic closure $\bar{k}$ the flag variety $\boldsymbol{\mathcal{F}}(G, \mathcal{N})$ is the product

$$\boldsymbol{\mathcal{F}}(G, \mathcal{N})_{\bar{k}} = \prod\nolimits_{j=1}^{t} \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,\bar{k}}, \mathcal{N}_j)_{\bar{k}},$$

where $\mathcal{N}_j$ is the $\mathrm{PGL}_{\ell+1,\bar{k}}$-conjugacy class of ν_j. Let $\nu_1 = (y_1^{(n_1)}, \ldots, y_r^{(n_r)})$ with $y_1 > y_2 > \cdots > y_r$ and $n_i \geq 1$, $i = 1, \ldots, r$. The corresponding period domain is then given by

$$\boldsymbol{\mathcal{F}}(G, \mathcal{N})^{ss}_{\bar{k}} = \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_1)^{ss}_{\bar{k}} \times \prod\nolimits_{j\geq 2} \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_j)_{\bar{k}}.$$

In the case $\sum_{i\neq 1} x_1^{[i]} < -x_2^{[1]}$, we have

$$\begin{aligned} \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_1)^{ss}(\bar{k}) \;&=\; \{\mathcal{F}^\bullet \in \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_1)(\bar{k}) \mid \mathcal{F}^{y_1} \text{ is not contained in} \\ &\qquad \text{any } k'\text{-rational hyperplane}\}. \end{aligned}$$

For $\sum_{i\neq 1} x_{\ell+1}^{[i]} > -x_\ell^{[1]}$, we have

$$\begin{aligned} \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_1)^{ss}(\bar{k}) \;&=\; \{\mathcal{F}^\bullet \in \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_1)(\bar{k}) \mid \mathcal{F}^{y_r} \text{ does not contain} \\ &\qquad \text{any } k'\text{-rational line }\}. \end{aligned}$$

Proof of Theorem 1 in the general case: The proof is the same as in the absolutely simple case and uses Proposition 3.2. □

We finish this paper by considering a non-trivial example.

Example 3.3. Let $G = \mathrm{Res}_{k'/k}\mathrm{PGL}_{2,k'}$ with $|k' : k| = 2$. Then ν corresponds to a tuple $(\nu_1, \nu_2) \in (\mathbb{Q}^2)^0_+ \times (\mathbb{Q}^2)^0_+$. Let $\nu_1 = (x_1 \geq x_2)$ and $\nu_2 = (y_1 \geq y_2)$. Then $x_2 = -x_1 \leq 0$ and $y_2 = -y_1 \leq 0$. If $\nu_1 \neq \nu_2$ then we may assume after changing ν_1 and ν_2 that $-x_2 > y_1$. Note that we allow $\nu_2 = (0, 0)$ to be trivial. Thus $\boldsymbol{\mathcal{F}} = \mathbb{P}^1 \times \mathbb{P}^j, j = 0, 1$, depending on whether ν_2 is trivial or not. Then $E = k$ and the period domain is given by

$$\boldsymbol{\mathcal{F}}^{ss} = \Omega^2_{k'} \times \mathbb{P}^j.$$

In particular, we get $\pi_1(\boldsymbol{\mathcal{F}}^{ss}) = \pi_1(\Omega^2_{k'})$. In the case $\nu_1 = \nu_2$ we get $E = k'$ and

$$\boldsymbol{\mathcal{F}}^{ss} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta(\mathbb{P}^1(k')),$$

where $\Delta : \mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ denotes the diagonal morphism. Here we have $\pi_1(\boldsymbol{\mathcal{F}}^{ss}) = \{1\}$.

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