On travelling-wave solutions for a moving-boundary problem of Hele-Shaw type

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Abstract

We discuss a 2D moving-boundary problem for the Laplacian with Robin boundary conditions in an exterior domain. It arises as model for Hele-Shaw flow of a bubble with kinetic undercooling regularization and is also discussed in the context of models for electrical streamer discharges.

The corresponding evolution equation is given by a degenerate, nonlinear transport problem with nonlocal lower-order dependence. We identify the local structure of the set of travelling-wave solutions in the vicinity of trivial (circular) ones. We find that there is a unique nontrivial travelling wave for each velocity near the trivial one. Therefore, the trivial solutions are unstable in a comoving frame.

The degeneracy of our problem is reflected in a loss of regularity in the estimates for the linearization. Moreover, there is an upper bound for the regularity of its solutions. To prove our results, we use a quasilinearization by differentiation, index results for degenerate ordinary differential operators on the circle, and perturbation arguments for unbounded Fredholm operators.

Keywords: *degenerate transport equation, moving-boundary problem, Hele-Shaw flow, kinetic undercooling, electrical streamer discharges*

MSC: 35R35, 76D27

1. Introduction and problem formulation

This paper is concerned with the following moving-boundary problem: One seeks a family of bounded moving domains $t \mapsto \Omega(t) \subset \mathbb{R}^2$, $t \geq 0$, with outer normal $n = n(t)$ and boundary $\Gamma(t)$ and functions $\phi = \phi(\cdot, t)$ defined on $\mathbb{R}^2 \setminus \overline{\Omega(t)}$ such that

$$\left.\begin{aligned} \Delta\phi &= 0 && \text{in } \mathbb{R}^2 \setminus \overline{\Omega(t)}, \\ \phi - \ell\partial_n\phi &= 0 && \text{on } \Gamma(t), \\ \nabla\phi - \vec{e}_1 &= o(|x|^{-1}) && \text{for } |x| \to \infty, \\ V_n &= \partial_n\phi. && \end{aligned}\right\} \tag{1.1}$$

Here, $\vec{e}_1$ is the unit vector in x_1 direction and V_n is the normal velocity of the moving boundary $\Gamma(t)$. The initial domain $\Omega(0) = \Omega_0$ is given and the parameter ℓ is a nonnegative constant. Note that we have to consider an exterior problem here. Due to $\ell \geq 0$, it can be shown by standard arguments that the fixed time problem $(1.1)_1$-$(1.1)_3$ is uniquely solvable for arbitrary (sufficiently smooth) $\Omega(t)$ [12]. The evolution of $\Omega(t)$ is volume preserving as

$$\frac{d}{dt}\int_{\Omega(t)} dx = \int_{\Gamma(t)} V_n\, ds = \int_{\Gamma(t)} \partial_n \phi\, ds = 0. \tag{1.2}$$

It is well known that for $\ell = 0$ the problem is strongly linearly ill-posed [6]. (The problem is first order forward or backward parabolic, depending on the sign of V_n which changes along $\Gamma(t)$.) Introducing $\ell > 0$ is a regularization technique as replacing in this way a Dirichlet boundary condition by a Robin condition changes the type of the problem to a transport problem with nonlocal lower order terms; this will be discussed below in more detail.

Problem (1.1) arises as model for the motion of a bubble in an exterior Hele-Shaw flow with so-called kinetic undercooling regularization [6, 11], driven by a prescribed flow velocity at infinity. For this problem, short time well-posedness results in Sobolev spaces of sufficiently high order can be established as in [10]. (A bounded drop is considered there instead of a bubble. However, the proof remains valid without essential changes.)

Furthermore, (1.1) is also discussed as a model for certain electrical discharge processes [2, 8, 9]. Such discharge processes are observed in various natural and experimental settings, e.g. as precursors of lightnings, and are sometimes referred to as electrical streamers. In this context, $\Omega(t)$ represents the domain of total ionization and ϕ is the electric potential in the non-ionized, charge-free phase. The regularization parameter ℓ is related to the thickness of the interface between these two phases. However, the (near) circular geometry we discuss here is different from the typically elongated shape of real streamers, and studying this geometry in the context of streamer modelling is motivated by the assumption that the dynamics and stability properties of the streamer are determined by the behavior near its tip [2]. There an ionization front advances into the nonionized domain. In this situation, the evolution law $(1.1)_4$ has been found from the discussion of a plane situation [1]. For further comments on our results with respect to stability properties for streamer models we refer to the Conclusions section at the end of the paper.

To remove the inhomogeneous term at infinity we write $\phi = \psi + x_1$ and obtain

$$\left.\begin{aligned} \Delta\psi &= 0 && \text{in } \mathbb{R}^2 \setminus \overline{\Omega(t)}, \\ \psi - \ell\partial_n\psi &= -x_1 + \ell n_1 && \text{on } \Gamma(t), \\ |\nabla\psi| &= o(|x|^{-1}) && \text{for } |x| \to \infty, \\ V_n &= \partial_n\psi + n_1. && \end{aligned}\right\} \tag{1.3}$$

It is not hard to see that $(1.3)_3$ implies boundedness of ψ, and this implies in turn even $|\nabla\psi| = O(|x|^{-2})$, cf. [12] §V.26. Therefore, we can equivalently replace $(1.3)_3$ by the more usual demand that ψ is bounded. In this case, the problem $(1.3)_1$, $(1.3)_2$ is known to be uniquely solvable for fixed Ω.

Our main concern here will be the existence of "travelling-wave solutions", i.e. solutions of the type $\Omega(t) = \Omega_0 + \vec{\nu}t$, where $\vec{\nu} = (\nu_1, \nu_2) \in \mathbb{R}^2$ is the velocity of the travelling wave. After introduction of a corresponding moving coordinate system we find the stationary free boundary problem

$$\left.\begin{aligned} \Delta\psi &= 0 && \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ \psi - \ell\partial_n\psi &= -x_1 + \ell n_1 && \text{on } \Gamma, \\ \psi &= O(1) && \text{for } |x| \to \infty, \\ \partial_n\psi + (\vec{e}_1 - \vec{\nu})n &= 0 && \text{on } \Gamma, \end{aligned}\right\} \tag{1.4}$$

where both the shape Ω of the moving domain and its velocity are a priori unknown.

We start by collecting some invariance properties of (1.4) for given, fixed $\vec{\nu}$ that will be used later. Assume in the following that (Ω, ψ, ℓ) is a solution to (1.4).

- Translation invariance: For any $a \in \mathbb{R}^2$, $(\Omega + a, T_a\psi, \ell)$ is a solution to (1.4), where
$$\Omega + a := \{x + a \,|\, x \in \Omega\}, \quad T_a\psi(x) := \psi(x - a).$$
- Scaling invariance: For any $R > 0$, $(R\Omega, S_R\psi, R\ell)$ is a solution to (1.4), where
$$R\Omega := \{Rx \,|\, x \in \Omega\}, \quad S_R\psi(x) := R\psi(x/R).$$
- Point reflection symmetry: $(-\Omega, \psi^-, \ell)$ is a solution to (1.4), where
$$-\Omega := \{-x \,|\, x \in \Omega\}, \quad \psi^-(x) := -\psi(-x).$$
- If $\vec{\nu} \parallel \vec{e}_1$ then one also has reflection symmetry with respect to the x_1-axis: $(\Omega^\times, \psi^\times, \ell)$ is a solution to (1.4), where
$$\Omega^\times := \{(x_1, -x_2) \,|\, (x_1, x_2) \in \Omega\}, \quad \psi^\times(x_1, x_2) := \psi(x_1, -x_2).$$
Together with the point reflection symmetry this implies also symmetry with respect to the x_2-axis.

It is easy to check that (1.4) has a solution for any $\ell \geq 0$ given by circles of radius 1, moving in direction $\vec{e}_1$ with speed $2/(\ell + 1)$:

$$\Omega = B_1(0) := \{|x| < 1\}, \quad \vec{\nu} = \vec{\nu}_0 := \frac{2}{\ell + 1}\vec{e}_1, \quad \psi = \frac{\ell - 1}{\ell + 1}\frac{x_1}{|x|^2}. \tag{1.5}$$

Being interested in the structure of the set of solutions $(\Omega, \vec{\nu})$ to (1.4) near (1.5) we first note that due to the translation and scaling invariance, (1.5) is an element of a trivial three-parameter family of solutions (with ℓ fixed)

$$\Omega = B_R(a) := \{|x - a| < R\}, \quad \vec{\nu} = \frac{2R}{\ell + R}\vec{e}_1, \quad \psi = R^2\frac{\ell - R}{\ell + R}\frac{x_1 - a_1}{|x - a|^2},$$

$R > 0$, $a \in \mathbb{R}^2$.

To exclude the corresponding degrees of freedom from our analysis we demand that Ω has the area of the unit circle and its geometric center is at 0:

$$\int_\Omega d\xi = \pi, \quad \int_\Omega \xi \, d\xi = 0. \tag{1.6}$$

Both demands are unessential as far as the shapes of possible travelling waves are concerned, i.e. up to translation and scaling. (Recall, however, that scaling entails a change of ℓ.)

Note, furthermore, that in the case $\ell = 0$ there is a one-parameter family of nontrivial solutions to (1.4), (1.6) given by ellipses, moving in direction of e_1 with varying speed, depending on the ratio of the axes:

$$\Omega := \{(x_1, x_2) \,|\, x_1^2/a^2 + a^2 x_2^2 < 1\}, \quad \vec{\nu} = (1 + a^2)\vec{e}_1.$$

The central result of the present paper is a proof of the fact that also for the regularized problem with $\ell > 0$, nontrivial solutions to (1.4), (1.6) exist whose smoothness appears to decrease as ℓ increases, i.e. while using $\ell > 0$ regularizes the evolution problem, it deregularizes travelling-wave solutions.

In fact, we will identify the set of all domains that appear in travelling-wave solutions near (1.5) as a two-dimensional submanifold in a suitable Sobolev space whose order decreases with increasing ℓ. This local submanifold can be parameterized by the corresponding velocities $\vec{\nu}$. (Note, however, that we strictly prove the nonsmoothness only for the linearized version of our problem, cf. Remark 2.4 below.)

This paper is organized as follows: In Section 2 we rewrite our problem as a nonlocal operator equation for an unknown function u which represents radial perturbations of the unit circle. We describe the properties of the nonlocal solution operator for the exterior Robin problem and announce our main result. In Section 3 we discuss the linearized problem at the trivial solution. While this might be of independent interest for a qualitative understanding of the problem, the results are also crucial as a basis of the perturbation arguments for unbounded Fredholm operators that are used later. Section 4 starts out with identifying a quasilinear structure for a differentiated version of our operator equation. Then some results on degenerate linear differential operators on the unit circle are derived which form the starting point for the proof of the crucial estimates on the linearized operator in Lemma 4.5. Finally, the proof of the main result is given by a fixed point iteration.

The main technical difficulty here is the fact that due to the degeneracy of the first order differential operators that form the principal part in the linearization, elliptic theory does not directly apply, and one has to cope with a loss of regularity. Moreover, the domains of definition of these operators are dependent on the point around which the linearization is carried out. These problems are overcome here by "quasilinearization by differentiation" and working with two different norms for the fixed point iteration.

We want to point out that we deliberately have refrained from using complex analysis and conformal mapping techniques that are widely used for problems of our type. In our approach, Fourier techniques can be applied and interpreted more straightforwardly, and it is more easily generalized to the 3D problem we intend to discuss in a forthcoming paper.

2. Transformation and main result

We restrict ourselves to domains that are star-shaped with respect to 0 and reformulate our problem as a nonlinear, nonlocal operator equation on the unit circle $S := \partial B_1(0)$.

To shorten notations, in the following we often identify functions $u = u(x_1, x_2)$ defined on S with 2π-periodic functions on $\mathbb{R}$ via $u(\theta) = u(\cos\theta, \sin\theta)$, $\theta \in \mathbb{R}$.

Define for positive continuous functions $u : S \to \mathbb{R}$

$$\Omega_u := \big\{x \in \mathbb{R}^2 \;\big|\; 0 < |x| < u(x/|x|)\big\} \cup \big\{(0,0)\big\}, \quad \Gamma_u := \partial\Omega_u.$$

Then (1.4), (1.6) can be written as

$$F(u, \vec{\nu}) := \ell\vec{\nu} \cdot n(u) - x_1(u) + A_\ell(u)\big[x_1(u) - \ell n_1(u)\big] = 0 \text{ on } S, \tag{2.1}$$

$$\int_0^{2\pi} u^2(\theta)\, d\theta = 2\pi, \qquad \int_0^{2\pi} u^3(\theta) \sin\theta\, d\theta = \int_0^{2\pi} u^3(\theta)\cos\theta\, d\theta = 0, \tag{2.2}$$

where

$$x(u)(\theta) = u(\theta)(\cos\theta, \sin\theta), \qquad \theta \in [0, 2\pi]$$

is a parameterization of Γ_u, and

$$n(u)(\theta) = \big(u^2(\theta) + u'^2(\theta)\big)^{-1/2}\left((u(\theta)\sin\theta)', -(u(\theta)\cos\theta)'\right)$$

is the exterior unit normal to Ω_u in the point $x(u)(\theta) \in \Gamma_u$. Furthermore, the operator $A_\ell(u)$ acting on (sufficiently smooth) real-valued functions ϕ defined on S is given by $A_\ell(u)\phi = \psi \circ x(u)$, where $\psi : \mathbb{R}^2 \setminus \overline{\Omega}_u \to \mathbb{R}$ is the solution to the exterior Robin problem

$$\left.\begin{aligned} \Delta\psi &= 0 && \text{in } \mathbb{R}^2 \setminus \overline{\Omega}_u, \\ \psi - \ell\partial_n\psi &= \phi \circ x(u)^{-1} && \text{on } \Gamma_u, \\ \psi &= O(1) && \text{for } |x| \to \infty. \end{aligned}\right\} \tag{2.3}$$

In other words, $A_\ell(u)$ is (the transformed version of) an exterior Robin-Dirichlet operator for the Laplacian. In particular, for $u = \mathbf{1}$ (i.e. $u \equiv 1$ on S) and

$$\phi(\theta) = a_0 + \sum_{k=1}^{\infty}\big(a_k\cos(k\theta) + b_k\sin(k\theta)\big)$$

we get

$$\big(A_\ell(\mathbf{1})\phi\big)(\theta) = a_0 + \sum_{k=1}^{\infty}\big(a_k\cos(k\theta) + b_k\sin(k\theta)\big)/(1 + k\ell). \tag{2.4}$$

Observe that $u = \mathbf{1}$, $\vec{\nu} = \vec{\nu}_0$ is a solution to (2.1), (2.2) corresponding to (1.5). Moreover, the volume conservation (1.2) reappears now as

$$\int_S F(u, \vec{\nu})(\theta)\big(u^2(\theta) + u'^2(\theta)\big)^{1/2}\, d\theta = 0. \tag{2.5}$$

We collect some properties of A_ℓ that will be needed in the sequel to study the mapping properties of F and its derivatives. As usual, the standard L^2-based Sobolev space of order $s \in \mathbb{R}$ on S will be denoted by $H^s(S)$. Furthermore, for $s > 1/2$ let

$$H^s_+(S) := \{u \in H^s(S) \,|\, u > 0\}.$$

Lemma 2.1. *(Properties of A_ℓ)*
Let $s > 3/2$, $t \in [1/2, s]$. Then $A_\ell \in C^\infty\left(H^s_+(S), \mathscr{L}\left(H^{t-1}(S), H^t(S)\right)\right)$ with

$$\|A_\ell^{(k)}(u)\{h_1, \dots, h_k\}\phi\|_t \le C_k\|h_1\|_s \cdots \|h_k\|_s\|\phi\|_{t-1}, \quad k = 1, 2, \dots$$

where the constants C_k are independent of u, as long as u varies in a set U of the form

$$U = \big\{u \in H^s(S) \;\big|\; \|u\|_s \le M \text{ and } u \ge \alpha \text{ on } S\big\}$$

with fixed $M, \alpha > 0$. Moreover, $A'_\ell(u)\{h\}\phi$ is explicitly given by

$$\begin{aligned} A'_\ell(u)\{h\}\phi = A_\ell(u)\big[\big(-\partial_r\psi \circ x(u)\big)h + \ell\big(\big(\partial_r\partial_n\psi \circ x(u)\big)h \\ + (\nabla\psi \circ x(u)) \cdot n'(u)\{h\}\big)\big] + \big(\partial_r\psi \circ x(u)\big)h \end{aligned} \tag{2.6}$$

where $\psi = A_\ell(u)\phi \circ x(u)^{-1}$ and $\partial_r := |x|^{-1}x \cdot \nabla$ denotes the derivative in radial direction.

Remark 2.2. Note that $u \in H^s(S)$, $\phi \in H^{s-1}(S)$ imply only $A_\ell(u)\phi \in H^s(S)$ and consequently

$$\partial_r\psi \circ x(u) \in H^{s-1}(S), \quad \partial_n\partial_r\psi \circ x(u) \in H^{s-2}(S),$$

so the summands on the right side of the representation formula (2.6) are both in $H^{s-1}(S)$ only while their sum is in $H^s(S)$. However, (2.6) will be used to explicitly calculate the linearization of F around $u = \mathbf{1}$.

We refrain from giving the details of the proof and refer the reader instead to Section 4 of [4] where the result is implicitly contained in the proof of Lemma 4.5; cf. in particular Eqns. (3.6), (4.14), (4.19), and (4.23) there. (In the problem discussed in [4], the domain is bounded and $\ell = 1$, however, this is not essential.)

Observe, moreover, that the structure of the linearization can be formally verified by "variation of the domain" in the BVP (2.3) which gives

$$\left.\begin{aligned} \Delta\psi' &= 0 && \text{in } \mathbb{R}^2 \setminus \overline{\Omega}_u, \\ \psi' - \ell\partial_n\psi' &= -\partial_r\psi h + \ell\big(\partial_r\partial_n\psi h + \nabla\psi \cdot n'(u)\{h\}\big) && \text{on } \Gamma_u, \\ \psi' &= O(1) && \text{for } |x| \to \infty. \end{aligned}\right\}$$

and using $A'_\ell(u)\{h\}\phi = \psi' \circ x(u) + \big(\partial_r\psi \circ x(u)\big)h$. Here ψ' has to be interpreted formally as derivative of a function defined on a changing domain. For a general approach to the problem of "variation of a boundary value problem with respect to the domain" see Ch. 2 in [5].

Using Lemma 2.1 we see that

$$F(\cdot, \vec{\nu}) \in C^\infty\big(H^s_+(S), H^{s-1}(S)\big), \qquad s > 3/2.$$

Now we are in position to formulate our main result about the existence of travelling-wave solutions near the moving circles given by (1.5).

Theorem 2.3. *Fix $\ell \in (0, 2/7)$ and let $s \in \mathbb{N}$ with $4 \le s < \frac{1}{\ell} + \frac{1}{2}$. Then there exist $\delta > 0$, $\varepsilon > 0$ such that for $|\vec{\nu} - \vec{\nu}_0| \le \varepsilon$ the equations* (2.1), (2.2) *have a unique solution in the set $\{u \in H^s(S) \mid \|u - \mathbf{1}\|_s \le \delta\}$. Moreover, we have*

$$u = \mathbf{1} + \ell(\vec{\nu}_0 - \vec{\nu}) \cdot (\varphi_1, \varphi_2) + o\big(|\vec{\nu} - \vec{\nu}_0|\big) \text{ in } H^{s-1} \tag{2.7}$$

with functions $\varphi_1, \varphi_2 \in H^s(S)$ given by the formulas (3.6), (3.7) *below. The corresponding domain Ω_u is point-symmetric with respect to* 0, *i.e. $\Omega_u = -\Omega_u$. If additionally $\vec{\nu} \parallel \vec{e}_1$ then Ω_u is symmetric with respect to both coordinate axes.*

This theorem gives a complete characterization of the structure of the set of solutions to (1.4) near (1.5) and reveals some remarkable features of the problem. First, it shows that the solutions given by translating circles have no (Ljapunov) stability as in any neighborhood of a circle there are shapes of travelling-wave solutions that move with different velocities. In such a situation, it seems hard to predict whether in the nonstationary problem there is convergence to a travelling-wave solution and if so, to which one.

Remark 2.4. We will see below that φ_1 and φ_2 are actually nonsmooth functions whose smoothness depends on ℓ. This reflects the degenerate character of the problem. More precisely, the smoothness increases towards C^∞ as $\ell \to 0$. This is in accordance with the fact that for $\ell = 0$ nontrivial solutions are given by ellipses. Thus, as mentioned above already, while introducing $\ell > 0$ regularizes the evolution problem, at the same time it appears to deregularize the (nontrivial) travelling waves.

To make the last point more precise, observe the following: Although Theorem 2.3 does not yield a strict result on the nonsmoothness of nontrivial solutions, it ensures that the mapping $\vec{\nu} \mapsto u$ cannot be differentiable at $\vec{\nu} = \vec{\nu}_0$ with values in $H^\sigma(S)$ for any $\sigma > \frac{1}{1} + \frac{1}{2}$, see Remark 3.2 below.

Remark 2.5. Although the case $\ell \approx 1$ is outside the scope of our analysis, we will use this particularly simple case to illustrate our results in an informal way by explicitly describing a subfamily of nontrivial travelling waves near circles. If $\ell = 1$ and $\vec{\nu} = \vec{e}_1$ then (2.1) reduces to the purely geometric problem of finding domains Ω such that $n_1 - x_1 = \text{const}$ on the boundary. By translational invariance we can demand $n_1 = x_1$ and $\int_\Omega x_2 \, dx = 0$. It is not hard to see that there is a one-parameter family of domains

$$\Omega(c) := \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, |x_1| < 1,\ |x_2| < (1 - x_1^2)^{1/2} + c \right\}, \qquad c \in (-1, \infty),$$

satisfying these demands. Of course, $\Omega(c)$ does not have the prescribed area for $c \neq 0$. However, if we rescale by an appropriate factor we get domains $\tilde{\Omega}(c)$ satisfying (1.4) with some $\ell = \ell(c)$ (and appropriate ψ) as well as (1.6). If $|c|$ is small then $\tilde{\Omega}(c)$ represents a domain near the unit circle moving as travelling wave with velocity $\vec{e}_1$ while the unit circle itself moves with the slightly different velocity $2\vec{e}_1/(\ell(c) + 1)$.

3. Linearization at the trivial solution

The first step to solve the equations (2.1), (2.2) locally near $u = \mathbf{1}$, $\vec{\nu}_0$ is to discuss the solvability of the linearized equations. From Lemma 2.1 and the facts that $x(\mathbf{1})$ is the identity on S and $\partial_r = \partial_n$ there it is not hard to calculate the linearization

$$\hat{L} := D_u F(\mathbf{1}, \vec{\nu}_0) \in \mathscr{L}\big(H^s(S), H^{s-1}(S)\big).$$

We get

$$\begin{aligned} \hat{L}h = {} & \frac{2\ell}{\ell + 1} h' \sin\theta - h\cos\theta + \partial_n \psi h \\ & + A_\ell(\mathbf{1}) \big[-\partial_n \psi h + \ell\big(\partial_n^2 \psi h + \nabla\psi \cdot n'(\mathbf{1})\{h\}\big) + h\cos\theta - \ell h' \sin\theta \big], \end{aligned}$$

where ψ satisfies (2.3) with $u \equiv 1$ and $\phi = (1-\ell)x_1$, i.e.

$$\psi = \frac{1-\ell}{\ell+1}\frac{x_1}{|x|^2}.$$

(We have abused notation here by writing $\sin\theta$ and $\cos\theta$ for the functions sin and cos. Moreover, note that n' denotes a Frechet derivative with respect to h while h' denotes the spatial derivative of h.) Consequently,

$$\hat{L}h = \frac{2}{\ell+1}\big(\ell h'\sin\theta - h\cos\theta + A_\ell(\mathbf{1})\big[(1+\ell-\ell^2)h\cos\theta - \ell^2 h'\sin\theta\big]\big). \tag{3.1}$$

Note that due to (2.2) we can restrict our attention to perturbations h of the trivial solution $u \equiv 1$ where the zeroth and first Fourier coefficients vanish. Using a corresponding Fourier representation

$$h(\theta) = \sum_{k=2}^{\infty}\big(a_k\cos(k\theta) + b_k\sin(k\theta)\big) \tag{3.2}$$

and the identities

$$\begin{aligned} \sin\theta\big(\cos(k\theta)\big)' &= \frac{k}{2}\Big(\cos\big((k+1)\theta\big) - \cos\big((k-1)\theta\big)\Big),\\ \cos\theta\cos(k\theta) &= \frac{1}{2}\Big(\cos\big((k+1)\theta\big) + \cos\big((k-1)\theta\big)\Big) \end{aligned}$$

together with their counterparts for $\sin(k\theta)$ and (2.4) we find

$$\begin{aligned} (\hat{L}h)(\theta) = \frac{\ell}{\ell+1}\sum_{k=2}^{\infty}\Big[&\big(\alpha_k\cos\big((k+1)\theta\big) + \beta_k\cos\big((k-1)\theta\big)\big)a_k\\ &+ \big(\alpha_k\sin\big((k+1)\theta\big) + \beta_k\sin\big((k-1)\theta\big)\big)b_k\Big] \end{aligned}$$

where

$$\alpha_k = (k-1)\left(1 - \frac{1}{1+(k+1)\ell}\right), \quad \beta_k = (k-1)\left(1 + \frac{1}{1+(k+1)\ell}\right). \tag{3.3}$$

With this we get

$$(\hat{L}h)(\theta) = \frac{\ell}{\ell+1}\sum_{k=1}^{\infty}\big(f_k\cos(k\theta) + g_k\sin(k\theta)\big)$$

where the coefficients f_k and g_k are given by

$$f_1 = -\beta_2 a_2, \quad f_2 = -\beta_3 a_3, \quad f_k = \alpha_{k-1}a_{k-1} - \beta_{k+1}a_{k+1}, \tag{3.4}$$

$$g_1 = -\beta_2 b_2, \quad g_2 = -\beta_3 b_3, \quad g_k = \alpha_{k-1}b_{k-1} - \beta_{k+1}b_{k+1}, \tag{3.5}$$

$k = 3, 4, \ldots$. From this we easily obtain

Lemma 3.1. *The equation*

$$\hat{L}h = \alpha\cos\theta + \beta\sin\theta$$

has the solution

$$h = \alpha\varphi_1 + \beta\varphi_2$$

which is unique within the class given by (3.2)*. Here,*

$$\varphi_1(\theta) := \frac{\ell+1}{\ell}\sum_{k=1}^{\infty} c_k \cos(2k\theta), \quad \varphi_2(\theta) := \frac{\ell+1}{\ell}\sum_{k=1}^{\infty} c_k \sin(2k\theta) \tag{3.6}$$

with coefficients given by

$$c_1 = -\frac{1}{\beta_2}, \quad c_k = -\frac{\alpha_2}{\beta_2}\cdot\frac{\alpha_4}{\beta_4}\cdots\frac{\alpha_{2k-2}}{\beta_{2k-2}}\cdot\frac{1}{\beta_{2k}} \quad (k \geq 2). \tag{3.7}$$

Remark 3.2. From (3.3), (3.7) we find $c_k \sim k^{-(1+1/\ell)}$, hence in fact $\varphi_1, \varphi_2 \in H^s(S)$ for $s < \frac{1}{\ell} + \frac{1}{2}$ and $\varphi_1, \varphi_2 \notin H^\sigma(S)$ for $\sigma > \frac{1}{\ell} + \frac{1}{2}$.

Remark 3.3. Note that

$$\varphi_1(\theta) + i\varphi_2(\theta) = C_\ell \Phi_\ell(e^{2i\theta})$$

where Φ_ℓ is a certain generalized hypergeometric function of type ${}_3F_2$ whose parameters depend on ℓ. Observe that the argument $e^{2i\theta}$ lies on the boundary of the domain of convergence of Φ_ℓ. This gives another possibility to understand how the smoothness of φ_1, φ_2 depends on ℓ.

Remark 3.4. A function u (near $\mathbf{1}$) on S satisfies $\Omega_u = -\Omega_u$ if and only if it has period π. Therefore, the fact that φ_1 and φ_2 have period π is a consequence of the invariance of (1.4) under point reflections which holds of course for the linearization as well. Similarly, the symmetry with respect to the axes in the case $\vec{\nu} \parallel \vec{e}_1$ implies that φ_1 has a pure cosine series.

Remark 3.5. Observe additionally that

$$\hat{L}[\cos\theta] = \hat{L}[\sin\theta] = 0, \quad \hat{L}[\mathbf{1}] = -\frac{2\ell^2}{(\ell+1)^2}\cos\theta. \tag{3.8}$$

This expresses the previously described invariances of the problem with respect to translations and scaling on the level of the linearization. Moreover, together with Lemma 3.1 this implies $\dim\ker\hat{L} = 2$.

Lemma 3.1 ensures the unique solvability of the linearization of (2.1), (2.2) around $(\mathbf{1}, \vec{\nu}_0)$, given by

$$\left.\begin{array}{rcl} \ell(\vec{\nu}-\vec{\nu}_0)\cdot(\cos\theta,\sin\theta) + \hat{L}(u-\mathbf{1}) & = & 0, \\ \langle u-\mathbf{1},\mathbf{1}\rangle_{L^2(S)} = \langle u-\mathbf{1},\sin\rangle_{L^2(S)} = \langle u-\mathbf{1},\cos\rangle_{L^2(S)} & = & 0, \end{array}\right\} \tag{3.9}$$

and provides a representation for the solution which formally resembles the one given in Theorem 2.3. Some examples are given in Figure 1.

4. Linearization near the trivial solution and proof of the main result

Throughout this section, let ℓ and s be as in the assumptions of Theorem 2.3.

It is important to realize that in our problem the main result for the nonlinear problem does not immediately follow from the discussion of the linearization at the trivial solution and then applying the Implicit Function theorem. The reason for this is the nonellipticity of the operator $\hat{L}$ which results in a loss of regularity in the corresponding a priori

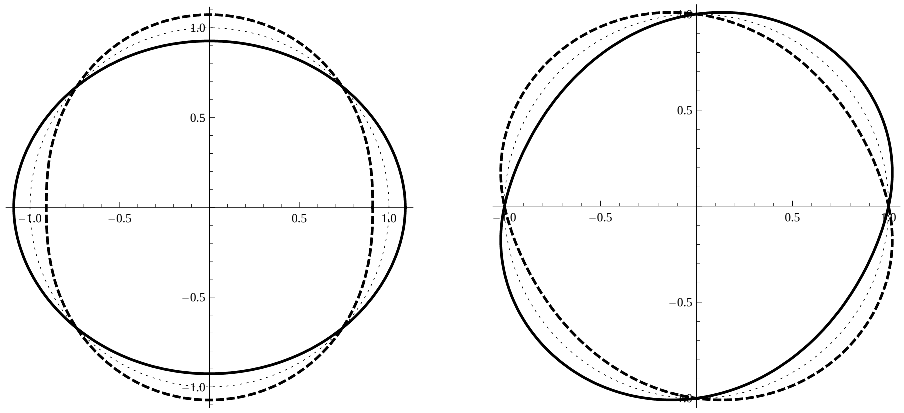

Figure 1: Domains corresponding to solutions of (3.9) for $\ell = 0.25$.
Left: $\vec{\nu} - \vec{\nu}_0 = \pm 0.1\,\vec{e}_1$; Right: $\vec{\nu} - \vec{\nu}_0 = \pm 0.1\,\vec{e}_2$.

The domains corresponding to the $+$ and $-$ sign are drawn with solid and broken lines, respectively. The dotted circle line in both pictures represents the trivial solution.

estimates. To explain this, we remark that up to terms of differentiation order zero we have

$$\hat{L} \sim \hat{L}_0 := \gamma_0 D, \qquad D := \frac{d}{d\theta}, \quad \gamma_0(\theta) := \frac{2\ell}{\ell+1}\sin\theta. \tag{4.1}$$

The degeneracy of $\hat{L}$ results from the fact that γ_0 changes sign on S, and so its kernel contains functions that are nonsmooth at the zeros of γ_0. Put more abstractly, the domain of the $H^{\sigma-1}(S)$-realization of both $\hat{L}$ and $\hat{L}_0$ are larger than $H^\sigma(S)$ if ℓ is sufficiently small, as seen from Remark 3.2. However, this realization is Fredholm. Moreover, we mention the fact, visible from Remark 3.2 as well, that for fixed ℓ the index of $\hat{L}$ as an operator in the scale $H^\sigma(S)$ depends on σ. For more general results on pseudodifferential operators on closed curves with degenerate symbols we refer to [3], Ch. 5. (Note that the results given there are not directly applicable here due to the nonsmoothness of the coefficients in our problem.)

Due to the degeneracy of the problem, it will be necessary to consider the linearization of F not only at $(\mathbf{1}, \vec{\nu}_0)$ but also at points nearby. The domains of definition will depend on this point, therefore the necessary estimates cannot be obtained by standard perturbation arguments for Fredholm operators but have to be proved directly.

Moreover, note that (2.1) is a fully nonlinear equation. As a remedy, we consider the differentiated equation

$$DF(\mathbf{1} + v, \vec{\nu}) = 0. \tag{4.2}$$

This is equivalent to (2.1) as $F \equiv \mathit{const}$ implies $F \equiv 0$ because of (2.5). Our further strategy ist to write (4.2) as a quasi-linear second-order equation and then to apply the usual iteration scheme to solve the resulting equations, based on a priori estimates for the linearization. In our case this procedure yields boundedness of the iteration sequence in H^s and contraction in H^{s-1}.

Assume $v \in H^s(S)$, $\vec{\nu} \in \mathbb{R}^2$ such that $\|v\|_s$ and $|\vec{\nu} - \vec{\nu}_0|$ are small. There is a smooth function $G = G(\vec{\nu}, \theta, \xi, \zeta)$ such that

$$\big(\ell\vec{\nu} \cdot n(\mathbf{1}+v) - x_1(\mathbf{1}+v)\big) = G(\vec{\nu}, \theta, v, v').$$

Differentiating with respect to θ we get

$$D\big(\ell\vec{\nu} \cdot n(\mathbf{1}+v) - x_1(\mathbf{1}+v)\big) = G_\theta(\vec{\nu}, \theta, v, v') + G_\xi(\vec{\nu}, \theta, v, v')v' + G_\zeta(\vec{\nu}, \theta, v, v')v''.$$

Writing

$$G_\theta(\vec{\nu}, \theta, \xi, \zeta) = G_\theta(\vec{\nu}, \theta, 0, 0) + \xi G_1(\vec{\nu}, \theta, \xi, \zeta) + \zeta G_2(\vec{\nu}, \theta, \xi, \zeta)$$

with smooth functions $G_i = G_i(\vec{\nu}, \theta, \xi, \zeta)$, $i = 1, 2$, we obtain the representation

$$D\big(\ell\vec{\nu} \cdot n(\mathbf{1}+v) - x_1(\mathbf{1}+v)\big) = \ell\vec{\nu}(-\sin\theta, \cos\theta) + \sin\theta + L_1(\vec{\nu}, v)v,$$

where $L_1(\vec{\nu}, v)w$ is given by

$$L_1(\vec{\nu}, v)w := G_1(\vec{\nu}, v, v')v + \big(G_2(\vec{\nu}, v, v') + G_\xi(\vec{\nu}, v, v')\big)w' + G_\zeta(\vec{\nu}, v, v')w''. \tag{4.3}$$

Further, setting

$$H(v) := DA_\ell(\mathbf{1}+v)\big[x_1(\mathbf{1}+v) - \ell n_1(\mathbf{1}+v)\big] \tag{4.4}$$

we have

$$H(v) = DA_\ell(\mathbf{1})\big[(1-\ell)\cos\theta\big] + L_2(v)v = \frac{\ell-1}{\ell+1}\sin\theta + L_2(v)v$$

where

$$L_2(v)w := \int_0^1 H'(\tau v)\{w\}\, d\tau. \tag{4.5}$$

Thus, defining linear operators

$$L(v, \vec{\nu}) \in \mathscr{L}\big(H^{s+1}(S), H^{s-1}(S)\big), \quad L(v, \vec{\nu})w := L_1(v, \vec{\nu})w + L_2(v)w \tag{4.6}$$

with L_1, L_2 given by (4.3)–(4.5) we obtain

$$\begin{aligned} DF(\mathbf{1}+v, \vec{\nu}) &= \ell\vec{\nu} \cdot (-\sin\theta, \cos\theta) + \frac{2\ell}{\ell+1}\sin\theta + L(v, \vec{\nu})v \\ &= \ell(\vec{\nu} - \vec{\nu}_0) \cdot (-\sin\theta, \cos\theta) + L(v, \vec{\nu})v. \end{aligned}$$

Note that $L(0, \vec{\nu}_0)w = D\hat{L}w$. Further, from the above construction and Lemma 2.1 one checks that we have a representation

$$L(v, \vec{\nu})w = D\hat{L}w + a(v, v', \vec{\nu})w'' + R(v, \vec{\nu})w \tag{4.7}$$

with a function $a = a(v, v', \vec{\nu}) \in H^{s-1}(S)$ and an operator

$$R = R(v, \vec{\nu}) \in \mathscr{L}\big(H^s(S), H^{s-1}(S)\big)$$

such that $\|a\|_{s-1}$ and $\|R\|_{\mathscr{L}(H^s(S), H^{s-1}(S))}$ are small.

As a preparation for our investigation of the operators $L(v, \vec{\nu})$ we start with a key estimate on degenerate first order differential operators on S. For given coefficient functions γ, b we define $L_0 = L_0(\gamma, b)$ by

$$L_0(\gamma, b)u := \gamma Du + bu.$$

At first we will consider operators of this form on the real axis. Note that

$$\langle L_0(\gamma,b)u,u\rangle = \int_{\mathbb{R}} (b-\tfrac{1}{2}\gamma')u^2\,dx, \qquad u\in H^1(\mathbb{R}), \tag{4.8}$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the standard inner product of $L^2(\mathbb{R})$. For positive integer $\sigma\le s-1$ and $u\in H^{\sigma+1}(\mathbb{R})$, $\gamma,b\in H^{s-1}(\mathbb{R})$ we have, moreover,

$$D^\sigma L_0(\gamma,b)u = L_0(\gamma,b+\sigma\gamma')D^\sigma u + Tu \tag{4.9}$$

where the remainder term Tu consists of a sum of terms of the form

$$D^\alpha\gamma D^\beta u, \quad \alpha+\beta=\sigma+1,\ \beta\le\sigma-1,\ \alpha\le\sigma$$

or

$$D^\alpha b D^\beta u, \quad \alpha+\beta=\sigma,\ \beta\le\sigma-1.$$

Estimating

$$\|D^\alpha\gamma D^\beta u\|_0 \le C\begin{cases} \|\gamma\|_3\|u\|_{\sigma-1} & (\beta=\sigma-1),\\ \|\gamma\|_\alpha\|u\|_{\beta+1} & (\beta\le\sigma-2)\end{cases}$$

and analogously for the terms containing b we find

$$\|Tu\|_0\le C\|u\|_{\sigma-1}, \tag{4.10}$$

where the constant C depends only on $\|\gamma\|_{s-1}$, $\|b\|_{s-1}$.

We will consider L_0 now on a finite interval, say $(-1,1)$, and restrict our attention to the situation in which γ is nonincreasing and changes sign near 0 while b is strictly positive. In this situation, the ordinary differential equation $L_0u=f$ can be solved straightforwardly on the two maximal intervals on which γ does not vanish. If this approach is used, the estimates ensuring the regularity of the solution have to be proved by means of a Hardy inequality. We have chosen a different approach via a Galerkin method which is more easily generalizable to higher dimensional problems.

Lemma 4.1. *Assume $\sigma\in[1,s-1]$ integer, $\gamma,b\in H^{s-1}(-1,1)$, $\delta\in(0,\frac{1}{2})$. Assume $\gamma'\le 0$ on $(-1,1)$, $\gamma(-\delta)>0$, $\gamma(\delta)<0$, and*

$$b+(\sigma-\tfrac{1}{2})\gamma'\ge\mu>0 \quad \text{on } (-\delta,\delta) \tag{4.11}$$

for a positive constant μ. Then for all $f\in H^\sigma(-1,1)$ with $\operatorname{supp} f\subset(-\delta,\delta)$ there is precisely one $u\in H^\sigma(-1,1)$ satisfying $L_0(\gamma,b)u=f$ and $\operatorname{supp} u\subset(-\delta,\delta)$. It satisfies an estimate

$$\|u\|_\sigma\le C\|f\|_\sigma$$

with C depending only on $\mu,\delta,\sigma,\|\gamma\|_{s-1}$, and $\|b\|_{s-1}$.

Proof. Extend $\gamma|_{(-\delta,\delta)}$, $b|_{(-\delta,\delta)}$ to $\widetilde{b},\widetilde{\gamma}\in H^{s-1}(\mathbb{R})$ such that

$$\widetilde{\gamma}'\le 0, \quad \widetilde{b}+(\sigma-\tfrac{1}{2})\widetilde{\gamma}'\ge\mu>0 \qquad \text{on } \mathbb{R}$$

and

$$\begin{aligned} \widetilde{b} &= b^0>0 && \text{on } \mathbb{R}\setminus(-2\delta,2\delta),\\ \widetilde{\gamma} &= \gamma^->0 && \text{on } (-\infty,-2\delta),\\ \widetilde{\gamma} &= \gamma^+<0 && \text{on } (2\delta,\infty), \end{aligned}$$

where b^0, $\gamma^\pm$ are suitable constants. Our assumptions and (4.8)–(4.10) imply that for $u \in H^{\sigma+1}(\mathbb{R})$

$$\langle L_0(\widetilde{\gamma}, \widetilde{b})u, u\rangle \geq \mu \|u\|_0^2 \tag{4.12}$$

and

$$\begin{aligned}\langle D^\sigma L_0(\widetilde{\gamma}, \widetilde{b})u, D^\sigma u\rangle &\geq \langle L_0(\widetilde{\gamma}, \widetilde{b} + \sigma\widetilde{\gamma}')D^\sigma u, D^\sigma u\rangle - C\|u\|_{\sigma-1}\|u\|_\sigma \\ &\geq \mu \|D^\sigma u\|_0^2 - C\|u\|_{\sigma-1}\|u\|_\sigma \\ &\geq \frac{\mu}{2}\|u\|_\sigma^2 - \lambda \|u\|_0^2 \end{aligned} \tag{4.13}$$

for some $\lambda > 0$, where a standard interpolation inequality has been used. Define the continuous bilinear form

$$[\cdot\,,\,\cdot] : H^{\sigma-1}(\mathbb{R}) \times H^{\sigma+1}(\mathbb{R}) \longrightarrow \mathbb{R}$$

by

$$[v, w] := -\langle D^{\sigma-1}v, D^{\sigma+1}w\rangle + \lambda\langle v, w\rangle.$$

Observe that $[\cdot\,,\,\cdot]$ is compatible with the norm of $H^\sigma(\mathbb{R})$, in particular, we have

$$|[v, w]| \leq C\|v\|_\sigma \|w\|_\sigma$$

for all $v \in H^\sigma(\mathbb{R})$, $w \in H^{\sigma+1}(\mathbb{R})$. Moreover, $[\cdot\,,\,\cdot]$ is easily seen to be nondegenerate, i.e. $v \in H^{\sigma-1}(\mathbb{R})$ and $[v, w] = 0$ for all $w \in H^{\sigma+1}(\mathbb{R})$ imply $v = 0$. From (4.12) and (4.13) we get

$$[L_0(\widetilde{\gamma}, \widetilde{b})u, u] \geq \tfrac{\mu}{2}\|u\|_\sigma^2, \qquad u \in H^{\sigma+1}(\mathbb{R})$$

By Galerkin approximations and a weak convergence argument as in [7] we find that for any $f \in H^\sigma(\mathbb{R})$ there is a $u \in H^\sigma(\mathbb{R})$ such that $L_0(\widetilde{\gamma}, \widetilde{b})u = f$ on $\mathbb{R}$, hence $L(\gamma, b)u = f$ on $(-\delta, \delta)$. We also get the estimate

$$\|u\|_{H^\sigma(\mathbb{R})} \leq C\|f\|_\sigma. \tag{4.14}$$

Moreover, by solving $L_0(\widetilde{\gamma}, \widetilde{b})u = f$ as an ordinary differential equation with constant coefficients on $\mathbb{R} \setminus (-2\delta, 2\delta)$ we find

$$u(x) = \begin{cases} c_- \exp(-b^0 x/\gamma^-) & \text{for } x \in (-\infty, -2\delta), \\ c_+ \exp(-b^0 x/\gamma^+) & \text{for } x \in (2\delta, \infty). \end{cases}$$

Therefore, $u \in H^\sigma(\mathbb{R})$ implies $c_\pm = 0$. Furthermore, solving the nondegenerate first-order equation $L_0(\widetilde{\gamma}, \widetilde{b})u = f$ on the intervals $(-2\delta, -\delta)$ and $(\delta, 2\delta)$ and using $u(\pm 2\delta) = 0$ yields $u = 0$ on these intervals as well. Therefore $L_0(\gamma, b)u = f$ on $(-1, 1)$, and the estimate of the lemma follows from (4.14).

To see uniqueness, assume $L_0(\gamma, b)u_1 = L_0(\gamma, b)u_2$ for $u_{1,2} \in H^\sigma(-1, 1)$, $\operatorname{supp} u_{1,2} \subset (-\delta, \delta)$. Let $\widetilde{u} := u_1 - u_2$ in $(-1, 1)$ and $\widetilde{u} := 0$ elsewhere. Then $\widetilde{u} \in H^1(\mathbb{R})$, $L_0(\widetilde{b}, \widetilde{\gamma})\widetilde{u} = 0$, and thus $\widetilde{u} = 0$ by (4.8). □

We define now γ_0 as in (4.1) and

$$b_0 = -\frac{2}{\ell + 1}\cos\theta, \quad b_1 = b_0 + \gamma_0' \tag{4.15}$$

and consider operators $L_0(\gamma, b_1)$ on S with $\|\gamma - \gamma_0\|_\sigma$ small.

Recall that the $H^\sigma(S)$-realization of $L_0(\gamma, b_1)$, denoted by $\overline{L}_0(\gamma, b_1)$, is defined as the restriction of this operator from $H^2(S)$ to the domain of definition

$$\operatorname{dom} \overline{L}_0(\gamma, b_1) := \{u \in H^\sigma(S) \mid L_0(\gamma, b_1)u \in H^\sigma\}.$$

We will define realizations of related operators analogously, without explicit mentioning.

Lemma 4.2. *Assume $\sigma \in [1, s-1]$ integer. There are constants $\varepsilon, C > 0$ such that for all $\gamma \in H^\sigma$ with $\|\gamma - \gamma_0\|_\sigma < \varepsilon$ the equation*

$$L_0(\gamma, b_1)u = f \tag{4.16}$$

has a solution $u \in H^\sigma(S)$ for arbitrary $f \in H^\sigma(S)$, and any solution u satisfies

$$\|u\|_\sigma \le C(\|f\|_\sigma + \|u\|_{\sigma-1}). \tag{4.17}$$

Moreover, the $H^\sigma(S)$-realization of $L_0(\gamma, b_1)$ is a Fredholm operator of index 2.

Proof. Choosing δ sufficiently small and defining

$$S_\delta := S \setminus \big((-\delta, \delta) \cup (\pi - \delta, \pi + \delta)\big)$$

we choose ε small enough to ensure that $|\gamma|$ is uniformly positive on S_δ. Consequently, we can solve $L(\gamma, b_1)v = f$ on S_δ as a nondegenerate linear first order ODE and find a solution $v \in H^{\sigma+1}(S_\delta)$ satisfying an estimate

$$\|v\|_{\sigma+1} \le C\big(\|f\|_\sigma + |v(\pi/2)| + |v(3\pi/2)|\big)$$

with C depending on ε only.

Let $\rho \in C^\infty(S)$ be a cutoff function with $\rho \equiv 1$ in $S_{2\delta}$, $\rho \equiv 0$ in $(-\delta, \delta) \cup (\pi - \delta, \pi + \delta)$. Setting $u = \rho v + w$ we find

$$L_0(\gamma, b_1)w = f - L_0(\gamma, b_1)[\rho v] =: \widetilde{f}, \tag{4.18}$$

where $\operatorname{supp} \widetilde{f} \subset (-2\delta, 2\delta) \cup (\pi - 2\delta, \pi + 2\delta)$. As we have ε small and $\ell(\sigma - \frac{1}{2}) < 1$ it is easy to check that on $(-2\delta, 2\delta)$ the operator $-L_0(\gamma, b_1)$ and on $(\pi - 2\delta, \pi + 2\delta)$ the operator $L_0(\gamma, b_1)$ satisfies the assumptions of Lemma 4.1 with δ replaced by 2δ. This lemma ensures the solvability of (4.18) with $w \in H^\sigma(S)$, $\operatorname{supp} w \subset (-2\delta, 2\delta) \cup (\pi - 2\delta, \pi + 2\delta)$ and an estimate

$$\|w\|_\sigma \le C\|\widetilde{f}\|_\sigma \le C\big(\|f\|_\sigma + \|v\|_{\sigma+1}\big).$$

Therefore

$$\|u\|_\sigma \le C\big(\|f\|_\sigma + |v(\pi/2)| + |v(3\pi/2)|\big) = C\big(\|f\|_\sigma + |u(\pi/2)| + |u(3\pi/2)|\big)$$

which implies (4.17). This estimate, together with the surjectivity, implies that the $H^\sigma(S)$-realization of $L_0(\gamma, b_1)$ is Fredholm.

If $f = 0$, the above construction provides a two-dimensional solution space, parameterized by $u(\pi/2)$ and $u(3\pi/2)$. On the other hand, if we assume $f = 0$, $u(\pi/2) = u(3\pi/2) = 0$ then from solving $L(\gamma, b_1)u = 0$ on S_δ we get $\operatorname{supp} u \subset (-2\delta, 2\delta) \cup (\pi - 2\delta, \pi + 2\delta)$, and thus $u = 0$ from the uniqueness result of Lemma 4.1. This implies $\operatorname{ind} \overline{L}_0(\gamma, b_1) = 2$. □

Returning to (3.1) we find

$$\hat{L}w = L_0(\gamma_0, b_0)w + L_3(\gamma_0 Dw) + L_4 w,$$

where $L_3, L_4 \in \mathscr{L}\big(H^\sigma(S), H^{\sigma+1}(S)\big)$ are given by

$$L_3 w := -\ell A_\ell(\mathbf{1})w, \quad L_4 w := \frac{2(1+\ell-\ell^2)}{\ell+1} A_\ell(\mathbf{1})[\cos\theta w].$$

Using (4.9) and the fact that D and L_3 commute we find

$$D\hat{L}w = (I+L_3)L_0(\gamma_0, b_1)Dw + Kw,$$

where $K \in \mathscr{L}\big(H^\sigma(S), H^\sigma(S)\big)$ is given by

$$Kw := b_0' w - DL_3(b_0 w) + DL_4 w.$$

Defining

$$\Lambda : H^{s-1}(S) \times \mathscr{L}\big(H^s(S), H^{s-1}(S)\big) \longrightarrow \mathscr{L}\big(H^{s+1}(S), H^{s-1}(S)\big)$$

by

$$\Lambda(a, B) := (I+L_3)L_0(\gamma_0 + a, b_1)D + B + K \tag{4.19}$$

we get from (4.7)

$$L(v, \vec{\nu}) = \Lambda(a, R - L_3 M(a) D^2),$$

where $M(a)$, given by $M(a)w := aw$, denotes the pointwise multiplication operator.

The next lemma provides an a priori estimate for operators of the type $\Lambda(a, B)$, provided a and B are small.

Lemma 4.3. *Let $t \in [2, s]$ be integer. There are constants $\delta > 0$, $C > 0$ such that for any $a \in H^{s-1}(S)$, $B \in \mathscr{L}\left(H^t(S), H^{t-1}(S)\right)$ with*

$$\|a\|_{s-1}, \ \|B\|_{\mathscr{L}(H^t(S), H^{t-1}(S))} < \delta \tag{4.20}$$

we have, for any $u \in H^{t+1}(S)$,

$$\|u\|_t \le C\left(\|\Lambda(a,B)u\|_{t-1} + \|u\|_{t-1}\right). \tag{4.21}$$

Proof. The facts that L_3 is compact on $H^{t-1}(S)$ and $H^{t-1}(S) \hookrightarrow\hookrightarrow H^{t-2}(S)$ imply

$$\|w\|_{t-1} \le C\big(\|(I+L_3)w\|_{t-1} + \|w\|_{t-2}\big), \qquad w \in H^{t-1}(S).$$

Thus, for sufficiently small δ we get from (4.17) and (4.20)

$$\begin{aligned} \|u\|_t &\le C\left(\|Du\|_{t-1} + \|u\|_{t-1}\right) & (4.22)\\ &\le C\left(\|L_0(\gamma_0 + a, b_1)Du\|_{t-1} + \|u\|_{t-1}\right) & (4.23)\\ &\le C\left(\|(I+L_3)L_0(\gamma_0 + a, b_1)Du\|_{t-1} + \|u\|_{t-1}\right) & (4.24)\\ &\le C\left(\|\Lambda(a,B)u\|_{t-1} + \|u\|_{t-1} + \delta\|u\|_t\right), & (4.25) \end{aligned}$$

and (4.21) follows for δ sufficiently small. $\square$

Assume $\|a\|_{s-1}$ small again and $B = 0$ for the moment. Let us denote by $\overline{L_0}(\gamma_0 + a, b_0)$ the $H^t(S)$-realization of $L_0(\gamma_0 + a, b_0)$ and by $\operatorname{dom} L_0(\gamma_0 + a, b_0)$ its domain of definition. Similarly, let $\overline{\Lambda}(a, 0)$ denote the $H^{t-1}(S)$-realization of $\Lambda(a, 0)$ and $\operatorname{dom}\Lambda(a, 0)$ its domain. As usual, we equip $\operatorname{dom} L_0(\gamma_0 + a, b_0)$ and $\operatorname{dom}\Lambda(a, 0)$ with the corresponding graph norms. In analogy to Lemma 4.2, we have that $\overline{L}_0(\gamma_0 + a, b_0)$ is Fredholm and

$$\operatorname{ind} \overline{L}_0(\gamma_0 + a, b_0) = 2.$$

Furthermore, recalling (4.19) we have $\operatorname{dom}\Lambda(a,0)=\operatorname{dom}L_0(\gamma_0+a,b_0)$ and

$$\overline{\Lambda}(a,0)=D(I+L_3)\overline{L}_0(\gamma_0+a,b_0)-(I+L_3)M(b_0')+K, \tag{4.26}$$

where $I+L_3\in\mathscr{L}\big(H^t(S),H^t(S)\big)\cap\mathscr{L}\big(H^{t-1}(S),H^{t-1}(S)\big)$ and $D\in\mathscr{L}\big(H^t(S),H^{t-1}(S)\big)$ are Fredholm operators of index 0, so the first term on the right in (4.26) is a Fredholm operator of index 2. The last two terms represent bounded operators on $H^t(S)$, and due to

$$\operatorname{dom}L_0(\gamma_0+a,b_0)\hookrightarrow H^t(S)\hookrightarrow\hookrightarrow H^{t-1}(S)$$

these operators are compact from $\operatorname{dom}\Lambda(a,0)$ to $H^{t-1}(S)$. Therefore $\overline{\Lambda}(a,0)$ is Fredholm and

$$\operatorname{ind}\overline{\Lambda}(a,0)=2.$$

To deal with the remaining degrees of freedom we introduce extended operators. For $t\in[2,s]$ integer, let

$$\begin{aligned}P^t&:=H^{s-1}(S)\times\mathscr{L}\big(H^t(S),H^{t-1}(S)\big)\times\big(L^2(S)\big)^3,\\ X^{t-1}&:=H^{t-1}(S)\times\mathbb{R}^3\end{aligned}$$

and define

$$\widetilde{L}:P^t\longrightarrow\mathscr{L}\big(H^{t+1}(S)\times\mathbb{R},X^{t-1}\big)$$

by

$$\widetilde{L}(a,B,z)(u,c):=\big(\Lambda(a,B)u+c,\langle 1+z_1,u\rangle_{L^2},\langle\cos+z_2,u\rangle_{L^2},\langle\sin+z_3,u\rangle_{L^2}\big).$$

From Lemma 4.3 one straightforwardly gets a corresponding a priori estimate

$$\|u\|_t\le C\big(\|\widetilde{L}(p)(u,c)\|_{X^{t-1}}+\|u\|_{t-1}+|c|\big) \tag{4.27}$$

with C independent of u and p, provided $p=(a,B,z)$ small in P^t.

Let $B=0$ first and write

$$\widetilde{L}(a,0,z)=\widetilde{L}_0(a)+\widetilde{K}(z),$$

where

$$\widetilde{L}_0(a)(u,c)=(\Lambda(a,0)u,0)$$

and observe that $\widetilde{K}(z)$ has finite rank. Clearly, the X^{t-1}-realization of $\widetilde{L}_0(a)$ is a Fredholm operator of index 0, and therefore also (with obvious notation)

$$\operatorname{ind}\overline{\widetilde{L}}(a,0,z)=0. \tag{4.28}$$

In particular, from Lemma 3.1 we additionally get that $\widetilde{L}(0)$ is injective, and (4.27) may therefore be sharpened to

$$\|u\|_t+|c|\le C\|\widetilde{L}(0)(u,c)\|_{X^{t-1}}. \tag{4.29}$$

The next lemma generalizes this estimate via a perturbation argument to all sufficiently small p.

Lemma 4.4. *Let $t\in[2,s]$ be integer. There are constants $\delta,C^*>0$ such that for all*

$p \in P^t$ *with* $\|p\|_{P^t} < \delta$ *and all* $(u,c) \in H^{t+1}(S) \times \mathbb{R}$ *one has*

$$\|u\|_t + |c| \leq C^* \|\widetilde{L}(p)(u,c)\|_{X^{t-1}}. \tag{4.30}$$

Proof. Assume the opposite. Then there would exist sequences (u_n, c_n) in $H^{t+1}(S) \times \mathbb{R}$, (p_n) in P^t such that

$$\|u_n\|_t + |c_n| = 1, \quad p_n \to 0 \text{ in } P^t,$$

and

$$\widetilde{L}(p_n)(u_n, c_n) \to 0 \text{ in } X^{t-1}. \tag{4.31}$$

Then (4.27) implies

$$\|u_n\|_{t-1} + |c_n| \geq c > 0 \tag{4.32}$$

for n sufficiently large. On the other hand, after restriction to a subsequence if necessary,

$$(u_n, c_n) \rightharpoonup (u^*, c^*) \text{ weakly in } H^t(S) \times \mathbb{R}, \tag{4.33}$$

$$(u_n, c_n) \to (u^*, c^*) \text{ in } H^{t-1}(S) \times \mathbb{R}. \tag{4.34}$$

Now (4.33) implies

$$\widetilde{L}(p_n)(u_n, c_n) \rightharpoonup \widetilde{L}(0)(u^*, c^*) \text{ weakly in } X^{t-2},$$

therefore $(u^*, c^*) = 0$ by (4.31) and the injectivity of $\widetilde{L}(0)$. In contradiction to this, (4.34) and (4.32) imply $(u^*, c^*) \neq 0$. □

While (4.30) ensures injectivity for the operators $\widetilde{L}(p)$, we get surjectivity from (4.28) only if $B = 0$. To prove surjectivity of $\widetilde{\widetilde{L}}(p)$ in the general case, note that

$$\widetilde{L}(p)(u,c) = f, \quad p = (a, B, z) \text{ small}, \ f \in X^{t-1},$$

is solvable by the iteration procedure

$$\widetilde{L}(a, 0, z)(u_n, c_n) = f - (Bu_{n-1}, 0),$$

for which convergence in $H^t \times \mathbb{R}$ is ensured by (4.30) for $\|B\| < 1/(2C^*)$ by a straightforward contraction argument.

Returning to the original notation (4.3)–(4.6) we summarize our results as follows:

Lemma 4.5. *There are constants* $C, \delta > 0$ *such that for all* $v \in H^s(S)$, $\vec{\nu} \in \mathbb{R}^2$, $w \in \left(L^2(S)\right)^3$ *satisfying*

$$\|v - \mathbf{1}\|_s, \ |\vec{\nu} - \vec{\nu}_0|, \ \|w_1 - \mathbf{1}\|_0, \ \|w_2 - \cos\|_0, \ \|w_3 - \sin\|_0 < \delta$$

and all $f \in H^{s-1}$, $g \in \mathbb{R}^3$ *the problem*

$$L(v, \vec{\nu})u = f + c, \tag{4.35}$$

$$\langle w_i, u \rangle_{L^2} = g_i, \quad i = 1, 2, 3 \tag{4.36}$$

has a unique solution $(u, c) \in H^s \times \mathbb{R}$ *satisfying*

$$\|u\|_t + |c| \leq C\big(\|f\|_{t-1} + |g|\big), \tag{4.37}$$

$2 \leq t \leq s$.

Moreover, we have

$$\|L(v,\vec{\nu})w - L(\tilde{v},\vec{\mu})w\|_{s-2} \le C\big(\|v-\tilde{v}\|_{s-1} + |\vec{\nu}-\vec{\mu}|\big)\|w\|_s \tag{4.38}$$

for $v,\tilde{v} \in V_\delta := \{v \in H^s(S) \mid \|v\|_s < \delta\}$, $|\vec{\nu}-\vec{\nu}_0|, |\vec{\mu}-\vec{\nu}_0|$ small, $w \in H^s(S)$.

To see this, one shows the corresponding estimates for L_1 and L_2 separately. The estimate for L_1 follows from (4.3) and standard results concerning superposition operators induced by smooth functions on spaces that are Banach algebras with respect to pointwise multiplication. To find the estimate for L_2 we recall (4.4) and (4.5) to find

$$L_2(v)w - L_2(\tilde{v})w = \int_0^1 \int_0^1 H''(\tau(\tilde{v} + t(v-\tilde{v})))\{v-\tilde{v}, w\}\, dt d\tau,$$

with

$$\begin{aligned} &H''(v)\{w_1,w_2\} \\ &\quad = DA_\ell''(\mathbf{1}+v)\{w_1,w_2\}[(\mathbf{1}+v)(\cos\theta - \ell n_1(v))] \\ &\qquad + DA_\ell'(\mathbf{1}+v)\{w_1\}[w_2(\cos\theta - \ell n_1(v))] - DA_\ell'(\mathbf{1}+v)\{w_1\}[\ell(\mathbf{1}+v)(n_1'(v)\{w_2\}] \\ &\qquad - DA_\ell(\mathbf{1}+v)[\ell w_1 n_1'(v)\{w_2\}] - DA_\ell(\mathbf{1}+v)[\ell(\mathbf{1}+v)n_1''(v)\{w_1,w_2\}] + \ldots, \end{aligned}$$

where the dots stand for three more summands obtained by interchanging w_1 and w_2 in the nonsymmetric terms. For all these terms, the estimates follow straightforwardly from Lemma 2.1 with $t = s-2$. Thus, (4.38) is proved.

Now we can complete the proof of our main result.

Proof of Theorem 2.3. As already mentioned above, we replace the equations (2.1), (2.2) by

$$\begin{aligned} L(v,\vec{\nu})v &= f, \\ \big\langle w_i(v), v\big\rangle_{L^2} &= 0, \quad i=1,2,3. \end{aligned}$$

with $v := u - \mathbf{1}$, $f := \ell(\vec{\nu}_0 - \vec{\nu})\cdot(-\sin\theta,\cos\theta)$ and

$$w_1(v) := \mathbf{1} + v/2, \quad w_2(v) := (1+v+v^2/3)\cos, \quad w_3(v) := (1+v+v^2/3)\sin.$$

Starting with $v_0 \equiv 0$ in view of Lemma 4.5 we can define iteratively functions $v_1, v_2, \ldots$ and real numbers $c_1, c_2, \ldots$ such that

$$\begin{aligned} L(v_n,\vec{\nu})v_{n+1} &= f + c_{n+1}, \\ \big\langle w_i(v_n), v_{n+1}\big\rangle_{L^2} &= 0, \quad i=1,2,3. \end{aligned}$$

Assuming $\delta > 0$ sufficiently small, we obtain from (4.37)

$$\|v_{n+1}\|_s \le C|\vec{\nu}-\vec{\nu}_0| \le \delta,$$

provided $C\varepsilon \le \delta$, hence

$$v_n \in V_\delta \;\text{ for }\; n = 1,2,\ldots. \tag{4.39}$$

Further, we have

$$\begin{aligned} L(v_n,\vec{\nu})(v_{n+1}-v_n) &= \big(L(v_{n-1},\vec{\nu}) - L(v_n,\vec{\nu})\big)v_n + c_{n+1} - c_n, \\ \big\langle w_i(v_n), v_{n+1}-v_n\big\rangle_{L^2} &= \big\langle w_i(v_{n-1}) - w_i(v_n), v_n\big\rangle_{L^2}, \quad i=1,2,3. \end{aligned}$$

Consequently we obtain from (4.37) with $t = s-1$ and (4.38)

$$\begin{aligned}
|c_{n+1} - c_n| &+ \|v_{n+1} - v_n\|_{s-1} \\
&\le C\big(\|(L(v_{n-1}, \vec{\nu}) - L(v_n, \vec{\nu}))v_n\|_{s-2} + \|w_i(v_{n-1}) - w_i(v_n)\|_0 \|v_n\|_0\big) \\
&\le C\|v_n - v_{n-1}\|_{s-1}\|v_n\|_s \\
&\le C\|v_n - v_{n-1}\|_{s-1}|\vec{\nu} - \vec{\nu}_0|,
\end{aligned}$$

hence for ε sufficiently small

$$|c_{n+1} - c_n| + \|v_{n+1} - v_n\|_{s-1} \le \tfrac{1}{2}\|v_n - v_{n-1}\|_{s-1}, \quad n = 1, 2, \dots.$$

Consequently, after restriction to a suitable subsequence, there exist $c \in \mathbb{R}$ and $v \in V_\delta$ with

$$c_n \to c, \quad v_n \rightharpoonup v \text{ in } H^s(S), \quad v_n \to v \text{ in } H^{s'}(S) \text{ for } s' < s$$

for $n \to \infty$, where $\rightharpoonup$ denotes weak convergence again. This implies

$$L(v_n, \vec{\nu})v_n \rightharpoonup L(v, \vec{\nu})v \text{ in } H^{s-2}(S), \quad w_i(v_n) \to w_i(v) \text{ in } L^2(S)$$

for $n \to \infty$, hence

$$\begin{aligned}
L(v, \vec{\nu})v &= f + c, \\
\big\langle w_i(v), v \big\rangle_{L^2} &= 0, \quad i = 1, 2, 3.
\end{aligned}$$

As $L(v, \vec{\nu})v$ and f have mean value zero, we conclude $c = 0$. This shows the existence statement of Theorem 2.3.

To prove uniqueness, assume

$$\begin{aligned}
L(z, \vec{\nu})z &= L(v, \vec{\nu})v, \\
\langle w_i(z), z\rangle_{L^2} &= \langle w_i(v), v\rangle_{L^2}, \quad i = 1, 2, 3,
\end{aligned}$$

for some $z \in V_\delta$. Then

$$\begin{aligned}
L(z, \vec{\nu})(v - z) &= \big(L(v, \vec{\nu}) - L(z, \vec{\nu})\big)v, \\
\langle w_i(z), v - z\rangle_{L^2} &= \langle w_i(z) w_i(v), v\rangle_{L^2},
\end{aligned}$$

and by (4.38) and (4.37) we find

$$\|v - z\|_{s-1} \le C\delta\|v - z\|_{s-1}$$

implying $v = z$ for δ small. The statements on the symmetry of Ω_u follow immediately from the uniqueness result and the invariance properties discussed in Section 1.

Finally, to get (2.7), let $\hat{v} := \ell(\vec{\nu}_0 - \vec{\nu}) \cdot (\phi_1, \phi_2)$. Then, by Lemma 3.1,

$$L(0, \vec{\nu}_0)\hat{v} = f, \quad \langle w_i(0), \hat{v}\rangle_{L^2} = 0, \; i = 1, 2, 3,$$

and therefore $\hat{v} \in V_\delta$ and

$$\begin{aligned}
L(v, \vec{\nu})(v - \hat{v}) &= \big(L(0, \vec{\nu}_0) - L(v, \vec{\nu})\big)v, \\
\langle w_i(v), v - \hat{v}\rangle_{L^2} &= \langle w_i(0) - w_i(v), \hat{v}\rangle_{L^2}.
\end{aligned}$$

Applying again (4.38) and (4.37) together with some straightforward estimates concerning the auxiliary conditions we get from this

$$\|v - \hat{v}\|_{s-1} \leq C\big(\|v\|_{s-1} + |\vec{\nu} - \vec{\nu}_0|\big)\|\hat{v}\|_s \leq C|\vec{\nu} - \vec{\nu}_0|^2.$$

□

5. Conclusions

Let us compare the (in)stability results found here to the results in [2], where stability properties of the linearization are discussed in the case $\ell = 1$ by complex analysis methods. The results given there amount to decay of solutions to the linearized problem in the L^∞-norm, but not in stronger ones. In such a situation, linearized stability of the time-dependent problem does not necessarily imply nonlinear stability, as is seen from our results, at least as long as no further assumptions on e.g. higher smoothness of initial data are made. Moreover, it is by no means clear whether a similar behavior of the linearization persists for $\ell < 1$, and the delicate stability properties in the near circular geometry should also be interpreted as a cautionary sign concerning the applicability of the simplified models to real streamer phenomena.

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