On the irreducibility of locally analytic principal series representations

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Abstract. Let $\mathbf{G}$ be a p-adic connected reductive group with Lie algebra $\mathfrak{g}$. For a parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ and a finite-dimensional locally analytic representation V of $\mathbf{P}$, we study the induced locally analytic $\mathbf{G}$-representation $W = \mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(V)$. Our result is the following criterion concerning the topological irreducibility of W. If the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V'$ associated to the dual representation V' is irreducible then W is topologically irreducible as well.

Contents

1. Introduction 1
2. Distribution algebras and locally analytic representations 3
2.1. Distribution algebras 3
2.2. Norms and completions of distribution algebras 4
2.3. The closure of the enveloping algebra 9
2.4. Locally analytic representations 10
3. Representations induced from a parabolic subgroup 11
3.1. The setting and statement of the main result 11
3.2. The structure of the proof 12
3.3. Parahoric subgroups and their distribution algebras 15
3.4. Modules for the completed distribution algebras 21
3.5. The main result 27
4. Examples 29
4.1. Irreducibility of Verma modules 29
4.2. Restriction of Scalars 30
References 31

1. Introduction

One of the principal methods for constructing representations of reductive groups is to induce representations of parabolic subgroups which come by inflation from representations of Levi factors. This applies for example to the theory of algebraic representations as well as to the theory of smooth representations of p-adic reductive groups. In this paper we consider parabolically induced representations in the theory of locally analytic representations of p-adic reductive groups. A systematic framework to study locally analytic representations of p-adic groups was developed in the recent years mainly by P. Schneider and J. Teitelbaum, cf. [ST1], [ST2].

Algebraic representations and smooth representations, as well as tensor products of these, provide first examples of such locally analytic representations, but there are much more. For instance, the representations which are locally analytically induced from representations of parabolic subgroups. Locally analytic principal series representations for $\mathrm{SL}_2(L)$, L a finite extension of $\mathbb{Q}_p$, were already defined and studied by Y. Morita in [Mo]. They were later reconsidered for the group GL_2 in [ST1] (for $L = \mathbb{Q}_p$) and in [KS] (for arbitrary L).

In this paper we prove a general criterion for the irreducibility of parabolically induced locally analytic representations. In his thesis H. Frommer [Fr] studied locally analytic representations of $\mathbf{G} = \mathfrak{G}(\mathbb{Q}_p)$, where $\mathfrak{G}$ is a split reductive group over $\mathbb{Q}_p$, which are induced from finite-dimensional representations V of a parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. Here V is defined over some field extension K of $\mathbb{Q}_p$. His main theorem is a criterion for the (topological) irreducibility of the induced representation $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(V)$ in terms of a canonically associated Verma module. The crucial idea is to compare the structure of the dual space $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(V)'$ as a module over the distribution algebra $D(G, K)$, where $G \subset \mathbf{G}$ is a maximal compact subgroup, with the structure of an associated Verma module over the universal enveloping algebra $U(\mathfrak{g})$. In order to pass from $D(G, K)$ to $U(\mathfrak{g})$ one needs a technical result about their relation and Frommer only showed this for $L = \mathbb{Q}_p$. This is the reason for the restriction to the base field $\mathbb{Q}_p$. Later J. Kohlhaase [K] proved this technical result for arbitrary finite extensions L over $\mathbb{Q}_p$, which we use here to generalize Frommer's theorem to the case of a not necessarily split reductive group over an arbitrary extension of $\mathbb{Q}_p$.

In order to state our main result let $\mathfrak{G}$ be a connected reductive group over L, and let $\mathfrak{P} \subset \mathfrak{G}$ be a parabolic subgroup. We consider a locally analytic representation V of the group $\mathbf{P} = \mathfrak{P}(L)$ which comes by inflation from a Levi subgroup and put $\mathbf{G} = \mathfrak{G}(L)$. Here V is a vector space over a discretely valued extension K which contains L and whose absolute value induces the given absolute value on L. Then we have:

Main result: *Suppose* $\dim_K(V) < \infty$. *Then the induced locally analytic representation* $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(V)$ *is topologically irreducible if* $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V'$ *is irreducible as a module over the universal enveloping algebra* $U(\mathfrak{g})$.

Our overall strategy of proof follows basically Frommer's treatment. However, we found that an essential argument in [Fr], stating that certain distribution algebras are integral domains, is not obvious, as it is claimed there. We reprove this in section 3.3.

But beside the fact that it is desirable to have such a criterion for the irreducibility in general, a motivation was provided by the concrete example of certain $\mathbb{Q}_p$-analytic principal series representations of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ (regarded as a group over $\mathbb{Q}_p$). The interest in these representations comes from conjectural relations to two-dimensional

crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_{p^2})$. Such induced locally analytic representations play an important role in the the p-adic Langlands program, cf. [BS].

In the last section we consider a particular example, namely the case where $\mathfrak{G}$ comes by restriction of scalars from a group which is split over L.

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Notation. We let p be a prime number and denote by L a finite extension of $\mathbb{Q}_p$. The normalized p-adic (logarithmic) valuation is denoted by v_p (i.e. $v_p(p) = 1$). We let K be a complete discretely valued field extension of L, whose absolute value restricts to the absolute value of L. The rings of integers are denoted by $\mathfrak{o}_L$ and $\mathfrak{o}_K$, respectively. For notions and notation in the context of non-archimedean functional analysis we refer to [S].

2. Distribution algebras and locally analytic representations

2.1. **Distribution algebras.** In this section we recall some definitions and results about algebras of distributions attached to locally analytic groups, cf. [ST1], [ST2]. We consider a locally L-analytic group H and denote by

$$C^{an}(H,K) = C^{an}_L(H,K)$$

the locally convex K-vector space of locally L-analytic functions on H as defined in [ST1]. The strong dual

$$D(H,K) = D_L(H,K) := (C^{an}_L(H,K))'_b$$

is the topological algebra of K-valued distributions on H. If furthermore H is compact, then $D(H,K)$ has the structure of a Fréchet algebra. The multiplication $\delta_1 * \delta_2$ of distributions $\delta_1, \delta_2 \in D(H,K)$ is defined by

$$\delta_1 * \delta_2(f) = (\delta_1 \hat{\otimes} \delta_2)((h_1,h_2) \mapsto f(h_1h_2))\,,$$

where the distribution $\delta_1 \hat{\otimes} \delta_2 \in D(H \times H, K)$ has the property that for functions $f_1, f_2 \in C^{an}(H,K)$, one has

$$(\delta_1 \hat{\otimes} \delta_2)((h_1,h_2) \mapsto f_1(h_1)f_2(h_2)) = \delta_1(f_1)\delta_2(f_2)\,.$$

The universal enveloping algebra $U(\mathfrak{h})$ of the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ of H acts naturally on $C^{an}(H,K)$. On elements $\mathfrak{x} \in \mathfrak{h}$, this action is given by

$$(\mathfrak{x}f)(h) = \frac{d}{dt}(t \mapsto f(\exp(-t\mathfrak{x})h))|_{t=0}\,.$$

This gives rise to an embedding of $U(\mathfrak{h})_K := U(\mathfrak{h}) \otimes_L K$ into $D(H,K)$:

$$U(\mathfrak{h})_K \hookrightarrow D(H,K)\,,\ \mathfrak{z} \mapsto (f \mapsto (\dot{\mathfrak{z}}f)(1))\,.$$

Here $\mathfrak{z} \mapsto \dot{\mathfrak{z}}$ is the unique K-linear anti-automorphism of $U(\mathfrak{h})_K$ which induces multiplication by -1 on $\mathfrak{h}$.

2.2. **Norms and completions of distribution algebras.**

2.2.1. *p-valuations and global charts.* Let H be a compact locally $\mathbb{Q}_p$-analytic group. Recall that a map

$$\omega : H - \{1\} \longrightarrow (\frac{1}{p-1}, \infty) \subset \mathbb{R}$$

is called a *p-valuation* (cf. [L], III.2.1.2) if the following conditions hold for all $g, h \in H$:

i) $\omega(gh^{-1}) \geq \min\{\omega(g), \omega(h)\}$,

ii) $\omega(g^{-1}h^{-1}gh) \geq \omega(g) + \omega(h)$,

iii) $\omega(g^p) = \omega(g) + 1$.

As usual one puts $\omega(1) = \infty$ and interprets the above inequalities in the obvious sense, if a term $\omega(1)$ occurs. Let ω be a p-valuation on H. The above conditions imply that for any $\nu > 0$ the sets

$$H_\nu = \{h \in H \,|\, \omega(h) \geq \nu\} \text{ and } H_{\nu^+} = \{g \in H \,|\, \omega(g) > \nu\}$$

are normal subgroups of H. We put

$$\mathrm{gr}(H) = \bigoplus\nolimits_{\nu>0} H_\nu / H_{\nu^+}\,.$$

The commutator induces a Lie bracket on $\mathrm{gr}(H)$ which gives $\mathrm{gr}(H)$ the structure of a Lie algebra over $\mathbb{F}_p$. The map ϵ defined by

$$\epsilon : \mathrm{gr}(H) \to \mathrm{gr}(H)\,,\ \epsilon(gH_{\nu^+}) = g^p H_{(\nu+1)^+}$$

is an $\mathbb{F}_p$-linear map on $\mathrm{gr}(H)$, which gives $\mathrm{gr}(H)$ the structure of a graded Lie algebra over $\mathbb{F}_p[\epsilon]$, cf. [L], III.2.1.1. It is a free $\mathbb{F}_p[\epsilon]$-module, whose rank is equal to the dimension of H as a $\mathbb{Q}_p$-analytic group, cf. loc. cit., III.3.1.3/7/9.

If $(h_1 H_{\omega(h_1)^+}, \ldots, h_d H_{\omega(h_d)^+})$ is a basis of $\mathrm{gr}(H)$ over $\mathbb{F}_p[\epsilon]$, then the elements $h_1, \ldots, h_d$ form a topological generating system of H, and the following map

$$\mathbb{Z}_p^d \to H\,,\ (a_1, \ldots, a_d) \mapsto h_1^{a_1} \cdot \ldots \cdot h_d^{a_d}$$

is well-defined and a homeomorphism. Moreover,

$$\omega(h_1^{a_1} \cdot \ldots \cdot h_d^{a_d}) = \min\{\omega(h_i) + v_p(a_i) \,|\, i = 1, \ldots, d\}\,.$$

The sequence $(h_1, \ldots, h_d)$ is called a *p-basis* of H.

2.2.2. *Uniform pro-p groups.* We recall some definitions and results about pro-p groups, cf. [DDMS], ch. 3, 4. In this section H will be a pro-p group which is equipped with its topology as a pro-finite group. H is called *powerful* if p is odd (resp. $p = 2$) and $H/\overline{H^p}$ (resp. $H/\overline{H^4}$ if $p = 2$) is abelian. Here, $\overline{H^p}$ (resp. $\overline{H^4}$) is the closure of the subgroup generated by the p-th (resp. fourth) powers of its elements. If H is topologically finitely generated one can show that the subgroups H^p (resp. H^4) are open and hence automatically closed. The *lower p-series* $(P_i(H))_{i\geq 1}$ of an arbitrary pro-p group H is defined inductively by

$$P_1(H) = H\,,\ P_{i+1}(H) = \overline{P_i(H)^p [P_i(H), H]}\,.$$

If H is topologically finitely generated then the groups $P_i(H)$ are all open in H and form a fundamental system of neighborhoods of 1, cf. Prop. 1.16 in loc.cit. A pro-p group H is called *uniform* if it is topologically finitely generated, powerful and its lower p-series satisfies

$$(H : P_2(H)) = (P_i(H) : P_{i+1}(H))$$

for all $i \geq 1$. If H is a topologically finitely generated powerful pro-p group then $P_i(H)$ is a uniform pro-p group for all sufficiently large i, cf. loc.cit. 4.2. Moreover, any compact $\mathbb{Q}_p$-analytic group contains an open normal uniform pro-p subgroup, cf. loc.cit. 8.34.

2.2.3. *The canonical p-valuation on uniform groups.* Let H be a uniform pro-p group. It carries a distinguished p-valuation ω^{can} which is associated to the lower p-series and which we call the *canonical p-valuation*. In order to define it, we let $\varepsilon_p = 1$ if $p = 2$ and $\varepsilon_p = 0$ for odd p. For $h \neq 1$, we then put

$$\omega^{can}(h) = \varepsilon_p + \max\{i \geq 1 \,|\, h \in P_i(H)\}\,.$$

To verify that this gives indeed a p-valuation one makes use of the fact that $[P_i(H), P_j(H)] \subset P_{i+j}(H)$ for all $i, j \geq 1$, cf. Prop. 1.16 in [DDMS]. Also property (iii) follows from Prop. 2.7 in loc.cit. For $p = 2$ one has to use the stronger

statement that $[P_i(H), P_j(H)] \subset P_{i+j+1}(H)$ for all $i, j \geq 1$, cf. [Sch], proof of Prop. 2.1.

2.2.4. *Norms induced by p-valuations.* In this section we let H be a compact L-analytic group. The distribution algebra $D(H,K)$ is then a *Fréchet-Stein algebra* in the sense of [ST2], sec. 3. This means in particular that there exists a family of norms $\| \ \|_r$, $\frac{1}{p} < r < 1$, on $D(H,K)$ such that, if $D_r(H,K)$ denotes the completion of $D(H,K)$ with respect to $\| \ \|_r$, the convolution product $*$ on $D(H,K)$ extends by continuity to a product on $D_r(H,K)$. We recall briefly the construction of such a family of norms. This is done in three steps, cf. [ST2] (proof of Thm. 5.1).

Step 1. We denote by $R^L_{\mathbb{Q}_p} H$ the group H when considered as a locally $\mathbb{Q}_p$-analytic group, and we put $d = \dim(R^L_{\mathbb{Q}_p} H)$. Let $H_0 \subset R^L_{\mathbb{Q}_p} H$ be an open normal subgroup which is equipped with a p-valuation ω. For instance, by [DDMS], 8.34, we may choose H_0 to be an open normal uniform pro-p subgroup, and we may take for ω the canonical valuation associated to its lower p-series (cf. 2.2.3). Then we consider the graded group $\mathrm{gr}(H_0)$ (which depends on ω). A choice of a basis of $\mathrm{gr}(H_0)$ gives rise to a homeomorphism $\psi : \mathbb{Z}_p^d \to H_0$, cf. 2.2.1, which in turn induces an isomorphism of locally convex K-vector spaces

$$\psi^* : C^{an}(H_0, K) \xrightarrow{\simeq} C^{an}(\mathbb{Z}_p^d, K)$$

as well as an isomorphism of K-Banach spaces of continuous functions

$$\psi^* : C(H_0, K) \xrightarrow{\simeq} C(\mathbb{Z}_p^d, K)\,.$$

Using Mahler expansions ([L], III.1.2.4) we can express elements of $C(\mathbb{Z}_p^d, K)$ as series

$$f(x) = \sum\nolimits_{n \in \mathbb{N}_0^d} c_n \binom{x}{n}\,,$$

where $c_n \in K$ and

$$\binom{x}{n} := \binom{x_1}{n_1} \cdots \binom{x_d}{n_d}$$

for the multi-indices $x = (x_1, \ldots, x_d)$ and $n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$. Further, we have $|c_n| \to 0$ as $|n| = n_1 + \ldots + n_d \to \infty$. A continuous function $f \in C(\mathbb{Z}_p^d, K)$ is locally analytic if and only if $|c_n| r^{|n|} \to 0$ for some $r > 1$. Consider the group algebra $K[H_0]$ of H_0. By identifying elements of H_0 with Dirac distributions, we get an embedding

$$K[H_0] \hookrightarrow D(H_0, K)\,.$$

Put $b_i := h_i - 1 \in K[H_0]$ and set for $n \in \mathbb{N}_0^d$,

$$\mathbf{b}^n = b_1^{n_1} \cdots b_d^{n_d} .$$

Then we have $\mathbf{b}^n(f) = c_n$ for any continuous function $f \in C(H_0, K)$, where the c_n are the Mahler coefficients of $\psi^*(f) \in C(\mathbb{Z}_p^d, K)$. It follows that every distribution $\lambda \in D(H_0, K)$ has the shape

$$\lambda = \sum\nolimits_{n \in \mathbb{N}_0^d} d_n \mathbf{b}^n$$

where $\{d_n r^{|n|} \,|\, n \in \mathbb{N}_0^d\}$ is a bounded set for all $0 < r < 1$. The norm $\| \cdot \|_r$ on $D(H_0, K)$ is then defined by

$$\| \sum\nolimits_{n \in \mathbb{N}_0^d} d_n \mathbf{b}^n \|_r = \sup\{|d_n| r^{\tau(n)} \,|\, n \in \mathbb{N}_0^d\} .$$

Here, $\tau(n)$ is given by $\tau(n) = \sum_i n_i \omega(h_i)$. The Banach algebra $D_r(H_0, K)$ is defined to be the completion of $D(H_0, K)$ with respect to $\| \cdot \|_r$. Thus we obtain

$$D_r(H_0, K) = \{ \sum\nolimits_{n \in \mathbb{N}_0^d} d_n \mathbf{b}^n \in K[[b_1, \ldots, b_d]] \,|\, \lim_{|n| \to \infty} |d_n| r^{\tau(n)} = 0 \} .$$

Furthermore, for $\frac{1}{p} < r < 1$, the norm $\| \cdot \|_r$ is multiplicative, cf. [ST2] Thm. 4.5 and does not depend on the chosen basis (loc. cit., before Thm. 4.11). We obtain a projective system of noetherian Banach algebras such that

$$D(H_0, K) = \varprojlim_r D_r(H_0, K) .$$

Moreover, the transition maps

$$D_{r'}(H_0, K) \to D_r(H_0, K)$$

are flat for $\frac{1}{p} < r \leq r' < 1$.

Step 2. We extend the norm $\| \cdot \|_r$ on $D(H_0, K)$ to a norm on $D(R^L_{\mathbb{Q}_p} H, K)$ as follows. Let $\eta_1, \ldots, \eta_s$ be a system of coset representatives of H/H_0. Then the Dirac distributions $\delta_{\eta_1}, \ldots, \delta_{\eta_s}$ form a basis of $D(R^L_{\mathbb{Q}_p} H, K)$ over $D(H_0, K)$. Writing $\mu \in D(R^L_{\mathbb{Q}_p} H, K)$ as $\mu = \lambda_1 \delta_{\eta_1} + \ldots + \lambda_s \delta_{\eta_s}$, we put

$$q_r(\mu) = \max\{\|\lambda_i\|_r \,|\, 1 \leq i \leq s\} .$$

Again, we obtain a system of sub-multiplicative norms q_r, $\frac{1}{p} < r < 1$, and a projective system $D_r(R^L_{\mathbb{Q}_p} H, K)$ of K-Banach algebras fulfilling the conditions above, and we have $D(R^L_{\mathbb{Q}_p} H, K) = \varprojlim D_r(R^L_{\mathbb{Q}_p} H, K)$. The norms q_r do not depend on the chosen representatives. It is worth to remark at this point that the norms q_r are

in general not multiplicative, because H may contain non-trivial elements of finite order which causes the group ring, as well as the rings of distributions, to have non-trivial zero divisors[1].

Step 3. Locally L-analytic functions on H are obviously locally $\mathbb{Q}_p$-analytic functions on $R^L_{\mathbb{Q}_p}H$, and hence there is a canonical map

$$C^{an}_L(H,K) \hookrightarrow C^{an}_{\mathbb{Q}_p}(R^L_{\mathbb{Q}_p}H,K)$$

which is a closed embedding (cf. the proof of Thm. 5.1 in [ST2]). This map induces by duality a continuous surjection

$$D_{\mathbb{Q}_p}(R^L_{\mathbb{Q}_p}H,K) \twoheadrightarrow D_L(H,K)$$

on the distribution algebras. We denote the induced residue norm on $D_L(H,K)$ respectively on the completion $D_r(H,K)$ by $\bar{q}_r$. Once again, we have $D_L(H,K) = \varprojlim D_r(H,K)$.

Although we will mostly consider the case where the parameter r lies strictly between $\frac{1}{p}$ and 1, we point out that the norms $\bar{q}_r$ are defined, as norms of K-vector spaces, for any $r \in (0,1)$. This is used later in the proof of 3.4.7, cf. 3.4.6.

Remark 2.2.5. We keep the notation from the preceding paragraph. In the recipe for the norm $\bar{q}_r$ on $D_L(H,K)$ given above, we could have carried out steps two and three in reverse order with the same resulting norm. To be precise, let $\overline{\|\cdot\|}_r$ be the quotient norm on $D_L(H_0,K)$ induced by the surjection $D_{\mathbb{Q}_p}(R^L_{\mathbb{Q}_p}H_0,K) \twoheadrightarrow D_L(H_0,K)$. We have again

$$D_L(H,K) = \bigoplus_{i=1}^{s} D_L(H_0,K) * \delta_{\eta_i}$$

and can hence consider the maximum norm $\hat{q}_r$ on $D_L(H,K)$ induced from this decomposition and the norm $\overline{\|\cdot\|}_r$ on $D_L(H_0,K)$. Then it is not difficult to check that $\hat{q}_r$ coincides with the norm $\bar{q}_r$ defined above, cf. [Sch], Lemma 4.4.

[1] In [Fr] it is mistakenly stated that these norms were multiplicative so that the corresponding completions would be integral domains. As a consequence, some completed distribution algebras were claimed to be obviously integral domains.

2.3. **The closure of the enveloping algebra.**

2.3.1. *Definition of $U_r(\mathfrak{h}, H_0)$.* Let H be a compact locally L-analytic group and let $H_0 \subset H$ be an open normal subgroup, which is equipped with a p-valuation ω. Associated to ω there is a norm $\bar{q}_r$ on $D(H, K)$ as defined in 2.2.4. We write

$$U_r(\mathfrak{h}, H_0) \subset D_r(H_0, K)$$

for the topological closure of $U(\mathfrak{h}) \otimes_L K$ in $D_r(H_0, K)$. The following theorem by Kohlhaase generalizes a result of Frommer who considered the case of $\mathbb{Q}_p$-analytic groups. It is this basic technical result which we use to generalize Frommer's irreducibility criterion.

Theorem 2.3.2. *(*[K]*, 1.3.5, 1.4.2.) Let H be a compact locally L-analytic group of dimension d with Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ and $r \in p^{\mathbb{Q}}$ with $\frac{1}{p} < r < 1$.*

(i) There exists an open L-analytic subgroup H_0 of H, and a $\mathbb{Z}_p$-lattice $\Lambda \subset \mathrm{Lie}(H)$ with the following properties:

1. *there is an L-basis $\mathfrak{X} = (\mathfrak{X}_1, \ldots, \mathfrak{X}_d)$ of $\mathfrak{h}$ and a $\mathbb{Z}_p$-basis $\xi_1, \ldots, \xi_m$ of $\mathfrak{o}_L$ such that $(\xi_1\mathfrak{X}_1, \ldots, \xi_m\mathfrak{X}_d)$ is a $\mathbb{Z}_p$-basis of Λ;*
2. *the corresponding canonical coordinates of the second kind[2] give an isomorphism $\Lambda \to R^L_{\mathbb{Q}_p} H_0$ of locally $\mathbb{Q}_p$-analytic manifolds;*
3. *H_0 is a uniform pro-p group.*

(ii) Let $H_0 \subset H$ be an open subgroup as in (i) which we assume furthermore to be normal. Then, if we equip $R^L_{\mathbb{Q}_p} H_0$ with its canonical valuation (2.2.3), $D_r(H_0, K)$ is a free, finitely generated module over the noetherian subalgebra $U_r(\mathfrak{h}, H_0)$.

(iii) Let the normal subgroup $H_0 \subset H$ be as in (i), and let $\mathfrak{X} = (\mathfrak{X}_1, \ldots, \mathfrak{X}_d)$ be the L-basis of $\mathfrak{h}$ as above. Then there is a norm ν_r on $U_r(\mathfrak{h}, H_0)$ which is equivalent to $\bar{q}_r$, such that $U_r(\mathfrak{h}, H_0)$ consists of exactly those series

$$\sum\nolimits_{n \in \mathbb{N}_0^d} d_n \mathfrak{X}^n$$

for which

$$\lim_{|n| \to \infty} |d_n| \nu_r(\mathfrak{X}^n) = 0\,,$$

and $\nu_r(\sum_n d_n \mathfrak{X}^n) = \sup_n |d_n| \nu_r(\mathfrak{X}^n)$.

[2] cf. [Bou], Ch. III, §4.3.

Corollary 2.3.3. *Let H be a compact locally L-analytic group, and let $H_0 \subset H$ be an open normal subgroup as in 2.3.2 (ii). Suppose that $D_r(H,K)$ is an integral domain. Then any non-zero left ideal I of $D_r(H,K)$ has non-zero intersection with $U_r(\mathfrak{h}, H_0)$.*

Proof. Compare [Fr], Corollary 5. Since H is compact, $D_r(H,K)$ is a finite $D_r(H_0,K)$-module, and so $D_r(H,K)$ is finite over $U_r(\mathfrak{h},H_0)$. Therefore, there exists for any $F \in D_r(H,K)$, a polynomial $P = \sum_{i=0}^{N} a_i X^i$, $a_i \in U_r(\mathfrak{h},H_0)$ with $P(F)=0$. Because we assumed $D_r(H,K)$ to be an integral domain, we may find a polynomial with $a_0 \neq 0$. So, if $F \in I$ then

$$0 \neq a_0 = -\sum_{i=1}^{N} a_i F^i \in I \cap U_r(\mathfrak{h},H_0)\,.$$

□

2.4. **Locally analytic representations.** We conclude this section by recalling some facts of locally analytic representations. Let H be a locally L-analytic group, V a locally convex K-vector space, and $\rho : H \to \mathrm{GL}_K(V)$ a homomorphism. Then ρ (or the pair (V,ρ)) is called a *locally analytic representation* of H if the topological K-vector space V is barrelled, the homomorphism ρ is continuous, and the orbit maps $\rho_v : H \to V$, $h \mapsto \rho(h)(v)$, are locally analytic maps for all $v \in V$, cf. [ST1], sec. 3. If V is of compact type, i.e., a compact inductive limit of Banach spaces, the strong dual V_b' is a nuclear Fréchet space and a separately continuous left $D(H,K)$-module. The module structure is given as follows:

$$D(H,K) \otimes_K V_b' \to V_b'\,,\ \delta \otimes \varphi \mapsto (v \mapsto \delta(g \mapsto \varphi(\rho(g^{-1})v)))\,.$$

This functor gives an equivalence of categories

$$\left\{\begin{array}{c}\text{locally analytic } H\text{-representations on } K\text{-vector spaces of compact type with continuous linear } H\text{-maps}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{separately continuous } D(H,K)\text{-modules on nuclear Fréchet spaces with continuous } D(H,K)\text{-module maps}\end{array}\right\}$$

In particular, V is topologically irreducible if V_b' is a simple $D(H,K)$-module.

For any closed subgroup H' of H and any locally analytic representation V of H', we denote by $\mathrm{Ind}_{H'}^{H}(V)$ the induced locally analytic representation. We recall the definition:

$$\mathrm{Ind}_{H'}^{H}(V) \;:=\; \Big\{ f \in C^{an}(H,V) \,|\, f(h' \cdot h) = h' \cdot f(h)\ \forall h' \in H', \forall h \in H \Big\}.$$

The group H acts on this vector space by $(h \cdot f)(x) = f(xh)$. We have a Frobenius reciprocity in the category of locally analytic representations (see [Fe] Theorem 4.2.6):

$$\mathrm{Hom}_H(W, \mathrm{Ind}_{H'}^H V) \cong \mathrm{Hom}_{H'}(\mathrm{Res}_{H'}^H W, V)\,.$$

Here, $\mathrm{Res}_{H'}^H(W)$ denotes as usual the restriction, viewing W via the embedding $H' \hookrightarrow H$ as a H'-representation. It is easy to check that the following map is an isomorphism of $D(H,K)$-modules:

$$D(H,K)\hat{\otimes}_{D(H',K)}V' \to (\mathrm{Ind}_{H'}^H V)'_b\,,\ \delta \otimes \varphi \mapsto \delta \cdot \varphi\,,$$

with $(\delta \cdot \varphi)(f) = \delta(g \mapsto \varphi(f(g^{-1})))$.

3. Representations induced from a parabolic subgroup

3.1. The setting and statement of the main result.

3.1.1. We consider a connected reductive algebraic group $\mathfrak{G}$ over the finite extension L of $\mathbb{Q}_p$. Let $\mathfrak{S} \subset \mathfrak{G}$ be a maximal split torus over L. Fix a minimal parabolic subgroup $\mathfrak{P}_0$ of $\mathfrak{G}$ which contains $\mathfrak{S}$ and denote by $\mathfrak{U}_0$ its unipotent radical. The choice of $\mathfrak{P}_0$ determines subsets $\Phi^+ \supset \Delta$ of positive and simple roots, respectively, in the root system Φ of $\mathfrak{G}$ with respect to $\mathfrak{S}$. Let $\mathfrak{P}$ be a parabolic subgroup containing $\mathfrak{P}_0$ with unipotent radical $\mathfrak{U}$. Let $\mathfrak{M}$ be the Levi subgroup of $\mathfrak{P}$ containing $\mathfrak{S}$. Let W be the Weyl group of $\mathfrak{G}$ with respect to $\mathfrak{S}$, and let $W_P \subset W$ be the Weyl group of the Levi subgroup $\mathfrak{M}$. Finally, we denote by Φ^{red} the set of reduced roots of Φ.

We denote the corresponding groups of L-valued points by bold letters:

$$\mathbf{G} = \mathfrak{G}(L)\,,\ \mathbf{P} = \mathfrak{P}(L)\,,\ \mathbf{S} = \mathfrak{S}(L)\,,\quad \text{etc.}$$

which we consider as locally L-analytic groups. The Lie algebras will be denoted by gothic letters, i.e.,

$$\mathfrak{g} = \mathrm{Lie}(\mathbf{G})\,,\ \mathfrak{p} = \mathrm{Lie}(\mathbf{P})\,,\ \mathfrak{s} = \mathrm{Lie}(\mathbf{S})\,,\quad \text{etc.}$$

3.1.2. We let

$$\rho : \mathbf{P} \longrightarrow \mathbf{M} \longrightarrow \mathrm{GL}(V)$$

be a representation of $\mathbf{P}$ which comes by inflation from a locally L-analytic representation of $\mathbf{M}$ on a *finite-dimensional* K-vector space V. We are interested in the locally analytic induced representation

$$\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\rho)$$

and our main result gives a criterion for the topological irreducibility of this locally L-analytic representation of $\mathbf{G}$ in terms of the generalized Verma module

$$m(\rho) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \rho' .$$

Here ρ' is the derived representation of $\mathfrak{p}$ on the dual space $V' = \mathrm{Hom}_K(V, K)$. Our main result then is

Theorem 3.1.3. *If $m(\rho)$ is a simple $U(\mathfrak{g})$-module, then $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\rho)$ is a topologically irreducible representation.*

3.2. **The structure of the proof.**

3.2.1. *Reduction to simply connected groups.* The first step is to reduce to the case where $\mathfrak{G}$ is a semi-simple simply connected algebraic group. In fact, let $\tilde{\mathfrak{G}}$ be the simply connected cover of the derived group $\mathfrak{G}_{\mathrm{der}}$ of $\mathfrak{G}$. Denote by $\tilde{\mathfrak{P}} \subset \tilde{\mathfrak{G}}$ the parabolic subgroup which maps onto $\mathfrak{P} \cap \mathfrak{G}_{\mathrm{der}}$. From the identification $\mathfrak{G}/\mathfrak{P} \cong \tilde{\mathfrak{G}}/\tilde{\mathfrak{P}}$ we deduce an identification of representations of $\tilde{\mathbf{G}}$

$$\mathrm{Ind}_{\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}}(\tilde{\rho}) = \mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\rho) .$$

Here $\tilde{\rho}$ is the representation of $\tilde{\mathbf{P}}$ given by the composition of the natural map $\tilde{\mathbf{P}} \to \mathbf{P}$ and ρ. The action of $\tilde{\mathbf{G}}$ on the right comes from the homomorphism $\tilde{\mathbf{G}} \to \mathbf{G}$. It follows that $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\rho)$ is a (topologically) irreducible $\mathbf{G}$-representation if $\mathrm{Ind}_{\tilde{\mathbf{P}}}^{\tilde{\mathbf{G}}}(\tilde{\rho})$ is a (topologically) irreducible $\tilde{\mathbf{G}}$-representation. The same argument applies to the Verma modules. Therefore, we will assume from now on that $\mathfrak{G}$ is semi-simple and simply connected.

3.2.2. *Passage to representations of compact groups.* The second step is to reduce the analysis of the induced representation to a question about representations of compact groups. To this end we fix a special maximal compact subgroup $G \subset \mathbf{G}$. From the Iwasawa decomposition $\mathbf{G} = G \cdot \mathbf{P}$ (cf. [Ca], sec. 3.5), we deduce an isomorphism of G-representations

$$\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\rho)|_G \xrightarrow{\simeq} \mathrm{Ind}_{G \cap \mathbf{P}}^{G}(\rho) .$$

Then we let $P \subset G$ be a parahoric subgroup of the same type as $\mathbf{P}$, and put $P^+ = P \cap \mathbf{P} = G \cap \mathbf{P}$. We note that every element of the Weyl group W of $\mathfrak{G}$ with respect to $\mathfrak{S}$ has a representative in G (cf. [BT1], 4.2.3), and hence we identify W with $(N_{\mathbf{G}}(\mathbf{S}) \cap G)/(Z_{\mathbf{G}}(\mathbf{S}) \cap G)$. The same remark applies to the Weyl group $W_{\mathbf{P}}$. From the Bruhat-Tits decomposition we deduce that

$$G = \coprod_{w \in W_{\mathbf{P}} \backslash W / W_{\mathbf{P}}} P w P^+ .$$

(cf. [Ca] 3.5 in the Iwahori case, the general case follows easily from [BT2], 5.2.10, since $\mathfrak{G}$ is semi-simple and simply connected). This in turn shows that as representations of P we have

$$\mathrm{Ind}_{G \cap \mathbf{P}}^{G}(\rho)|_P \cong \bigoplus_{w \in W_{\mathbf{P}} \backslash W / W_{\mathbf{P}}} \mathrm{Ind}_{P_w^+}^{P}(\rho^w) ,$$

where $P_w^+ = P \cap w P^+ w^{-1}$. For $w \in W_{\mathbf{P}} \backslash W / W_{\mathbf{P}}$, we let $M^w(\rho)$ be the $D(P,K)$-module dual to $\mathrm{Ind}_{P \cap P_w^+}^{P}(\rho^w)$, i.e.,

$$M^w(\rho) = \left(\mathrm{Ind}_{P_w^+}^{P}(\rho^w)\right)_b' = D(P,K) \otimes_{D(P_w^+,K)} V_w' ,$$

where V_w denotes the vector space V equipped with the action ρ^w of P_w^+ given by $\rho^w(h) = \rho(w^{-1} h w)$. We get

$$\left(\mathrm{Ind}_{G \cap \mathbf{P}}^{G}(\rho)\right)_b' = \bigoplus_{w \in W_{\mathbf{P}} \backslash W / W_{\mathbf{P}}} M^w(\rho) .$$

Assuming that all $D(P,K)$-modules $M^w(\rho)$ are simple we show in 3.5.1 that they are pairwise non-isomorphic. This implies that the left hand side is a simple $D(G,K)$-module, which in turn shows that $\mathrm{Ind}_{G \cap \mathbf{P}}^{G}(\rho)$ is a topologically irreducible G-representation.

3.2.3. *From $D(P,K)$-modules to $D_r(P,K)$-modules.* Because the representation V is finite-dimensional, the canonical map $V_w' \to D_r(P_w^+, K) \otimes_{D(P_w^+,K)} V_w'$ is an isomorphism for $r < 1$ sufficiently close to 1, cf. 3.4.2. Hence

$$\begin{aligned} M_r^w(\rho) \;&:= D_r(P,K) \,\hat{\otimes}_{D(P,K)} M^w(\rho) = D_r(P,K) \,\hat{\otimes}_{D(P_w^+,K)} V_w' \\ &= D_r(P,K) \hat{\otimes}_{D_r(P_w^+,K)} \left(D_r(P_w^+,K) \hat{\otimes}_{D(P_w^+,K)} V_w' \right) \\ &= D_r(P,K) \hat{\otimes}_{D_r(P_w^+,K)} V_w' . \end{aligned}$$

By [ST2], Lemma 3.9, $M^w(\rho)$ is a simple $D(P,K)$-module if $M_r^w(\rho)$ is a simple $D_r(P,K)$-module for a sequence of r's tending to 1, so that we are done if we show that all $M_r^w(\rho)$ are simple $D_r(P,K)$-modules, assuming the simplicity of $m(\rho)$ as an $U(\mathfrak{g})$-module.

In order to study the modules $M_r^w(\rho)$ we use the Iwahori product decomposition, cf. 3.3.2,

$$P = U_w^- \cdot P_w^+ \,,$$

where $U_w^- = P \cap w\mathbf{U}^- w^{-1}$ and $\mathbf{U}^-$ is the unipotent radical of the parabolic subgroup opposite to $\mathbf{P}$. We choose an open uniform normal subgroup $P_0 \subset P$ such that

$$P_0 = U_{w,0}^- \cdot P_{w,0}^+ \quad \text{where} \quad U_{w,0}^- = U_w^- \cap P_0 \quad \text{and} \quad P_{w,0}^+ = P_w^+ \cap P_0$$

The completed distribution algebras of P (U_w^- and P_w^+, resp.) are defined by means of the canonical p-valuation on P_0 (its restriction to $U_{w,0}^-$ and $P_{w,0}^+$, resp.), cf. 2.2.4. For $r < 1$ sufficiently close to 1, we have a canonical isomorphism of Banach spaces

$$D_r(U_w^-, K) \hat{\otimes}_K D_r(P_w^+, K) \xrightarrow{\simeq} D_r(P, K) \,,$$

which in turn gives rise to a canonical isomorphism of $D_r(U_w^-, K)$-modules

$$D_r(U_w^-, K) \otimes_K V_w' \xrightarrow{\simeq} M_r^w(\rho) \,,$$

cf. 3.4.2. Using the integrality of the distribution algebra $D_r(U_w^-, K)$, cf. 3.3.5, together with corollary 2.3.3, we prove that any non-zero $D_r(P, K)$-submodule N of $M_r^w(\rho)$ has non-zero intersection with

$$m_r^w(\rho) := U_r(\mathfrak{u}_w^-, U_{w,0}^-) \otimes_K V_w' \,,$$

cf. 3.4.7. Using general results about orthogonal bases we can even infer that N has non-zero intersection with

$$m^w(\rho) = U(\mathfrak{u}_w^-) \otimes_K V_w' \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_w^+)} V_w' \,.$$

But it is not difficult to see that if $m(\rho) := m^1(\rho) = U(\mathfrak{g}) \otimes_{\mathfrak{p}} V'$ is a simple $U(\mathfrak{g})$-module, then $m^w(\rho)$ is a simple $U(\mathfrak{g})$-module for all w, and this implies that $M_r^w(\rho)$ is a simple $D_r(P, K)$-module for all w and all r sufficiently close to 1. Hence, by our previous remark, $M^w(\rho)$ is a simple $D(P, K)$-module for all w. From what we have said at the end of 3.2.2 it then follows that $\mathrm{Ind}_{G \cap \mathbf{P}}^G(\rho)$ is a topologically irreducible G-representation.

3.3. **Parahoric subgroups and their distribution algebras.**

3.3.1. Recall that by the reasoning in the beginning of sec. 3.2 we will assume $\mathfrak{G}$ to be semi-simple and simply connected. For the following compare [Ca], sec. 3.5. The torus $\mathfrak{S}$ determines an apartment $\mathcal{A}$ in the Bruhat-Tits building of $\mathfrak{G}$ over L. We fix a special vertex x_0 in the apartment $\mathcal{A}$. Then there is a unique conical chamber $\mathcal{C}$ in $\mathcal{A}$ having x_0 as apex and satisfying the following property: for every u in the unipotent radical $\mathbf{U}_0$ of $\mathbf{P}_0$ the intersection $\mathcal{C} \cap u\mathcal{C}$ contains a translate of $\mathcal{C}$. Moreover, there is a unique chamber C_0 in $\mathcal{C}$ having x_0 as one of its vertices. We let $G \subset \mathbf{G}$ be the stabilizer of x_0 and $P_0 \subset G$ the pointwise stabilizer of C_0. Let $\alpha^\perp \subset X_*(\mathfrak{S})_\mathbb{R}$ be the orthogonal complement under the natural pairing $X_*(\mathfrak{S})_\mathbb{R} \times X^*(\mathfrak{S})_\mathbb{R} \longrightarrow \mathbb{R}$. The subset

$$C = C_0 \cap \big(x_0 + \bigcap_{1 \le i \le t} \alpha_i^\perp\big)$$

is a facet of C_0 and we let P be its pointwise stabilizer. It is a parahoric subgroup of the same type as $\mathbf{P}$. Both groups correspond to a subset $I \subset \Delta$. Let $\mathbf{U}^-$ be the unipotent radical of the parabolic subgroup of $\mathbf{G}$ opposite to $\mathbf{P}$. For $w \in W_P \backslash W / W_P$, we put

$$P_w^+ = P \cap w\mathbf{P}w^{-1}\,,\ U_w^+ = P \cap w\mathbf{U}w^{-1}\,,\ U_w^- = P \cap w\mathbf{U}^-w^{-1}\,.$$

Lemma 3.3.2. *The multiplication map*

$$P_w^+ \times U_w^- \longrightarrow P$$

is an isomorphism of locally L-analytic manifolds. In particular, there are decompositions

$$P = P_w^+ \cdot U_w^- = U_w^- \cdot P_w^+\,.$$

Proof. Let $\mathbf{N}$ be the normalizer of $\mathbf{S}$ in $\mathbf{G}$. By [Ti], 3.1.1, we have a product decomposition

$$P = X(C)^+ \cdot \mathbf{N}^C \cdot X(C)^-\,,$$

where $X^+(C) = P \cap \mathbf{U}_0$, $X^-(C) = P \cap \mathbf{U}_0^-$ and $\mathbf{N}^C$ is the subset of $\mathbf{N}$ fixing C pointwise. For a root $\alpha \in \Phi^{\mathrm{red}}$, let $U_\alpha = \mathbf{U}_\alpha \cap P$. Then

$$X(C)^\pm = \prod_{\alpha \in (\Phi^{\mathrm{red}})^\pm} U_\alpha\,,$$

where the product is taken with respect to any ordering on the roots. Let $\Phi_I^{\mathrm{red}} \subset \Phi^{\mathrm{red}}$ be the set of roots generated by I. Then

$$M = \mathbf{M} \cap P = \big(\prod_{\alpha \in (\Phi_I^{\mathrm{red}})^+} U_\alpha \big) \cdot \mathbf{N}^C \cdot \big(\prod_{\alpha \in (\Phi_I^{\mathrm{red}})^-} U_\alpha \big)$$

and

$$U^+ := \mathbf{U}^+ \cap P = \prod_{\alpha \in (\Phi^{\mathrm{red}})^+ \setminus \Phi_I^{\mathrm{red}}} U_\alpha$$

resp.

$$U^- := \mathbf{U}^- \cap P = \prod_{\alpha \in (\Phi^{\mathrm{red}})^+ \setminus \Phi_I^{\mathrm{red}}} U_\alpha \,.$$

Thus

$$P = U^+ \cdot M \cdot U^- = P^+ \cdot U^- \,.$$

We have

$$U_w^+ = \big(\prod_{\substack{\alpha \in (\Phi^{\mathrm{red}})^+ \setminus \Phi_I^{\mathrm{red}} \\ w^{-1}\alpha > 0}} U_\alpha \big) \cdot \big(\prod_{\substack{\alpha \in (\Phi^{\mathrm{red}})^- \setminus \Phi_I^{\mathrm{red}} \\ w^{-1}\alpha > 0}} U_\alpha \big)$$

and

$$U_w^- = \big(\prod_{\substack{\alpha \in (\Phi^{\mathrm{red}})^+ \setminus \Phi_I^{\mathrm{red}} \\ w^{-1}\alpha < 0}} U_\alpha \big) \cdot \big(\prod_{\substack{\alpha \in (\Phi^{\mathrm{red}})^- \setminus \Phi_I^{\mathrm{red}} \\ w^{-1}\alpha < 0}} U_\alpha \big) \,.$$

Since M normalizes the factors of these groups, we get

$$P = U_w^+ \cdot M \cdot U_w^- = P_w^+ \cdot U_w^- \,.$$

□

3.3.3. *Defining norms on $D(P,K)$.* We start with the construction of the subgroup $P_0 \subset P$ mentioned in 3.2.3. By using Prop. 1.4.4 in [Cas], we choose an open normal subgroup $P_1 \subset P$ having Iwahori factorization with respect to the parabolic subgroup $w\mathbf{P}w^{-1}$. In particular, we get a decomposition $P_1 = (U_w^- \cap P_1)(P_w^+ \cap P_1)$. By possibly shrinking P_1, we may further suppose that it is contained in some uniform subgroup $P_2 \subset P$. The $\mathbb{Z}_p$-Lie algebra $L_{P_2} = (P_2, +)$, in the sense of [DDMS], 4.3, 9.4, is a powerful Lie algebra over $\mathbb{Z}_p$. This means that L_{P_2} is torsion-free, and for all $x, y \in L_{P_2}$ one has

$$[x,y] \in p^{1+\varepsilon_p} L_{P_2}\,,$$

where ε_p is defined as in 2.2.3. Note that there is a canonical identification (cf. [DDMS] ch. 9.5)

$$L_{P_2} \otimes \mathbf{Q}_p \cong \mathrm{Lie}(P_2) = \mathrm{Lie}(P_1) = \mathrm{Lie}(\mathbf{G})\,.$$

The inverse image L_{P_1} of $P_1 \subset P_2$ under the exponential map $\exp_{P_2} : L_{P_2} \xrightarrow{\simeq} P_2$ is a $\mathbb{Z}_p$-Lie subalgebra of L_{P_2}, and the same is true for

$$L_1' := \exp_{P_2}^{-1}(U_w^- \cap P_1)\,,\ L_2' := \exp_{P_2}^{-1}(P_w^+ \cap P_1)\,.$$

From the Iwahori factorization $P_1 = (U_w^- \cap P_1)(P_w^+ \cap P_1)$ we deduce that $L_{P_1} = L_1' \oplus L_2'$. For $m \in \mathbb{Z}_{>0}$ large enough the $\mathbb{Z}_p$-submodules $L_1 := p^m L_1'$ and $L_2 := p^m L_2'$ and their sum $L_0 := L_1 \oplus L_2$ are then in fact powerful $\mathbb{Z}_p$-Lie algebras. These in turn correspond to uniform pro-p subgroups

$$U_{w,0}^- := \exp_{P_2}(L_1) \subset P_0 := \exp_{P_2}(L_0)\,,\ P_{w,0}^+ := \exp_{P_2}(L_2) \subset P_0$$

of P (cf. [DDMS], Thm. 9.10). Note that $U_{w,0}^-$ (resp. $P_{w,0}^+$, resp. P_0) is equal to the group $(U_w^- \cap P_1)^{p^m}$ (resp. $(P_w^+ \cap P_1)^{p^m}$, resp. $P_1^{p^m}$), consisting of p^m powers of elements of $U_w^- \cap P_1$ (resp. $P_w^+ \cap P_1$, resp. P_1). They are therefore characteristic subgroups of $U_w^- \cap P_1$ (resp. $P_w^+ \cap P_1$, resp. P_1), and this implies that $U_{w,0}^-$ (resp. $P_{w,0}^+$, resp. P_0) is normal in U_w^- (resp. P_w^+, resp. P). Next we remark that L_1 and L_2 are actually $\mathfrak{o}_L$-modules, as $U_{w,0}^-$ (resp. $P_{w,0}^+$) are L-analytic groups. By possibly enlarging m we can assure that the pair $(L_1, U_{w,0}^- = \exp_{P_2}(L_1))$ has the properties (1)-(3) from Thm. 2.3.2 (this is an immediate consequence of the proof of Prop. 1.3.5 of [K]). We have $P_0 = U_{w,0}^- \cdot P_{w,0}^+$, and for the canonical p-valuation on P_0 it follows from the identity $P_i(P_0) = \exp(p^i L_0)$:

$$\omega^{can}(\exp(x)\exp(y)) = 1 + \varepsilon_p + \min\{a,b\} = \min\{\omega^{can}(\exp(x)), \omega^{can}(\exp(y))\}\,,$$

where a, b are such that $x \in p^a L_1 \setminus p^{a+1} L_1$ and $y \in p^b L_2 \setminus p^{b+1} L_2$ (with a (resp. b) being ∞ if $x = 0$ (resp. $y = 0$)). In order to define the norms q_r on the rings $D(U_w^-, K)$, $D(P_w^+, K)$ and $D(P, K)$, we work with the uniform normal subgroups $U_{w,0}^-$, $P_{w,0}^+$ and P_0, and the canonical p-valuations on these groups, following the recipe explained in 2.2.4.

Proposition 3.3.4. *(i) The decompositions $P = U_w^- \cdot P_w^+$, $P_0 = U_{w,0}^- \cdot P_{w,0}^+$ induce isomorphisms of topological K-vector spaces*

$$C^{an}_{\mathbb{Q}_p}(U_w^-, K)\hat{\otimes}_K C^{an}_{\mathbb{Q}_p}(P_w^+, K) \xrightarrow{\simeq} C^{an}_{\mathbb{Q}_p}(P, K)$$

$$C^{an}_{\mathbb{Q}_p}(U_{w,0}^-, K)\hat{\otimes}_K C^{an}_{\mathbb{Q}_p}(P_{w,0}^+, K) \xrightarrow{\simeq} C^{an}_{\mathbb{Q}_p}(P_0, K)$$

$$C^{an}_{L}(U_w^-, K)\hat{\otimes}_K C^{an}_{L}(P_w^+, K) \xrightarrow{\simeq} C^{an}_{L}(P, K)$$

$$C^{an}_{L}(U_{w,0}^-, K)\hat{\otimes}_K C^{an}_{L}(P_{w,0}^+, K) \xrightarrow{\simeq} C^{an}_{L}(P_0, K)\,.$$

(ii) We equip the rings $D_{\mathbb{Q}_p}(P_0, K)$, $D_{\mathbb{Q}_p}(U_{w,0}^-, K)$ and $D_{\mathbb{Q}_p}(P_{w,0}^+, K)$ with the norm $\|\cdot\|_r$, $\frac{1}{p} < r < 1$, associated to the canonical p-valuation. The rings $D_{\mathbb{Q}_p}(P, K)$, $D_{\mathbb{Q}_p}(U_w^-, K)$ and $D_{\mathbb{Q}_p}(P_w^+, K)$ carry the maximum norms q_r, and $D_L(P, K)$, $D_L(U_w^-, K)$, $D_L(P_w^+, K)$ as well as $D_L(P_0, K)$, $D_L(U_{w,0}^-, K)$ and $D_L(P_{w,0}^+, K)$ are equipped with the quotient norms $\bar{q}_r$. On the tensor products of these spaces we put the usual induced norm. Then the isomorphisms in (i) induce isometries of topological K-vector spaces

$$D_{\mathbb{Q}_p}(P, K) \xrightarrow{\simeq} D_{\mathbb{Q}_p}(U_w^-, K)\hat{\otimes}_K D_{\mathbb{Q}_p}(P_w^+, K)$$

$$D_{\mathbb{Q}_p}(P_0, K) \xrightarrow{\simeq} D_{\mathbb{Q}_p}(U_{w,0}^-, K)\hat{\otimes}_K D_{\mathbb{Q}_p}(P_{w,0}^+, K)$$

$$D_{L}(P, K) \xrightarrow{\simeq} D_{L}(U_w^-, K)\hat{\otimes}_K D_{L}(P_w^+, K)$$

$$D_{L}(P_0, K) \xrightarrow{\simeq} D_{L}(U_{w,0}^-, K)\hat{\otimes}_K D_{L}(P_{w,0}^+, K)\,.$$

(iii) The isometries in (ii) furnish isometries of the completions

$$D_r(P, K) \xrightarrow{\simeq} D_r(U_w^-, K)\hat{\otimes}_K D_r(P_w^+, K)$$

$$D_r(P_0, K) \xrightarrow{\simeq} D_r(U_{w,0}^-, K)\hat{\otimes}_K D_r(P_{w,0}^+, K)\,.$$

Proof. (i) is easily checked using the definitions of the locally convex topologies on the spaces of locally analytic functions (cf. [S] for the inductive limit topology on the space of locally analytic functions).
(ii) Let $h_1, \ldots, h_{d'}$ be a p-basis of $U_{w,0}^-$ and $h_{d'+1}, \ldots, h_d$ be a p-basis of $P_{w,0}^+$. For $u \in U_{w,0}^-$, $p \in P_{w,0}^+$ we have by the discussion in 3.3.3

$$\omega^{can}(up) = \min\{\omega^{can}(u), \omega^{can}(p)\}\,,$$

so that $h_1, \ldots, h_{d'}, h_{d'+1}, \ldots, h_d$ is a p-basis of P_0. Elements of $D_{\mathbb{Q}_p}(P_0, K)$ have a unique expansion as series of the form

$$\sum\nolimits_{n \in \mathbb{N}_0^d} d_n \mathbf{b}^n \,,$$

with $\mathbf{b}^n = (h_1 - 1)^{n_1} \cdot \ldots \cdot (h_d - 1)^{n_d}$. From this we deduce immediately that the canonical map

$$D_{\mathbb{Q}_p}(P_0, K) \xrightarrow{\simeq} D_{\mathbb{Q}_p}(U_{w,0}^-, K) \hat{\otimes}_K D_{\mathbb{Q}_p}(P_{w,0}^+, K)$$

is an isometry when equipped with the norm $\| \cdot \|_r$ on the left hand side and with the induced norm on the tensor product. Consider the canonical commutative diagram

$$\begin{array}{ccc} D_{\mathbb{Q}_p}(P_0, K) & \xrightarrow{\simeq} & D_{\mathbb{Q}_p}(U_{w,0}^-, K) \hat{\otimes}_K D_{\mathbb{Q}_p}(P_{w,0}^+, K) \\ \downarrow & & \downarrow \\ D_L(P_0, K) & \xrightarrow{\simeq} & D_L(U_{w,0}^-, K) \hat{\otimes}_K D_L(P_{w,0}^+, K) \end{array}$$

Let $\bar{q}_r$ denote the quotient norm on $D_L(U_{w,0}^-, K)$ as well as on $D_L(P_{w,0}^+, K)$, and let $\bar{q}_r \otimes \bar{q}_r$ be the induced norm on the tensor product. On the other hand, let $\| \cdot \|_r \otimes \| \cdot \|_r$ be the norm on $D_{\mathbb{Q}_p}(U_{w,0}^-, K) \hat{\otimes}_K D_{\mathbb{Q}_p}(P_{w,0}^+, K)$, and denote by $\overline{\| \cdot \|_r \otimes \| \cdot \|_r}$ the norm on $D_L(U_{w,0}^-, K) \hat{\otimes}_K D_L(P_{w,0}^+, K)$ induced by the surjection

$$D_{\mathbb{Q}_p}(U_{w,0}^-, K) \hat{\otimes}_K D_{\mathbb{Q}_p}(P_{w,0}^+, K) \twoheadrightarrow D_L(U_{w,0}^-, K) \hat{\otimes}_K D_L(P_{w,0}^+, K) \,.$$

By [BGR], Ch. 2.1, Prop. 6, we know that

$$\bar{q}_r \otimes \bar{q}_r = \overline{\| \cdot \|_r \otimes \| \cdot \|_r} \,,$$

which in turn shows that

$$D_L(P_0, K) \xrightarrow{\simeq} D_L(U_{w,0}^-, K) \hat{\otimes}_K D_L(P_{w,0}^+, K)$$

is an isometry. Finally, using Remark 2.2.5 we can conclude that

$$D_L(P, K) \xrightarrow{\simeq} D_L(U_w^-, K) \hat{\otimes}_K D_L(P_w^+, K)$$

is an isometry too.
(iii) This statement follows from (ii). □

Proposition 3.3.5. *For $r < 1$ sufficiently close to 1, the ring $D_r(U_w^-, K)$ is an integral domain.*

Proof. Let $\bar{q}_r$ be the norm on $D_r(U_w^-, K)$ defined by means of the uniform subgroup $U_{w,0}^-$ (using the canonical p-valuation on $U_{w,0}^-$) as explained in 2.2.4. The key idea of the proof is to embed $D_r(U_w^-, K)$ in another distribution ring of the form $D_{r'}(U, K)$ which is an integral domain. First we note that for any open subgroup U of $w\mathbf{U}^-w^{-1}$ there is an element s in the torus $\mathbf{S}$ such that $sU_w^-s^{-1}$ is contained in U. It follows from [Sch], Prop. 5.6, that there is an open compact uniform pro-p subgroup $U \subset w\mathbf{U}^-w^{-1}$ and a norm $\bar{q}_{r'}^U$, $\frac{1}{p} < r' < 1$, on $D(U, K)$ defined by means of the canonical p-valuation on U (as in 2.2.4) such that the completion $D_{r'}(U, K)$ is an integral domain[3]. For later use we remark that for $r' \leq r'' < 1$ the canonical map $D_{r''}(U, K) \to D_{r'}(U, K)$ is injective, so we may take r' as close to 1 as we wish. As we can embed U_w^- into U by conjugation we may in fact assume that U_w^- is contained in U. Because U_w^- is of finite index in U we have an embedding of topological rings

$$D(U_w^-, K) \hookrightarrow D(U, K)\,,\ \lambda \mapsto (f \mapsto \lambda(f|_{U_w^-}))\,.$$

We will show that this embedding induces an embedding $D_r(U_w^-, K) \hookrightarrow D_{r'}(U, K)$ for r sufficiently close to 1. Let $h_1, \ldots, h_{d'}$ be a p-basis of the uniform pro-p group $U_{w,0}^-$. Then we have $\bar{q}_{r'}^U(h_i - 1) < 1$ for $i = 1, \ldots, d'$. In particular, there is $r \in (\frac{1}{p}, 1)$ such that

$$\bar{q}_{r'}^U(h_i - 1) \leq r^{1+\varepsilon_p} = r^{\omega^{can}(h_i)}\,.$$

Because $\bar{q}_{r'}^U$ is multiplicative, we have for $\sum_{n\in\mathbb{N}_0^d} d_n\mathbf{b}^n \in D_{\mathbb{Q}_p}(U_{w,0}^-, K)$,

$$\bar{q}_{r'}^U(\sum_{n\in\mathbb{N}_0^d} d_n\mathbf{b}^n) \leq \sup_{n\in\mathbb{N}_0^d}\{|d_n|\bar{q}_{r'}^U(\mathbf{b}^n)\} \leq \sup_{n\in\mathbb{N}_0^d}\{|d_n|r^{\tau(n)}\} = \|\sum_{n\in\mathbb{N}_0^d} d_n\mathbf{b}^n\|_r\,.$$

This shows that in fact for all $\lambda \in D(U_{w,0}^-, K)$, one has $\bar{q}_{r'}^U(\lambda) \leq \bar{q}_r(\lambda)$. Hence there is for such r, a continuous map of the completions $D_r(U_{w,0}^-, K) \to D_{r'}(U, K)$, which in turn induces a continuous map $D_r(U_w^-, K) \to D_{r'}(U, K)$. The same argument as above shows that, if for some $r_0 \in (\frac{1}{p}, 1)$ we equip $D(U, K)$ with the maximum norm μ_{r_0} induced by $\bar{q}_{r_0}$ and the inclusion $D(U_{w,0}^-, K) \subset D(U, K)$, we find $r' \in (r_0, 1)$ such that the identity extends to a continuous map

$$D_{r'}(U, K) \to (D(U, K), \mu_{r_0})^\wedge\,,$$

where on the right hand side we have the completion of $D(U, K)$ with respect to μ_{r_0}. The composed map

$$D_r(U_w^-, K) \longrightarrow D_{r'}(U, K) \longrightarrow (D(U, K), \mu_{r_0})^\wedge$$

[3]Here we use the embedding of $D_{r'}(U, K)$ into the distribution algebra for smaller r', which is an integral domain by the result of Schmidt, cf. loc. cit..

maps $D_r(U_w^-, K)$ into the completion of $D(U_w^-, K)$ with respect to μ_{r_0}, but the latter is simply $D_{r_0}(U_w^-, K)$, and the induced map

$$D_r(U_w^-, K) \longrightarrow D_{r_0}(U_w^-, K)$$

is injective. So $D_r(U_w^-, K) \to D_{r'}(U, K)$ is injective as well, and hence $D_r(U_w^-, K)$ is an integral domain. □

Corollary 3.3.6. *Let $r \in (\frac{1}{p}, 1)$ be such that 3.3.5 holds. Then any non-zero left ideal I of $D_r(U_w^-, K)$ has non-zero intersection with $U_r(\mathfrak{u}_w^-, U_{w,0}^-)$.*

Proof. We already remarked in 3.3.3 that the lattice $L_1 \subset \mathfrak{u}_w^-$ and the subgroup $U_{w,0}^- = \exp(L_1)$ can be chosen to have the properties of the group H_0 in 2.3.2 (i). Using the above proposition 3.3.5 we can apply corollary 2.3.3 whose assertion is exactly the claim made above. □

3.4. **Modules for the completed distribution algebras.**

3.4.1. *The modules $M^w(\rho)$ and $m^w(\rho)$.* Let

$$\rho : \mathbf{P} \longrightarrow \mathbf{M} \longrightarrow \mathrm{GL}(V)$$

be the fixed locally analytic representation of $\mathbf{P}$ on V (equipped with the unique Hausdorff locally convex topology) from 3.1.2. The representation of P_w^+ which we get by restriction and conjugation will be denoted by (ρ^w, V_w), cf. 3.2.3. Consider its (strong) dual $V_w' := (V_w)_b'$, which is a $D(P_w^+, K)$-module. We recall our convention to denote Lie algebras of locally analytic groups by their corresponding gothic letters:

$$\mathfrak{p}_w^+ = \mathrm{Lie}(P_w^+) = w\,\mathrm{Lie}(P^+)w^{-1}\,,\ \mathfrak{u}_w^- = \mathrm{Lie}(U_w^-) = w\,\mathrm{Lie}(\mathbf{U}^-)w^{-1}\,,\ \mathfrak{s} = \mathrm{Lie}(\mathbf{S})\,.$$

Via the embedding $U(\mathfrak{p}_w^+) \hookrightarrow D(P_w^+, K)$ we view V_w' as a Lie algebra representation of $\mathfrak{p}_w^+$. We put

$$m^w(\rho) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_w^+)} V_w' \ \cong\ U(\mathfrak{u}_w^-)_K \otimes_K V_w'\,.$$

Next, consider the induced locally analytic representation $\mathrm{Ind}_{P_w^+}^P(\rho^w)$ and let

$$M^w(\rho) = (\mathrm{Ind}_{P_w^+}^P(\rho^w))_b' = D(P, K) \otimes_{D(P_w^+, K)} V_w'$$

be the corresponding $D(P, K)$-module. By 3.3.4, there is a canonical isomorphism of $D(U^-, K)$-modules

$$M^w(\rho) = D(P, K) \otimes_{D(P_w^+, K)} V_w' \ \simeq\ D(U_w^-, K) \otimes_K V_w'\,.$$

Thus we see that the natural map $m^w(\rho) \to M^w(\rho)$ is injective.

Proposition 3.4.2. *(i) For $r \in (\frac{1}{p}, 1)$ sufficiently close to 1 the canonical map*

$$V'_w \longrightarrow D_r(P^+_w, K) \otimes_{D(P^+_w,K)} V'_w \ , \ v \mapsto 1 \otimes v \, ,$$

is an isomorphism of K-vector spaces. Consequently, via this isomorphism, we can extend the continuous operation of $D(P^+_w, K)$ on V'_w to a continuous operation of $D_r(P^+_w, K)$ on V'_w.

(ii) When the finitely generated $D_r(P, K)$-module $D_r(P, K) \otimes_{D_r(P^+_w,K)} V'_w$ is equipped with its natural topology, the canonical map

$$D_r(U^-_w, K) \otimes_K V'_w \longrightarrow D_r(P, K) \otimes_{D_r(P^+_w,K)} V'_w$$

is a topological isomorphism.

Proof. (i) V'_w is a finite-dimensional K-vector space and hence finitely generated as $D(P^+_w, K)$-module. V_w is thus a strongly admissible representation of P^+_w in the sense of [ST1], §3. This implies that V'_w is a co-admissible $D(P^+_w, K)$-module in the sense of [ST2], §3 (cf. [ST2], paragraph following Prop. 6.4). As a co-admissible module the canonical map

$$\phi : V'_w \longrightarrow \varprojlim_{r<1} \Big(D_r(P^+_w, K) \otimes_{D(P^+_w,K)} V'_w \Big)$$

is an isomorphism of K-vector spaces. By [ST2], Theorem A, p. 152, for every $r \in (\frac{1}{p}, 1)$ the canonical map

$$\phi_r : V'_w \longrightarrow D_r(P^+_w, K) \otimes_{D(P^+_w,K)} V'_w$$

has dense image, and is therefore surjective. For $r_1 < r_2 < 1$ the map ϕ_{r_1} factors through ϕ_{r_2}. This implies that the family $\ker(\phi_r)$ is decreasing with increasing r. The intersection of all $\ker(\phi_r)$ is the zero space because ϕ is injective. As V'_w is finite-dimensional, there must be some $r \in (\frac{1}{p}, 1)$ such that $\ker(\phi_r)$ is zero. Hence, there is necessarily some $r \in (\frac{1}{p}, 1)$ such that the map $V'_w \longrightarrow D_r(P^+_w, K) \otimes_{D(P^+_w,K)} V'_w$ is bijective.

(ii) By Lemma 3.3.4 it follows that the map in (ii) is a bijection. If we give the right hand side its natural $D_r(P, K)$-module topology, i.e., the quotient topology induced by an arbitrary surjection of $D_r(P, K)$-modules

$$D_r(P, K)^m \to D_r(P, K) \otimes_{D_r(P^+_w,K)} V'_w \, ,$$

then the map in (ii) is a continuous bijective map, and hence, by the open mapping theorem, a homeomorphism. □

3.4.3. *The modules $M_r^w(\rho)$ and $m_r^w(\rho)$.* For the rest of this section, $r < 1$ denotes a real number sufficiently close to 1 such that the assertions of the propositions 3.4.2 and 3.3.5 do hold. We put

$$M_r^w(\rho) := D_r(P,K) \otimes_{D(P,K)} M^w(\rho)\,,$$

which we consider as a module over the Banach algebra $D_r(P,K)$. By the foregoing proposition, we have

$$\begin{aligned} D_r(P,K)\,\hat{\otimes}_{D(P,K)} M^w(\rho) &= D_r(P,K)\,\hat{\otimes}_{D(P_w^+,K)} V_w' \\ &= D_r(P,K)\hat{\otimes}_{D_r(P_w^+,K)} \left(D_r(P_w^+,K)\hat{\otimes}_{D(P_w^+,K)} V_w' \right) \\ &= D_r(P,K)\hat{\otimes}_{D_r(P_w^+,K)} V_w'\,. \end{aligned}$$

Using this and 3.3.4 we see that the map

$$D_r(U_w^-,K)\hat{\otimes}_K V_w' \longrightarrow M_r^w(\rho)$$

is an isomorphism of $D_r(U_w^-,K)$-modules. Therefore, the natural map

$$M^w(\rho) \longrightarrow M_r^w(\rho)$$

is an embedding. We denote the topological closure of $m^w(\rho)$ in $M_r^w(\rho)$ by $m_r^w(\rho)$. Because $U_r(\mathfrak{u}_w^-, U_{w,0}^-)$ is the topological closure of $U(\mathfrak{u}_w^-)_K$ in $D_r(U_w^-,K)$ we have

$$m_r^w(\rho) = U_r(\mathfrak{u}_w^-, U_{w,0}^-) \otimes_K V_w'\,.$$

Since the maximal split torus $\mathbf{S}$ is contained in $\mathbf{P}$, we have a natural action of $U(\mathfrak{s})$ on $m^w(\rho)$. This action extends by continuity to the closure $m_r^w(\rho)$. For a weight $\lambda \in \mathfrak{s}^*$ and an arbitrary Lie algebra representation W of $\mathfrak{s}$ we denote as usual by W_λ the weight space of λ. We have the following result:

Lemma 3.4.4. *For any $\lambda \in \mathfrak{s}^*$ the weight space $m_r^w(\rho)_\lambda$ is finite-dimensional.*

Proof. We already remarked in 3.3.3 that the lattice $L_1 \subset \mathfrak{u}_w^-$ and the subgroup $U_{w,0}^- = \exp(L_1)$ can be chosen to have the properties of the group H_0 in 2.3.2 (i). Thus the assertions (ii) and (iii) of 2.3.2 do hold. By the proof of [K], Prop. 1.3.5, the basis $\mathfrak{X} = (\mathfrak{X}_1, \ldots, \mathfrak{X}_d)$ of $\mathrm{Lie}(U_w^-) = \mathfrak{u}_w^-$ can be chosen to consist of weight vectors, i.e., there are $\alpha_1, \ldots, \alpha_d \in \mathfrak{s}^*$ such that

$$\mathrm{ad}(\mathfrak{x})(\mathfrak{X}_i) = \alpha_i(\mathfrak{x})\mathfrak{X}_i$$

for all $\mathfrak{x} \in \mathfrak{s}$ and $1 \leq i \leq d$. By 2.3.2 (iii), the closure $U_r(\mathfrak{u}_w^-, U_{w,0}^-)$ of $U(\mathfrak{u}_w^-) \otimes_L K$ in $D_r(U_{w,0}^-, K)$, cf. 2.3.1, consists exactly of those series

$$\sum\nolimits_{n \in \mathbb{N}_0^d} d_n \mathfrak{X}^n$$

for which

$$\lim_{|n| \to \infty} |d_n| \nu_r(\mathfrak{X}^n) = 0\,.$$

Here, ν_r is a norm on $U_r(\mathfrak{u}_w^-, U_{w,0}^-)$ which is equivalent to $\bar{q}_r$ and $\nu_r(\sum_n d_n \mathfrak{X}^n) = \sup_n |d_n| \nu_r(\mathfrak{X}^n)$. Let $v_1, \ldots, v_k$ be a basis of V_w' which consists of weight vectors: $\mathfrak{x} \cdot v_j = \gamma_j(\mathfrak{x}) v_j$ for all $\mathfrak{x} \in \mathfrak{s}$, $1 \leq j \leq k$. If

$$\mu = \sum\nolimits_{n \in \mathbb{N}_0^d, 1 \leq j \leq k} d_{n,j} \mathfrak{X}^n \otimes v_j \;\in\; m_r^w(\rho) = U_r(U_0^-, K) \otimes_K V_w'$$

is an element of $m_r^w(\rho)_\lambda$, then

$$\lambda = \gamma_j + \sum\nolimits_{1 \leq i \leq k} n_i \alpha_i$$

if $d_{n,j} \neq 0$. This is an immediate consequence of the fact that the monomials $\{\mathfrak{X}^n\}_{n \in \mathbb{N}_0^d}$ are an orthogonal basis of $U_{r'}(U_0^-, K)$ with respect to the norm ν_r, cf. 2.3.2. Because the characters α_i do all occur in the Lie algebra $\mathfrak{u}_w^-$, there can only be finitely many possibilities to write λ as a sum as above. Therefore, the weight spaces $m_r^w(\rho)$ are all finite-dimensional. □

From $m_r^w(\rho)$ back to $m^w(\rho)$. From the result above we can deduce that closed $U(\mathfrak{s})$-invariant subspaces of $m_r^w(\rho)$ are in bijection with $U(\mathfrak{s})$-invariant subspaces of $m^w(\rho)$.

Proposition 3.4.5. *(i) We have an inclusion preserving bijection*

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{closed } U(\mathfrak{s})\text{-invariant} \\ \text{subspaces of } m_r^w(\rho) \end{array} \right\} & \xrightarrow{\sim} & \left\{ U(\mathfrak{s})\text{-invariant subspaces of } m^w(\rho) \right\} \\ W & \longmapsto & W \cap m^w(\rho) \end{array}$$

Any $U(\mathfrak{s})$-invariant subspace W of $m_r^w(\rho)$ is the direct sum of its weight components: $W = \oplus W_\lambda$.

(ii) Let $N \subset m_r^w(\rho)$ be a closed $U(\mathfrak{s})$-invariant subspace, $N \neq 0$. Then $N \cap m^w(\rho) \neq 0$. In particular, any weight vector for the action of $\mathfrak{s}$ lies already in $m^w(\rho)$.

Proof. (i) This statement follows from Lemma 3.4.4 and [Fe] 1.3.12.
(ii) This is an immediate consequence of (i). □

Lemma 3.4.6. *There is $r' \in (0,1)$ such that $U(\mathfrak{u}_w^-) \otimes_L K$ is dense in $D_{r'}(U_{w,0}^-, K)$.*

Proof. Let $(\mathfrak{x}_i)_i$ be a basis of the $\mathbb{Z}_p$-Lie algebra L_1 of $U_{w,0}^-$, with the notation as introduced in sec. 3.3.3. Then the elements $(\exp(\mathfrak{x}_i))_i$, for an arbitrary but fixed ordering, form a p-basis of $U_{w,0}^-$. Put $b_i = \exp(\mathfrak{x}_i) - 1 \in D(R^L_{\mathbb{Q}_p} U_{w,0}^-, K)$, so that the monomials $\mathbf{b}^n$ in the b_i form an orthogonal basis of $D_r(R^L_{\mathbb{Q}_p} U_{w,0}^-, K)$, as in sec. 2.2.4. For $0 < r' < 1$ sufficiently close to 0 the right hand side of the identity $b_i = \exp(\mathfrak{x}_i) - 1$ can be expanded as a convergent series

$$\sum\nolimits_{\nu \geq 1} \frac{\mathfrak{x}_i^\nu}{\nu!}$$

in $D_r(R^L_{\mathbb{Q}_p} U_{w,0}^-, K)$. This shows that we can approximate b_i by elements in $U(\mathfrak{u}_w^-)_K$. Hence $U(\mathfrak{u}_w^-) \otimes_L K$ is dense in $D_{r'}(U_{w,0}^-, K)$. □

Proposition 3.4.7. *Every non-trivial $U(\mathfrak{s})$-invariant $D_r(U_w^-, K)$-submodule of $M_r^w(\rho)$ has a non-trivial intersection with $m_r^w(\rho)$.*

Proof. (See also Proposition 11 of [Fr].) Fix a basis $v_1, \ldots, v_k$ of weight vectors in V_w' with respect to the action of $U(\mathfrak{s})$. We obtain $D_r(U_w^-, K)$-submodules

$$D_r(U_w^-, K) \otimes Kv_i \subset M_r^w(\rho)$$

which are $U(\mathfrak{s})$-invariant. Consider the projections

$$\mathrm{pr}_i : M_r^w(\rho) = D_r(U_w^-, K) \otimes_K V_w' \to D_r(U_w^-, K) \otimes_K Kv_i$$

and let $N \subset M_r^w(\rho)$ be a $D_r(U_w^-, K)$ submodule which is $U(\mathfrak{s})$-invariant. By defining

$$N^{(i)} = \bigcap_{1 \leq j < i} \ker(\mathrm{pr}_j) \cap N\,,$$

we obtain a descending filtration of $D_r(U_w^-, K) \times U(\mathfrak{s})$-modules

$$0 = N^{(k+1)} \subset N^{(k)} \subset \ldots \subset N^{(2)} \subset N^{(1)} = N\,.$$

Let $1 \leq i \leq k$ be the unique index with

$$0 = N^{(i+1)} \subset N^{(i)} \neq 0\,.$$

Identifying $D_r(U_w^-, K) \otimes_K Kv_i$ with $D_r(U_w^-, K)$ as $D_r(U_w^-, K)$-modules we see by Corollary 3.3.6 that

$$\mathrm{pr}_i(N^{(i)}) \cap (U_r(\mathfrak{u}_w^-, U_{w,0}^-) \otimes_K Kv_i) \neq 0.$$

By applying Corollary 3.4.5 we can infer that

$$\mathrm{pr}_i(N^{(i)}) \cap (U(\mathfrak{u}_w^-) \otimes_L Kv_i) \neq 0\,.$$

Therefore, there is an element $F \in N^{(i)}$ such that $\mathrm{pr}_i(F) \in U(\mathfrak{u}_w^-) \otimes_L Kv_i$ is a weight vector (again by 3.4.5). In order to prove the statement of our proposition, it suffices to show that $\mathrm{pr}_j(F) \in U_r(\mathfrak{u}_w^-, U_{w,0}^-) \otimes_K Kv_j$, $j = i+1, \ldots, k$. Suppose that there is an index $i < j \leq k$ such that $\mathrm{pr}_j(F) \notin U_r(\mathfrak{u}_w^-, U_{w,0}^-) \otimes_K Kv_j$.

We want to show that $\mathrm{pr}_j(F)$ can not be a weight vector for the action of $U(\mathfrak{s})$.

To this end, choose $r' \in (0, r)$ sufficiently small such that $U(\mathfrak{u}_w^-) \otimes_L K$ is dense in $D_{r'}(U_{w,0}^-, K)$, cf. 3.4.6. Then we have a commutative diagram of embeddings

$$\begin{array}{ccc} U_{r'}(\mathfrak{u}_w^-, U_{w,0}^-) \otimes_K Kv_j & \xrightarrow{=} & D_{r'}(U_{w,0}^-, K) \otimes_K Kv_j \\ \cup & & \cup \\ U_r(\mathfrak{u}_w^-, U_{w,0}^-) \otimes_K Kv_j & \hookrightarrow & D_r(U_{w,0}^-, K) \otimes_K Kv_j \end{array}$$

Therefore, we can consider $\mathrm{pr}_j(F)$ as an element of $U_{r'}(\mathfrak{u}_w^-, U_{w,0}^-) \otimes_K Kv_j$. If $\mathrm{pr}_j(F)$ was a weight vector, it would then automatically be an element of $U(\mathfrak{u}_w^-) \otimes_L Kv_j$, by 3.4.5 (ii). Hence, a fortiori, $\mathrm{pr}_j(F)$ would be in $U_r(\mathfrak{u}_w^-, U_{w,0}^-) \otimes_K Kv_j$.

Thus we have shown that $\mathrm{pr}_j(F)$ is not a weight vector. Hence we may choose $\lambda \in U(\mathfrak{s})$ such that $\lambda \cdot \mathrm{pr}_j(F)$ is not a scalar multiple of $\mathrm{pr}_j(F)$. Let $C_\lambda \in K$ be the scalar with

$$\lambda \cdot \mathrm{pr}_i(F) = C_\lambda \cdot \mathrm{pr}_i(F)\,.$$

Then the non-zero element $(\lambda - C_\lambda) \cdot F$ is contained in $N^{(i+1)} = 0$, which is a contradiction. □

Corollary 3.4.8. *If $m(\rho) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} V'$ is a simple $U(\mathfrak{g})$-module then:*

(i) $m^w(\rho) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_w^+)} V'_w$ is a simple $U(\mathfrak{g})$-module for every w.

(ii) $M_r^w(\rho)$ is a simple $D_r(P, K)$-module for every w.

Proof. (i) Note that the map

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} V' \longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_w^+)} V'_w\,,\ \mathfrak{z} \otimes v \mapsto \mathrm{ad}(w)(\mathfrak{z}) \otimes v\,,$$

is an isomorphism of the underlying vector spaces. It sends $U(\mathfrak{g})$-submodules to $U(\mathfrak{g})$-submodules. The left hand side is therefore a simple $U(\mathfrak{g})$-module if and only if the right hand side is a simple $U(\mathfrak{g})$-module.

(ii) Let $N \subset M_r^w(\rho)$ be a $D_r(P,K)$-submodule. It is automatically closed because $D_r(P,K)$ is noetherian and $M_r^w(\rho)$ is a finitely generated $D_r(P,K)$-module. By the previous proposition, we see that $N \cap m_r^w(\rho) \neq 0$. By 3.4.5 we get that $N \cap m^w(\rho) \neq 0$. Since $m^w(\rho)$ is a simple $U(\mathfrak{g})$-module, we obtain an inclusion $m^w(\rho) \subset N$. Thus we conclude that $1 \otimes V'_w$ is contained in N and therefore $N = M_r^w(\rho)$. □

Theorem 3.4.9. *If $m(\rho) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} V'$ is a simple $U(\mathfrak{g})$-module then*

$$M^w(\rho) = \left(\mathrm{Ind}_{P_w^+}^P(\rho^w)\right)'_b$$

is a simple $D(P,K)$-module for every w.

Proof. It follows from the definition of $M^w(\rho)$ that this is a finitely generated $D(P,K)$-module (using that V is finite-dimensional over K). Therefore, $\mathrm{Ind}_{P_w^+}^P(\rho^w)$ is a strongly admissible representation of P in the sense of [ST1], §3. This implies that V'_w is a co-admissible $D(P_w^+,K)$-module in the sense of [ST2], §3 (cf. [ST2], paragraph following Prop. 6.4). Now we can use [ST2], Lemma 3.9. and 3.4.8 to conclude that $M^w(\rho)$ is a simple $D(P,K)$-module. □

3.5. **The main result.** The last essential step, to show the topological irreducibility of the induced representation

$$\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\rho) \;\cong\; \mathrm{Ind}_{P^+}^G(\rho)$$

is to prove that the various $D(P,K)$-modules $M^w(\rho)$ are pairwise non-isomorphic.

Proposition 3.5.1. *Every homomorphism $M^{w'}(\rho) \to M^w(\rho)$ of $D(P,K)$-modules for $w \neq w'$ is zero.*

Proof. Our proof is a slight generalization of [Fr], Proposition 12. A homomorphism $M^{w'}(\rho) \to M^w(\rho)$ corresponds by duality, cf. 2.4, to a homomorphism

$$\mathrm{Ind}_{P_w^+}^P(\rho^w) \longrightarrow \mathrm{Ind}_{P_{w'}^+}^P(\rho^{w'})$$

of locally analytic P-representations. By Frobenius reciprocity this corresponds to a continuous $P_{w'}^+$-homomorphism

$$\mathrm{Ind}_{P_w^+}^P(\rho^w) \to V_{w'} = V\,.$$

From the decomposition $P = U_w^- \cdot P_w^+$ we deduce an isomorphism

$$\mathrm{Ind}_{P_w^+}^P(\rho^w) \xrightarrow{\simeq} C_L^{an}(U_w^-, V)$$

of representations of $U_w^- \cap P_{w'}^+$. Since $w \neq w'$ the intersection $w'\mathbf{U}(w')^{-1} \cap w\mathbf{U}^- w^{-1}$ contains a root group. Here, $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$. Therefore,

$$U := w'\mathbf{U}(w')^{-1} \cap U_w^- \cap P_{w'}^+$$

is a non-trivial L-analytic group of positive dimension. The group $w'\mathbf{U}(w')^{-1}$ acts trivially on $V_{w'}$, by assumption. Thus U acts trivially on $V_{w'}$. A homomorphism

$$\mathrm{Ind}_{P_w^+}^P(\rho^w) \longrightarrow \mathrm{Ind}_{P_{w'}^+}^P(\rho^{w'})$$

gives therefore rise to a continuous map

$$\phi : C_L^{an}(U_w^-, V) \longrightarrow V$$

which is U-equivariant, with U acting trivially on V. The canonical projection $U_w^- \to U\backslash U_w^-$ has a locally L-analytic section, so that we can find an isomorphism of locally L-analytic manifolds $U_w^- \simeq U \times U'$ with some compact L-analytic manifold U'. This isomorphism we may assume to be compatible with the action of U by left translation (acting trivially on U'). This in turn gives rise to an isomorphism

$$C_L^{an}(U_w^-, V) \xrightarrow{\simeq} C_L^{an}(U, K)\hat{\otimes}C_L^{an}(U', V)\,.$$

Let $\lambda : V \to K$ be a linear form and fix $g \in C_L^{an}(U', V)$. The map

$$\phi_{\lambda,g} : C_L^{an}(U, K) \to K\,,\ f \mapsto \phi_{\lambda,g}(f) = \lambda(\phi(fg))\,,$$

then has the property that for all $u \in U$:

$$\phi_{\lambda,g}(x \mapsto f(ux)) = \phi_{\lambda,g}(f)\,.$$

Because U does not have a p-adic Haar measure, there is no non-zero U-invariant continuous linear form on $C_L^{an}(U, K)$. Hence, we find that $\phi_{\lambda,g} = 0$. The functions of the form $f \cdot g$, $f \in C_L^{an}(U, K)$, $g \in C_L^{an}(U', V)$, span a dense subspace of $C_L^{an}(U_w^-, V)$. Hence we deduce that $\lambda \circ \phi = 0$ for all linear forms λ on V. This shows that ϕ is necessarily zero. □

Theorem 3.5.2. *Suppose $m(\rho)$ is simple as a $U(\mathfrak{g})$-module. Then*

(i) $M(\rho) = \left(\mathrm{Ind}_{P^+}^G(\rho)\right)_b'$ is simple as a $D(G, K)$-module. A fortiori, the representation $\mathrm{Ind}_{P^+}^G(\rho)$ is topologically irreducible.

(ii) $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\rho)$ is a topologically irreducible representation of $\mathbf{G} = \mathfrak{G}(L)$.

Proof. Consider the decomposition

$$\left(\mathrm{Ind}_{P^+}^G(\rho)\right)_b' = \bigoplus_{w \in W_P \backslash W / W_P} M^w(\rho)\,.$$

Each term $M^w(\rho)$ is by Theorem 3.4.9 a simple $D(P,K)$-module. Furthermore, the summands are not pairwise isomorphic by the previous proposition. Since the Weyl group permutes the summands transitively, $M(\rho)$ is a simple $D(G,K)$-module. By the relation between $D(G,K)$-modules and representations of G, we conclude that $\mathrm{Ind}_{P^+}^G(\rho)$ is a topologically irreducible representation of G, cf. 2.4. The second assertion follows immediately from (i). □

4. Examples

4.1. **Irreducibility of Verma modules.** Theorem 3.5.2 is of course only useful if one can determine whether the generalized Verma module $m(\rho) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \rho'$ is irreducible or not. To simplify this problem, one may pass to a finite extension $\tilde{K}$ of K such that the Lie algebra

$$\mathfrak{g}_{\tilde{K}} = \mathfrak{g} \otimes_L \tilde{K}$$

is split over $\tilde{K}$. Clearly, if $m(\rho) \otimes_K \tilde{K}$ is irreducible as module over $U(\mathfrak{g}_{\tilde{K}})$, then $m(\rho)$ is simple as $U(\mathfrak{g})$-module. If one can reduce the problem to a question about classical Verma modules, i.e. those which are induced from one-dimensional representations of a Borel subalgebra, one is in a particular simple situation. In the next paragraph we recall the set-up from [D], Ch.7.

Suppose $\mathfrak{g}$ is a split reductive Lie algebra over K. Let $\mathfrak{h}$ be Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{b}$ be a Borel subalgebra containing $\mathfrak{h}$, Φ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and $\Phi^+ \supset \Delta$ the set of positive and simple roots, respectively. Denote by $\delta = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ the sum of positive roots, as usual. Let $\{X_\beta, \beta \in \Phi; H_\alpha, \alpha \in \Delta\}$ be a Chevalley basis of $\mathfrak{g}_{\mathrm{der}} = [\mathfrak{g},\mathfrak{g}]$. For a character $\lambda \in \mathrm{Hom}_K(\mathfrak{h}, K)$ let K_λ be the one-dimensional representation of $\mathfrak{h}$ on K defined by λ and extend this to a representation of $\mathfrak{b}$ by letting the nilpotent radical act trivially.

Theorem 4.1.1. *(Bernstein, Gelfand, Gelfand) The Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} K_\lambda$ is a simple $U(\mathfrak{g})$-module if and only if $(\lambda+\delta)(H_\alpha) \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Delta$.*

We refer to [D], Thm. 7.6.24. Note that Dixmier uses the normalized induction, i.e. his λ is our $\lambda + \delta$.

Example 4.1.2. Let $\mathfrak{G} = \mathrm{GL}_{2,L}$ be the group GL_2 over L, $\mathfrak{P}$ the Borel subgroup of upper triangular matrices, $\mathfrak{S} \subset \mathfrak{P}$ the diagonal torus and

$$\chi : \mathbf{S} = \mathfrak{S}(L) \to K^* \ , \ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto \chi_1(t_1)\chi_2(t_2)\,,$$

where the characters $\chi_i : L^* \to K^*$, $i = 1, 2$, are locally L-analytic. We lift χ to a character of $\mathbf{P} = \mathfrak{P}(L)$, and denote by K_χ the corresponding representation on K. Consider the locally L-analytic representation $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(K_\chi)$. For $t \in L^*$ close to 1 one has $\chi_i(t) = t^{c_i}$ with $c_i = \frac{d}{dt}\chi_i(1) \in K$, $i = 1, 2$. The representation of $\mathfrak{p}$ on the dual space $(K_\chi)' \simeq K$ is given by the linear form

$$\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto -(c_1 t_1 + c_2 t_2)\,.$$

By Thm. 4.1.1 and Thm. 3.5.2 we conclude that $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(K_\chi)$ is topologically irreducible if $-(c_1 - c_2) \notin \mathbb{Z}_{\geq 0}$. This was shown also in [KS], cf. Thm. 3.1.6.

4.2. **Restriction of Scalars.** Let $\tilde{L}$ be a finite extension of L and denote by Γ the set of L-embeddings of $\tilde{L}$ into K. Suppose that the cardinality of Γ is equal to $[\tilde{L} : L]$. Let $\tilde{\mathfrak{G}}$ be a connected split reductive group over $\tilde{L}$, $\tilde{\mathfrak{S}}$ a maximal split torus and $\tilde{\mathfrak{P}}$ a Borel subgroup containing $\tilde{\mathfrak{S}}$. Then we consider the reductive group $\mathfrak{G} = \mathrm{Res}_L^{\tilde{L}}\,\tilde{\mathfrak{G}}$ over L. Put $\mathfrak{P} = \mathrm{Res}_L^{\tilde{L}}\,\tilde{\mathfrak{P}}$, $\mathfrak{T} = \mathrm{Res}_L^{\tilde{L}}\,\tilde{\mathfrak{S}}$, and

$$\mathbf{G} = \mathfrak{G}(L) = \tilde{\mathfrak{G}}(\tilde{L})\,,\ \mathbf{P} = \mathfrak{P}(L) = \tilde{\mathfrak{P}}(\tilde{L})\,,\ \mathbf{T} = \mathfrak{T}(L) = \tilde{\mathfrak{S}}(\tilde{L})\,.$$

Consider a locally L-analytic character $\chi : \mathbf{P} \to K^*$ which factors through $\mathbf{T}$, and let $d\chi \in \mathrm{Hom}_L(\mathfrak{t}, K)$ be the derivative of χ. Because $\mathfrak{t} = \mathfrak{s} \otimes_L \tilde{L}$ and by our assumption on K we have a canonical K-linear isomorphism

$$\mathrm{Hom}_L(\mathfrak{t}, K) = \mathrm{Hom}_L(\mathfrak{s}, K) \otimes_K \mathrm{Hom}_L(\tilde{L}, K)\,.$$

The set Γ is a basis for the set of L-vector space homomorphism $\mathrm{Hom}_L(\tilde{L}, K)$, and so we can write

$$d\chi = \sum_{\sigma \in \Gamma} d\chi_\sigma \otimes \sigma\,,$$

with uniquely determined L-linear maps $d\chi_\sigma : \mathfrak{s} \to K$.

The proof of the following assertion is an easy exercise:

Proposition 4.2.1. *(i) The map*

$$\tilde{L} \otimes_L K \longrightarrow \prod_{\sigma \in \Gamma} K\,,\ \xi \otimes \zeta \mapsto (\sigma(\xi)\zeta)_\sigma\,,$$

induces an isomorphism $U(\mathfrak{g}) \otimes_L K \simeq \prod_{\sigma\in\Gamma} U(\tilde{\mathfrak{g}}_K)$ *such that the* $U(\mathfrak{g}) \otimes_L K$*-module* $U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} K_{-d\chi}$ *becomes isomorphic to the direct sum of the modules*

$$U(\tilde{\mathfrak{g}}_K) \otimes_{U(\tilde{\mathfrak{p}}_K)} K_{-d\chi_\sigma}\,.$$

(ii) The $U(\mathfrak{g}) \otimes_L K$*-module* $U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} K_{-d\chi}$ *is irreducible if and only if for every* $\sigma \in \Gamma$ *the* $U(\tilde{\mathfrak{g}})$*-module* $U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{p}})} K_{-d\chi_\sigma}$ *is irreducible. If the latter condition is fulfilled, then* $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(K_\chi)$ *is a topologically irreducible representation.*

Example 4.2.2. With the notation introduced above we let $\tilde{\mathfrak{g}} = \mathrm{Lie}(\mathrm{GL}_{2,\tilde{L}})$ and hence we consider $\mathbf{G} = \mathrm{GL}_2(\tilde{L})$ as a locally L-analytic group. Let χ be a locally L-analytic character of

$$\mathbf{T} = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mid t_1, t_2 \in \tilde{L}^* \right\}.$$

For $t_1, t_2 \in \tilde{L}^*$ sufficiently close to 1 we can write

$$\chi \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \prod_{\sigma\in\Gamma} \sigma(t_1)^{c_{1,\sigma}} \sigma(t_2)^{c_{2,\sigma}}\,.$$

Then the representation $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(K_\chi)$ is topologically irreducible if for all $\sigma \in \Gamma$:

$$-(c_{1,\sigma} - c_{2,\sigma}) \notin \mathbb{Z}_{\geq 0}\,.$$

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