THE FUNDAMENTAL GROUP OF PERIOD DOMAINS OVER FINITE FIELDS

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Abstract. We determine the fundamental group of period domains over finite fields. This answers a question of M. Rapoport raised in [R].

1. Introduction

Period domains over finite fields are open subvarieties of flag varieties defined by a semi-stability condition. They were introduced and discussed by M. Rapoport in [R]. In this paper we determine their fundamental groups which answers a question raised in loc.cit.

Let G be a reductive group over a finite field k. We fix an algebraic closure $\overline{k}$ of k and denote by $\Gamma = \Gamma_k$ the corresponding absolute Galois group of k. Let $\mathcal{N}$ be a conjugacy class of $\mathbb{Q}$-1-PS of $G_{\overline{k}}$. We denote by $E = E(G, \mathcal{N})$ the reflex field of the pair $(G, \mathcal{N})$. This is a finite extension of k which is characterized by its Galois group $\Gamma_E = \{\sigma \in \Gamma \mid \nu \in \mathcal{N} \Longrightarrow \nu^\sigma \in \mathcal{N}\}$. Every $\mathbb{Q}$-1-PS ν induces via Tannaka formalism a $\mathbb{Q}$-filtration $\mathcal{F}_\nu$ over $\bar{k}$ of the forgetful fibre functor $\omega^G : \mathrm{Rep_k(G)} \to \mathrm{Vec}_k$ from the category of algebraic G-representations over k into the category of k-vector spaces. Two $\mathbb{Q}$-1-PS are called par-equivalent if they define the same $\mathbb{Q}$-filtration. There exists a smooth projective variety $\boldsymbol{\mathcal{F}}(G, \mathcal{N})$ over E with

$$\boldsymbol{\mathcal{F}}(G, \mathcal{N})(\overline{k}) = \{\nu \in \mathcal{N} \ \text{ modulo par-equivalence } \} \ .$$

The variety is a generalized flag variety for G_E. More precisely, by a lemma of Kottwitz [K], there is a $\mathbb{Q}$-1-PS $\nu \in \mathcal{N}$ which is defined over $E = E(G, \mathcal{N})$. Thus we may write $\boldsymbol{\mathcal{F}}(G, \mathcal{N}) = G_E/P$, where $P = P(\nu)$ is the parabolic subgroup of G_E attached to ν. Further, after fixing a maximal torus and a Borel subgroup in G, we may suppose that ν is contained in the closure $\bar{C}_{\mathbb{Q}}$ of the corresponding rational Weyl chamber $C_{\mathbb{Q}}$.

A point $x \in \boldsymbol{\mathcal{F}}(G, \mathcal{N})(\bar{k})$ is called semi-stable if the induced filtration $\mathcal{F}_x(\mathrm{Lie}(G)_{\bar{k}})$ on the adjoint representation $\mathrm{Lie}(G)_{\bar{k}} = \mathrm{Lie}(G) \otimes_k \bar{k}$ of G is semi-stable. The latter means that for all k-subspaces U of $\mathrm{Lie}(G)$, the following inequality is satisfied

$$\frac{1}{\dim U}\Big(\sum\nolimits_y y \cdot \dim \mathrm{gr}^y_{\mathcal{F}|U_{\bar{k}}}(U_{\bar{k}})\Big) \leq \frac{1}{\dim \mathrm{Lie}(G)}\Big(\sum\nolimits_y y \cdot \dim \mathrm{gr}^y_{\mathcal{F}}(\mathrm{Lie}(G)_{\bar{k}})\Big).$$

In [DOR] it is shown that there is an open subvariety $\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}$ of $\boldsymbol{\mathcal{F}}(G,\mathcal{N})$ parametrizing all semi-stable points, i.e. $\boldsymbol{\mathcal{F}}(G,\mathcal{N})(\bar{k})^{ss} = \boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}(\bar{k})$. This open subvariety $\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}$ is called the *period domain* to $(G,\mathcal{N})$.

The most prominent example of a period domain is the Drinfeld upper half plane $\Omega_k^{(\ell+1)} = \mathbb{P}_k^\ell \setminus \cup \mathbb{P}(H)$ where H runs through all k-rational hyperplanes of $k^{\ell+1}$. This space corresponds to the pair $(G,\mathcal{N})$ where $G = \mathrm{PGL}_{\ell+1,k}$ and $\nu = (x_1, x_2, \ldots, x_2) \in \bar{C}_{\mathbb{Q}}$ with $x_1 > x_2$ and $x_1 + \ell \cdot x_2 = 0$. Here we identify $\bar{C}_{\mathbb{Q}}$ as usual with $(\mathbb{Q}^{\ell+1})^0_+ = \{(x_1, \ldots, x_{\ell+1}) \in \mathbb{Q}^{\ell+1} \mid \sum_i x_i = 0, x_1 \geq x_2 \geq \ldots \geq x_{\ell+1}\}$. The period domain $\Omega_k^{(\ell+1)}$ is isomorphic to a Deligne-Lusztig variety and admits therefore interesting étale coverings, cf. [DL]. In [OR] it is shown that $\Omega_k^{(\ell+1)}$ is essentially the only period domain which is at the same time a Deligne-Lusztig variety.

Period domains only depend on their adjoint data, cf. [OR], [DOR]. More precisely, let G_{ad} be the adjoint group of G, and let $\mathcal{N}_{\mathrm{ad}}$ be the induced conjugacy class of $\mathbb{Q}$-1-PS of G_{ad}. Then

$$\boldsymbol{\mathcal{F}}(G,\mathcal{N})(\bar{k})^{ss} \xrightarrow{\sim} \boldsymbol{\mathcal{F}}(G_{\mathrm{ad}},\mathcal{N}_{\mathrm{ad}})(\bar{k})^{ss}\ .$$

Also if G splits into a product $G = \prod_i G$, the corresponding period domain splits into products, as well. Thus for formulating our main result, we may assume that G is k-simple adjoint. Hence there is an absolutely simple adjoint group G' over a finite extension k' of k with $G = \mathrm{Res}_{k'/k} G'$. In this case $\mathcal{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_t)$ is given by a tuple of conjugacy classes $\mathcal{N}_j$ of $\mathbb{Q}$-1-PS of $G'_{\bar{k}}$, where $t = |k' : k|$. Thus ν is given by a tuple of $\mathbb{Q}$-1-PS $\nu = (\nu_1, \ldots, \nu_t)$.

Our main result is the following. Let ℓ be the (absolute) rank of G'. We denote by π_1 the functor which associates to a variety its geometric fundamental group.

Theorem 1. *Let G be absolutely simple adjoint over k. Then $\pi_1(\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}) = \{1\}$ unless $G = \mathrm{PGL}_{\ell+1,k}$ and $\nu = (x_1 \geq x_2 \geq \ldots \geq x_{\ell+1}) \in (\mathbb{Q}^{\ell+1})^0_+$ with $x_2 < 0$ or $x_\ell > 0$. In the latter case we have $\pi_1(\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}) = \pi_1(\Omega_k^{(\ell+1)})$.*

More generally, let $G = \mathrm{Res}_{k'/k} G'$ be k-simple adjoint. Then $\pi_1(\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}) = \{1\}$ unless $G' = \mathrm{PGL}_{\ell+1,k'}$ and such that the following two conditions are satisfied. Write $\nu_i = (x_1^{[i]} \geq x_2^{[i]} \geq \ldots \geq x_{\ell+1}^{[i]}) \in (\mathbb{Q}^{\ell+1})^0_+$, $i = 1, \ldots, t$. Then there is a unique $1 \leq j \leq t$, such that

(i) $x_2^{[j]} < 0$ or $x_\ell^{[j]} > 0$.

(ii) $\sum_{i \neq j} x_1^{[i]} < -x_2^{[j]}$ if $x_2^{[j]} < 0$ resp. $\sum_{i \neq j} x_{\ell+1}^{[i]} > -x_\ell^{[j]}$ if $x_\ell^{[j]} > 0$.

In the latter case we have $\pi_1(\boldsymbol{\mathcal{F}}(G,\mathcal{N})^{ss}) = \pi_1(\Omega_{k'}^{(\ell+1)})$.

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2. Some preparations

In this section we recall some results concerning the relation of period domains to Geometric Invariant Theory (GIT).

Let G be a reductive group over k and let $\mathcal{N} = \{\nu\}$ be a conjugacy class of $\mathbb{Q}$-1-PS of $G_{\bar{k}}$. We abbreviate $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}(G, \mathcal{N})$. We fix an *invariant inner product* $(\ ,\ )$ on G over k. Recall that this is a positive-definite bilinear form $(\ ,\ )$ on $X_*(T)_{\mathbb{Q}}$ for any maximal torus T of G defined over $\overline{k}$. The following conditions are required:

(i) For $g \in G(\overline{k})$, the inner automorphism $\mathrm{Int}(g)$ induces an isometry

$$\mathrm{Int}(g) : X_*(T)_{\mathbb{Q}} \longrightarrow X_*(T^g)\ ,\ T^g = g \cdot T \cdot g^{-1}\ .$$

(ii) Any $\sigma \in \Gamma$ induces an isometry

$$\sigma : X_*(T)_{\mathbb{Q}} \longrightarrow X_*(T^{\sigma})_{\mathbb{Q}}\ .$$

The choice of such an inner invariant product induces together with the standard pairing $\langle\ ,\ \rangle : X_*(T)_{\mathbb{Q}} \times X^*(T)_{\mathbb{Q}} \to \mathbb{Q}$ an identification $X_*(T)_{\mathbb{Q}} \cong X^*(T)_{\mathbb{Q}}$ for all maximal tori T of G defined over $\overline{k}$. To the pair $(G, \mathcal{N})$ there is attached an ample homogeneous $\mathbb{Q}$-line bundle $\boldsymbol{\mathcal{L}}$ on $\boldsymbol{\mathcal{F}}$ given by

$$\boldsymbol{\mathcal{L}} = G \times^P \mathbb{G}_{a,-\nu^*}\ \ .$$

Here ν^* denotes the rational character of T which corresponds to ν under the above identification (it extends to a character of P). The following theorem of Totaro [To] describes the semi-stable points $\boldsymbol{\mathcal{F}}^{ss}$ inside $\boldsymbol{\mathcal{F}}$ via GIT. Here we denote by $\mu^{\boldsymbol{\mathcal{L}}}(x, \lambda)$ the slope of $x \in \boldsymbol{\mathcal{F}}(\bar{k})$ with respect to the 1-PS λ and the ample line bundle $\boldsymbol{\mathcal{L}}$ in the sense of GIT, cf. [MFK].

Theorem 2.1. *Let $x \in \boldsymbol{\mathcal{F}}(\bar{k})$. Then $x \in \boldsymbol{\mathcal{F}}^{ss}(\bar{k})$ if and only if for all 1-PS λ of G_{der} defined over k the Hilbert-Mumford inequality holds, i.e.*

$$\mu^{\boldsymbol{\mathcal{L}}}(x, \lambda) \geq 0\ .$$

Let $\Delta_k = \{\alpha_1, \ldots, \alpha_d\}$ be the set of relative simple roots with respect to a fixed maximal split torus $S \subset G$ and a Borel subgroup $B \subset G$ containing S. Note that G is quasi-split since k is a finite field. Let $T = Z(S)$ be the centralizer of S which is a maximal torus over k. We let Δ be the set of absolutely simple roots of G with respect to $T \subset B$. Then the relative simple roots are given by $\Delta_k = \{\alpha|S \mid \alpha \in \Delta, \alpha|S \neq 0\}$, cf. [Ti]. By conjugating ν with an element of the (absolute) Weyl group W, we may assume that ν is contained in the closure of the dominant Weyl chamber, i.e.,

$$\nu \in \bar{C}_{\mathbb{Q}} = \{\lambda \in X_*(T)_{\mathbb{Q}} \mid \langle \lambda, \alpha \rangle \geq 0\ \forall \alpha \in \Delta\}.$$

We denote by $(\omega_\alpha)_{\alpha\in\Delta} \subset X_*(T)_{\mathbb{Q}}$ the set of co-fundamental weights. Recall that they are defined by $(\omega_\alpha, \beta^\vee) = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$. For $1 \leq i \leq d$, let

$$\Psi(\alpha_i) = \{\beta \in \Delta \mid \beta|S = \alpha_i\}.$$

We set

$$\omega_i = \sum\nolimits_{\beta\in\Psi(\alpha_i)} \omega_\beta. \tag{2.1}$$

Up to multiplication by a positive scalar these are just the relative fundamental weights. In [O] we have shown[1] that in Theorem 2.1 it suffices to treat the vertices of the spherical Tits-complex [CLT] defined by Curtis, Lehrer and Tits. Thus

Proposition 2.2. *Let $x \in \boldsymbol{\mathcal{F}}(\bar{k})$. Then $x \in \boldsymbol{\mathcal{F}}^{ss}(\bar{k})$ iff for all $g \in G(k)$ and for all i the inequality $\mu^{\mathcal{L}}(x, \mathrm{Int}(g)\circ\omega_i) \geq 0$ is satisfied.*

We consider the closed complement $Y := \boldsymbol{\mathcal{F}} \setminus \boldsymbol{\mathcal{F}}^{ss}$ of $\boldsymbol{\mathcal{F}}^{ss}$. For any integer $1 \leq i \leq d$, we set

$$Y_i(\bar{k}) := \{x \in \boldsymbol{\mathcal{F}}(\bar{k}) \mid \mu^{\mathcal{L}}(x, \omega_i) < 0\}.$$

The sets $Y_i(\bar{k})$ are induced by closed subvarieties Y_i of Y which are defined over E. Let $P_i = P(\omega_i)$ be the parabolic subgroup corresponding to ω_i. If $n \in \mathbb{N}$ is some integer such that $n\omega_i \in X_*(T)$, then

$$P(\omega_i)(\bar{k}) = \{g \in G(\bar{k}) \mid \lim\nolimits_{t\to 0} \mathrm{Int}(n\omega_i(t)) \circ g \text{ exists in } G(\bar{k})\},$$

cf. [MFK]. This definition does not depend on n and P_i is defined over k since $\omega_i \in X_*(S)_{\mathbb{Q}}$. The natural action of G on $\boldsymbol{\mathcal{F}}$ restricts to an action of P_i on Y_i for every i. It is a consequence of Prop. 2.2 that we can write Y as the union

$$Y = \bigcup\nolimits_{i=1,\ldots,d} \bigcup\nolimits_{g\in G(k)} gY_i. \tag{2.2}$$

In [O] we proved that the varieties Y_i are unions of Schubert cells. More precisely, denote by $W_P \subset W$ the parabolic subgroup induced by P. We identify the elements of $W^P := W/W_P$ with representatives of shortest length in W.

Proposition 2.3. *We have*

$$\begin{aligned} Y_i &= \bigcup_{\substack{w\in W^P \\ (\omega_i, w\nu)>0}} P_i w P/P \\ &= \bigcup_{\substack{w\in W^P \\ (\omega_i, w\nu)>0}} BwP/P. \end{aligned}$$

[1] Actually, in loc.cit. we considered the dual basis of Δ_k which consists of certain positive multiples of $(\omega_i)_i$. This does not affect the statement.

The proof follows from the identity

$$\mu^{\mathcal{L}}(pw[\nu], \omega_i) = -(\omega_i, w\nu),$$

for all $p \in P_i(\bar{k})$, $w \in W$. Here $[\nu]$ denotes the point of $\boldsymbol{\mathcal{F}}(E)$ induced by ν.

We conclude by (2.2) that

$$\dim Y = \max_{i=1,\ldots,d} \dim Y_i.$$

On the other hand, each subvariety Y_i is a union of the Schubert cells BwP/P, $w \in W^P$, with $(\omega_i, w\nu) > 0$. The dimension of BwP/P is $\ell(w)$, cf. [Bo]. Thus we deduce that

$$\dim Y_i = \max \{\ell(w) \mid w \in W^P,\ (\omega_i, w\nu) > 0\}. \tag{2.3}$$

Let w_0 resp. w_0^P be the longest element of the Weyl group W resp. of W^P. Then $w_0 = w_0^P \cdot w_P$ where w_P is the longest element in W_P. In particular

$$w_0\nu = w_0^P \nu \tag{2.4}$$

and

$$\dim \boldsymbol{\mathcal{F}} = \ell(w_0^P). \tag{2.5}$$

We shall examine in the next section when it happens that $\dim Y = \dim \boldsymbol{\mathcal{F}} - 1$, i.e., $\operatorname{codim} Y = 1$.

3. The proof of Theorem 1

From now on we assume that G is k-simple adjoint, i.e., $G = \mathrm{Res}_{k'/k} G'$ for some finite extension k'/k of degree t, cf. [Ti]. Let ℓ be the (absolute) rank of G'. We start with the case where G is absolutely simple adjoint i.e., $k' = k$.

Proposition 3.1. *Let G be absolutely simple adjoint over k. Then* $\operatorname{codim} Y \geq 2$ *unless $G = \mathrm{PGL}_{\ell+1}$ and $\nu = (x_1 \geq x_2 \geq \ldots \geq x_\ell \geq x_{\ell+1}) \in (\mathbb{Q}^{\ell+1})^0_+$ with $x_2 < 0$ or $x_\ell > 0$.*

Proof. The elements of length $\ell(w_0) - 1$ in W are given by the expressions sw_0, where $s \in W$ is a simple reflection. We deduce from (2.3) - (2.5) that there is some integer $1 \leq i \leq d$ with $\operatorname{codim} Y_i = 1$, if and only if there is a simple reflection $s_\beta \in W$, $\beta \in \Delta$, with

$$(\omega_i, s_\beta w_0 \nu) > 0. \tag{3.1}$$

By the equivariance of (,) we get

$$(\omega_i, s_\beta w_0 \nu) = (s_\beta \omega_i, w_0 \nu). \tag{3.2}$$

1^{st} *case:* G is split.

Thus we have $\Delta_k = \Delta$. Further, by [Bou] ch. VI, 1.10, we have[2]

$$s_\beta \omega_i = \begin{cases} \omega_i & \text{if } \beta \neq \alpha_i \\ \omega_i - \alpha_i & \text{if } \beta = \alpha_i \end{cases}.$$

Since $w_0\nu \in -\bar{C}_{\mathbb{Q}}$ we get $(\omega_i, w_0\nu) < 0$. Thus we conclude that $\beta = \alpha_i$ is a necessary condition in order that (3.1) holds. Further, in this situation we get by (3.2) $(\omega_i, s_\beta w_0 \nu) > 0$ if and only if

$$(\omega_i, w_0\nu) > (\alpha_i, w_0\nu). \tag{3.3}$$

We start to investigate inequality (3.3) for the root system of type $A_\ell (\ell \geq 1)$. In this case the data is given as follows:

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1},\ i = 1, \ldots, \ell, \\ \omega_i &= \frac{1}{\ell+1}\big((\ell+1-i)^{(i)}, -i^{(\ell+1-i)}\big),\ i = 1, \ldots, \ell, \\ \bar{C}_{\mathbb{Q}} &= (\mathbb{Q}^{\ell+1})^0_+. \end{aligned}$$

Here in the definition of ω_i the exponent (j) means that we repeat the corresponding entry j times. Further, w_0 acts on $\mathbb{Q}^{\ell+1}$ via

$$w_0(x_1, x_2, \ldots, x_{\ell+1}) = (x_{\ell+1}, x_\ell, \ldots, x_1).$$

Let $\nu = (x_1 \geq x_2 \geq \ldots \geq x_{\ell+1}) \in (\mathbb{Q}^{\ell+1})^0_+$. Then

$$(\omega_i, w_0\nu) = x_{\ell+1} + \ldots + x_{\ell-i+2}$$

and

$$(\alpha_i, w_0\nu) = x_{\ell-i+2} - x_{\ell-i+1}.$$

Thus inequality (3.3) is satisfied if and only if

$$x_{\ell+1} + \ldots + x_{\ell-i+3} > -x_{\ell-i+1} \ \text{ if } \ 1 < i < \ell \tag{3.4}$$

resp.

$$x_\ell > 0 \ \text{ if } \ i = 1$$

resp.

$$x_2 < 0 \ \text{ if } \ i = \ell.$$

Let $1 < i < \ell$. Then

$$x_1 + \ldots + x_{\ell-i} + x_{\ell-i+2} \geq x_{\ell+1} + \ldots + x_{\ell-i+3} + x_{\ell-i+1}$$

[2] Here we make use of the identification $X_*(T)_{\mathbb{Q}} = X^*(T)_{\mathbb{Q}}$

as $x_{\ell-i+2} \geq x_{\ell-i+3}$, $x_{\ell-i} \geq x_{\ell-i+1}$ and $\sum_{j=1}^{\ell-i-1} x_j \geq 0$ resp. $\sum_{j=0}^{i-3} x_{\ell+1-j} \leq 0$. Thus (3.4) cannot be satisfied if $1 < i < \ell$ since the sum over all entries in ν vanishes. Hence the proof follows in the case of the root system $A_\ell (\ell \geq 1)$.

For the other split root systems, i.e., of type $B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4, G_2$, we proceed as follows. We write down $\nu = \sum_{i=1}^{\ell} n_i \omega_i$ as linear combination of the co-fundamental weights with non-negative coefficients $n_i \geq 0$. Note that $n_i = (\nu, \alpha_i^\vee)$, $i = 1, \ldots, \ell$. We get

$$w_0 \nu = -\sum_{j=1}^{\ell} n_j \omega_{\tau(j)}.$$

where τ is the opposition involution of $\{1, \ldots, \ell\}$, cf. [Ti]. In the case of B_ℓ, C_ℓ, $D_\ell (\ell \text{ even}), E_7, E_8, F_4, G_2$ we have $\tau = \text{id}$. For $D_\ell (\ell \text{ odd})$, we have $\tau = (\ell - 1, \ell)$. Finally in the case E_6 we have $\tau = (1,6)(2,5)(3,4)$. In all cases

$$(\omega_i, w_0 \nu) = -\sum_{j=1}^{\ell} n_j (\omega_i, \omega_{\tau(j)}).$$

and

$$(\alpha_i, w_0 \nu) = -n_{\tau^{-1}(i)} \cdot \frac{1}{2} \cdot (\alpha_i, \alpha_i)$$

as $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$. Since $(\omega_i, \omega_j) \geq 0$ for all i, j, cf. [Bou], ch. VI, 1.10, we get

$$(\omega_i, w_0 \nu) \leq -n_{\tau^{-1}(i)} \cdot (\omega_i, \omega_i). \tag{3.5}$$

Further one checks case by case by the explicit representation of the co-fundamental weights in loc.cit. p. 265-290, that

$$(\omega_i, \omega_i) \geq \frac{1}{2} \cdot (\alpha_i, \alpha_i) \text{ for } i = 1, \ldots, \ell.$$

Hence we get by using (3.5)

$$(\omega_i, w_0 \nu) \leq (\alpha_i, w_0 \nu).$$

Thus we deduce that the inequality (3.3) cannot be satisfied for root systems different from A_ℓ. Let us illustrate this argument for the root system of type G_2. Here the data is given by

$$\begin{aligned} \alpha_1 &= \epsilon_1 - \epsilon_2, \ \alpha_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3, \\ \omega_1 &= \epsilon_3 - \epsilon_2, \ \omega_2 = -\epsilon_1 - \epsilon_2 + 2\epsilon_3. \end{aligned}$$

Let $\nu = n_1 \omega_1 + n_2 \omega_2$ with $n_1, n_2 \geq 0$. We get $w_0 \nu = -n_1 \omega_1 - n_2 \omega_2$. Then

$$(\omega_1, w_0 \nu) = -n_1 (\omega_1, \omega_1) - n_2 (\omega_1, \omega_2) = -2n_1 - 3n_2$$

and

$$(\omega_2, w_0 \nu) = -n_1 (\omega_2, \omega_1) - n_2 (\omega_2, \omega_2) = -3n_1 - 6n_2.$$

Further, we compute

$$(\alpha_1, w_0\nu) = -n_1 \cdot \frac{1}{2} \cdot (\alpha_1, \alpha_1) = -n_1$$

and

$$(\alpha_2, w_0\nu) = -n_2 \cdot \frac{1}{2} \cdot (\alpha_2, \alpha_2) = -3n_2.$$

Hence

$$(\omega_1, w_0\nu) \leq -n_1(\omega_1, \omega_1) = -2n_1 \leq (\alpha_1, w_0\nu) = -n_1$$

and

$$(\omega_2, w_0\nu) \leq -n_2(\omega_2, \omega_2) = -6n_2 \leq (\alpha_2, w_0\nu) = -3n_2.$$

2^{nd} *case:* G is not split.

Recall that $\omega_i = \sum_{\beta \in \Psi(\alpha_i)} \omega_\beta$, cf. (2.1). We get

$$s_\beta \omega_i = \begin{cases} \omega_i & \text{if } \beta \notin \Psi(\alpha_i) \\ \omega_i - \beta & \text{if } \beta \in \Psi(\alpha_i) \end{cases}.$$

Again we conclude that $\beta \in \Psi(\alpha_i)$ is a necessary condition in order that (3.1) holds. Further $(\omega_i, s_\beta w_0\nu) > 0$, if and only if

$$(\omega_i, w_0\nu) > (\beta, w_0\nu). \tag{3.6}$$

Now we have

$$(\omega_i, w_0\nu) = \sum\nolimits_{\beta \in \Psi(\alpha_i)} (\omega_\beta, w_0\nu) \leq (\omega_\beta, w_0\nu) \ \text{ for all } \beta \in \Psi(\alpha_i).$$

Thus by the computation in the 1^{st} case, we conclude that a necessary condition in order that (3.6) holds is that the root system of $G_{\bar{k}}$ is of type $A_\ell (\ell \geq 1)$.

In this case the group $G = \mathrm{PU}_{\ell+1}$ is the projective unitary group of (absolute) rank ℓ and $d = [\frac{\ell+1}{2}]$, cf. [Ti]. The co-fundamental weights $(\omega_i)_i$ of $\mathrm{PU}_{\ell+1}$ are given as follows. Let $\Delta = \{\beta_1 = \epsilon_1 - \epsilon_2, \ldots, \beta_\ell = \epsilon_\ell - \epsilon_{\ell+1}\}$ be the set of standard simple roots of type A_ℓ. Then

$$\omega_i = \omega_{\beta_i} + \omega_{\beta_{\ell+1-i}},\ i = 1, \ldots, d-1$$

and

$$\omega_d = \begin{cases} \omega_{\beta_d} & \text{if } \frac{\ell+1}{2} \in \mathbb{Z} \\ \omega_{\beta_d} + \omega_{\beta_{d+1}} & \text{if } \frac{\ell+1}{2} \notin \mathbb{Z} \end{cases}.$$

Thus by the explicit computation in the $\mathrm{PGL}_{\ell+1}$-case, we see that if inequality (3.6) is satisfied, then we necessarily have $i = 1$ and $\beta = \beta_1$ or $\beta = \beta_\ell$. But we compute

$$(\omega_1, w_0\nu) = x_{\ell+1} - x_1$$

and

$$(\beta_1, w_0\nu) = x_{\ell+1} - x_\ell$$

resp.

$$(\beta_\ell, w_0\nu) = x_2 - x_1.$$

Hence we see that inequality (3.6) cannot be satisfied for $G = \mathrm{PU}_{\ell+1}$ either. $\square$

Next we determine explicitely the period domains for which the codimension of the closed complement is 1. So by Prop. 3.1 we may assume that $G = \mathrm{PGL}_{\ell+1,k}$ and $\nu = (x_1, x_2, \ldots, x_{\ell+1}) \in (\mathbb{Q}^{\ell+1})^0_+$. We rewrite ν in the shape $\nu = (y_1^{(n_1)}, \ldots, y_r^{(n_r)})$ with $y_1 > y_2 > \cdots > y_r$ and $n_i \geq 1$, $i = 1, \ldots, r$. Let $V = k^{\ell+1}$. Then $\boldsymbol{\mathcal{F}}(G, \mathcal{N})(\bar{k})$ is given by the set of filtrations

$$(0) \subset \mathcal{F}^{y_1} \subset \mathcal{F}^{y_2} \subset \ldots \subset \mathcal{F}^{y_r} = V_{\bar{k}}$$

with

$$\dim \mathcal{F}^{y_i} = n_1 + \cdots + n_i.$$

If $x_2 < 0$ then $n_1 = 1$ resp. if $x_\ell > 0$ then $n_r = 1$. In order to determine the period domain, one can replace in the definition of a semi-stable filtration the Lie Algebra $\mathrm{Lie}(G)$ by V, cf. [DOR]. Thus a point $\mathcal{F}^\bullet$ is semi-stable if for all k-subspaces U of V the following inequality is satisfied

$$\frac{1}{\dim U}\Big(\sum\nolimits_y y \cdot \dim \mathrm{gr}^y_{\mathcal{F}|U_{\bar{k}}}(U_{\bar{k}})\Big) \leq \frac{1}{\dim V}\Big(\sum\nolimits_y y \cdot \dim \mathrm{gr}^y_{\mathcal{F}}(V_{\bar{k}})\Big).$$

Then one computes easily that

$$\boldsymbol{\mathcal{F}}^{ss}(\bar{k}) = \{\mathcal{F}^\bullet \in \boldsymbol{\mathcal{F}}(\bar{k}) \mid \mathcal{F}^{y_1} \text{ is not contained in any } k\text{-rational hyperplane}\}$$

resp.

$$\boldsymbol{\mathcal{F}}^{ss}(\bar{k}) = \{\mathcal{F}^\bullet \in \boldsymbol{\mathcal{F}}(\bar{k}) \mid \mathcal{F}^{y_r} \text{ does not contain any } k\text{-rational line}\}.$$

Thus the projections

$$\begin{array}{ccccccc} \boldsymbol{\mathcal{F}} & \to & \mathbb{P}^\ell_k & \text{resp.} & \boldsymbol{\mathcal{F}} & \to & \check{\mathbb{P}}^\ell_k \\ \\ \mathcal{F}^\bullet & \mapsto & \mathcal{F}^{y_1} & & \mathcal{F}^\bullet & \mapsto & \mathcal{F}^{y_r} \end{array}$$

induce surjective proper maps

$$\boldsymbol{\mathcal{F}}^{ss} \to \Omega_k^{(\ell+1)} \quad \text{resp.} \quad \boldsymbol{\mathcal{F}}^{ss} \to \check{\Omega}_k^{(\ell+1)} \tag{3.7}$$

in which the fibres are generalized flag varieties.

Proof of Theorem 1 in the absolute simple case: The proof follows from Proposition 3.1 and the following facts on fundamental groups of algebraic varieties. If $\operatorname{codim} Y \geq 2$, then we get $\pi_1(\boldsymbol{\mathcal{F}}^{ss}) = \pi_1(\boldsymbol{\mathcal{F}}) = \{1\}$, since $\boldsymbol{\mathcal{F}}$ is simply connected, cf. [SGA1], ch.

XI, Cor. 1.2. If $\operatorname{codim} Y = 1$ we are in the situation (3.7). Then the statement follows from [SGA1] Cor. 6.11 since the fibres of the maps (3.7) are simply connected. Note that the fundamental groups of $\Omega_k^{(\ell+1)}$ and $\check{\Omega}_k^{(\ell+1)}$ are the same since both varieties are isomorphic. $\square$

Now we consider the general case of an k-simple adjoint group G.

Proposition 3.2. *Let $G = \mathrm{Res}_{k'/k} G'$ be k-simple adjoint. Then* $\operatorname{codim} Y \geq 2$ *unless $G' = \mathrm{PGL}_{\ell+1}$ and there is a unique $1 \leq j \leq t$, such that the following two conditions are satisfied. Let $\nu_j = (x_1^{[j]} \geq x_2^{[j]} \geq \ldots \geq x_{\ell+1}^{[j]}) \in (\mathbb{Q}^{\ell+1})_+^0$, $j = 1, \ldots, t$. Then*

(i) ν_j as in the absolutely simple case, i.e., with $x_2^{[j]} < 0$ or $x_\ell^{[j]} > 0$.

(ii) $\sum_{i \neq j} x_1^{[i]} < -x_2^{[j]}$ if $x_2^{[j]} < 0$ resp. $\sum_{i \neq j} x_{\ell+1}^{[i]} > -x_\ell^{[j]}$ if $x_\ell^{[j]} > 0$.

Proof. We conclude by the same argument as in the proof of Proposition 3.1, 2^{nd} case, that $\operatorname{codim} Y_i = 1$ if and only if there is a simple root $\beta \in \Psi(\alpha_i)$ such that

$$(\omega_i, w_0\nu) > (\beta, w_0\nu). \tag{3.8}$$

Let $\mathrm{Gal}(k'/k) = \{\sigma^j \mid 0 \leq j \leq t-1\}$ and denote by W' the Weyl group of G'. Since $G = \mathrm{Res}_{k'/k} G'$ we have $W = \prod_{j=1}^t W'$ and $w_0 = (w_0', \ldots, w_0') \in W$. Further, the natural restriction map $\Delta'_{k'} \to \Delta_k$ is bijective where $\Delta'_{k'} = \{\alpha'_1, \ldots, \alpha'_d\}$ is the set of relative simple roots of G' with respect to a maximal k'-split torus S' such that $S(k) \subset S'(k')$. It follows that $\omega_i = \sum_{j=0}^{t-1} \sigma^j \omega'_i$. Here $(\omega'_i)_i \in X_*(S')_{\mathbb{Q}}$ is defined with respect to $(\alpha'_i)_i \in X^*(S')_{\mathbb{Q}}$. Furthermore, Δ is formed by t copies of the set Δ' of absolute simple roots to G'. We conclude that for each $\beta \in \Psi(\alpha_i)$ there is an index $j(\beta) = j$, $1 \leq j \leq t$, with

$$(\beta, w_0\nu) = (\beta, w'_0\nu_j).$$

For all other indices $h \neq j$, we have $(\beta, w'_0\nu_h) = 0$. We compute

$$(\omega_i, w_0\nu) = \sum_{j=0}^{t-1} (\sigma^j\omega'_i, w_0\nu) \leq (\sigma^j\omega'_i, w_0\nu) = (\omega'_i, w'_0\nu_j). \tag{3.9}$$

Thus by the computation in the proof of Proposition 3.1 we conclude that a necessary condition in order that (3.8) holds is that G' is split and that the root system of G' is of type $A_\ell (\ell \geq 1)$.

So let $G' = \mathrm{PGL}_{\ell+1,k'}$. Then Δ is given by the set $\{\alpha_i^{[j]} \mid 1 \leq i \leq \ell, 1 \leq j \leq t\}$, where

$$\alpha_i^{[j]} = \epsilon_i^{[j]} - \epsilon_{i+1}^{[j]}.$$

Here $\epsilon_i^{[j]}$ is the appropriate coordinate function on $T_{\bar{k}} \cong \prod_{j=1}^t S_{\bar{k}}$, where S is the diagonal torus in $\mathrm{PGL}_{\ell+1,k'}$. Furthermore, the sets $\Psi(\alpha_i)$ are given by

$$\Psi(\alpha_i) = \{\alpha_i^{[j]} \mid 1 \leq j \leq t\}.$$

Let $\nu = (\nu_1, \ldots, \nu_t) \in \bar{C}_{\mathbb{Q}}$. We get $w_0\nu = (w_0'\nu_1, \ldots, w_0'\nu_t)$, where the entries are given by $w_0'\nu_j = (x_{\ell+1}^{[j]}, x_\ell^{[j]}, \ldots, x_1^{[j]})$, $j = 1, \ldots, t$. In the proof of Proposition 3.1 we have seen that if the inequalities (3.8) and (3.9) are satisfied then $\beta = \alpha_1^{[j]}$ and $x_\ell^{[j]} > 0$ resp. $\beta = \alpha_\ell^{[j]}$ and $x_2^{[j]} < 0$ for some integer j with $1 \le j \le t$.

Let $\beta = \alpha_1^{[j]}$ and $x_\ell^{[j]} > 0$. Then

$$(\omega_1, w_0\nu) = \sum_{i=1}^{t} x_{\ell+1}^{[i]}$$

and

$$(\beta, w_0\nu) = x_{\ell+1}^{[j]} - x_\ell^{[j]}.$$

Thus the inequality (3.8) is satisfied if and only if

$$\sum\nolimits_{i \ne j} x_{\ell+1}^{[i]} > -x_\ell^{[j]}.$$

Furthermore, we claim that the integer j is uniquely determined. In fact, suppose first that h is another integer with $1 \le h \le t$ and

$$\sum\nolimits_{i \ne h} x_{\ell+1}^{[i]} > -x_\ell^{[h]}.$$

Without loss of generality we may assume that $-x_\ell^{[j]} \le -x_\ell^{[h]}$. Then

$$-x_\ell^{[j]} \le -x_\ell^{[h]} < \sum\nolimits_{i \ne h} x_{\ell+1}^{[i]} \le x_{\ell+1}^{[j]} \le -x_\ell^{[j]},$$

which is a contradiction. Here the latter inequality follows from the fact that $x_{\ell+1}^{[j]} + x_\ell^{[j]} \le 0$, since $\nu_j \in (\mathbb{Q}^{\ell+1})_+^0$.

If in the opposite direction h is another integer with $1 \le h \le t$ and

$$\sum\nolimits_{i \ne h} x_1^{[i]} < -x_2^{[h]}$$

then

$$x_1^{[j]} \le \sum\nolimits_{i \ne h} x_1^{[i]} < -x_2^{[h]} \le -x_{\ell+1}^{[h]} \le -\sum\nolimits_{i \ne j} x_{\ell+1}^{[i]} < x_\ell^{[j]},$$

which is a contradiction, as well.

The case $\beta = \alpha_\ell^{(j)}$ and $x_2^{[j]} < 0$ behaves dually and yields $\sum_{i \ne j} x_1^{[i]} < -x_2^{[j]}$. □

Again we determine explicitly the period domains where the codimension of the closed complement is 1. So let $\nu = (\nu_1, \ldots, \nu_t) \in \bar{C}_{\mathbb{Q}}$ such that $\operatorname{codim} Y = 1$. After reindexing we may suppose that $\nu_1 \in (\mathbb{Q}^{\ell+1})_+^0$ is the vector with $\sum_{i \ne 1} x_1^{[i]} < -x_2^{[1]}$ or $\sum_{i \ne 1} x_{\ell+1}^{[i]} > -x_\ell^{[1]}$. Over the algebraic closure $\bar{k}$ the flag variety $\boldsymbol{\mathcal{F}}(G, \mathcal{N})$ is the product

$$\boldsymbol{\mathcal{F}}(G, \mathcal{N})_{\bar{k}} = \prod\nolimits_{j=1}^{t} \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,\bar{k}}, \mathcal{N}_j)_{\bar{k}},$$

where $\mathcal{N}_j$ is the $\mathrm{PGL}_{\ell+1,\bar{k}}$-conjugacy class of ν_j. Let $\nu_1 = (y_1^{(n_1)}, \ldots, y_r^{(n_r)})$ with $y_1 > y_2 > \cdots > y_r$ and $n_i \geq 1$, $i = 1, \ldots, r$. The corresponding period domain is then given by

$$\boldsymbol{\mathcal{F}}(G, \mathcal{N})^{ss}_{\bar{k}} = \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_1)^{ss}_{\bar{k}} \times \prod\nolimits_{j\geq 2} \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_j)_{\bar{k}}.$$

In the case $\sum_{i\neq 1} x_1^{[i]} < -x_2^{[1]}$, we have

$$\begin{aligned}\boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_1)^{ss}(\bar{k}) \;=\; & \{\mathcal{F}^\bullet \in \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_1)(\bar{k}) \mid \mathcal{F}^{y_1} \text{ is not contained in} \\ & \text{any } k'\text{-rational hyperplane}\}.\end{aligned}$$

For $\sum_{i\neq 1} x_{\ell+1}^{[i]} > -x_\ell^{[1]}$, we have

$$\begin{aligned}\boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_1)^{ss}(\bar{k}) \;=\; & \{\mathcal{F}^\bullet \in \boldsymbol{\mathcal{F}}(\mathrm{PGL}_{\ell+1,k'}, \mathcal{N}_1)(\bar{k}) \mid \mathcal{F}^{y_r} \text{ does not contain} \\ & \text{any } k'\text{-rational line }\}.\end{aligned}$$

Proof of Theorem 1 in the general case: The proof is the same as in the absolutely simple case and uses Proposition 3.2. □

We finish this paper by considering a non-trivial example.

Example 3.3. Let $G = \mathrm{Res}_{k'/k}\mathrm{PGL}_{2,k'}$ with $|k' : k| = 2$. Then ν corresponds to a tuple $(\nu_1, \nu_2) \in (\mathbb{Q}^2)^0_+ \times (\mathbb{Q}^2)^0_+$. Let $\nu_1 = (x_1 \geq x_2)$ and $\nu_2 = (y_1 \geq y_2)$. Then $x_2 = -x_1 \leq 0$ and $y_2 = -y_1 \leq 0$. If $\nu_1 \neq \nu_2$ then we may assume after changing ν_1 and ν_2 that $-x_2 > y_1$. Note that we allow $\nu_2 = (0,0)$ to be trivial. Thus $\boldsymbol{\mathcal{F}} = \mathbb{P}^1 \times \mathbb{P}^j, j = 0, 1$, depending on whether ν_2 is trivial or not. Then $E = k$ and the period domain is given by

$$\boldsymbol{\mathcal{F}}^{ss} = \Omega^2_{k'} \times \mathbb{P}^j.$$

In particular, we get $\pi_1(\boldsymbol{\mathcal{F}}^{ss}) = \pi_1(\Omega^2_{k'})$. In the case $\nu_1 = \nu_2$ we get $E = k'$ and

$$\boldsymbol{\mathcal{F}}^{ss} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta(\mathbb{P}^1(k')),$$

where $\Delta : \mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ denotes the diagonal morphism. Here we have $\pi_1(\boldsymbol{\mathcal{F}}^{ss}) = \{1\}$.

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