A justification for the thin film approximation of Stokes flow with surface tension

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Abstract

In the free boundary problem of Stokes flow driven by surface tension, we pass to the limit of small layer thickness. It is rigorously shown that in this limit the evolution is given by the well-known thin film equation. The main techniques are appropriate scaling and uniform energy estimates in Sobolev spaces of sufficiently high order, based on parabolicity.

Keywords: *Free boundary motion, degenerate parabolic evolution, lubrication approximation*

MSC: 35R35, 76D07

1. Introduction

In fluid flow problems where viscous forces dominate inertia, i.e. where the Reynolds number is negligibly small, the Navier-Stokes equations simplify to the linear Stokes equations. When instationary free boundary motion driven solely by surface tension is considered, the type of the problem changes significantly: while in the general case a coupled evolution of the bulk fluid velocity and the boundary of the fluid has to be considered ([3, 14], e.a.), in the Stokes flow problem a pure domain evolution results, i.e. at any instant of time the motion of the fluid domain is determined completely by its shape. A more detailed investigation of the conditions under which this simplification is applicable and an analysis of its role by means of formal asymptotics is given in [13]. A rigorous justification has been given by Solonnikov [16] who investigated the difference between the domain evolutions resulting from the complete and the quasistationary models. This difference is shown to consist of two parts: One represents an "exponential boundary layer" near $t = 0$, and the other vanishes uniformly in t (in appropriate norms) as the Reynolds number approaches zero.

During the last decades, the problem of Stokes flow driven by surface tension has been investigated by a variety of analytic methods. It is a common feature of them all that the problem is transformed to a fixed domain by a time dependent diffeomorphism and a nonlinear, nonlocal evolution equation for this diffeomorphism is derived.

More specifically, in the two-dimensional case, complex function theory has been applied to describe the evolution of the fluid domain by a time-dependent conformal map (e.g. [1, 2, 4, 10, 11]). This method allows the explicit construction of families of solutions, the investigation of the stability of equilibria, and the derivation of short-time existence and uniqueness results, albeit restricted to spaces of analytic functions.

To derive well-posedness and stability results for arbitrary dimension in spaces of functions of finite smoothness, one considers perturbations of the boundary of a fixed reference domain, described by a scalar function on this boundary. The moving boundary problem is then translated into a Cauchy problem for this perturbation function. This Cauchy problem is formulated in an appropriate function space on the reference manifold and can be investigated by various methods:

- Derivation of energy estimates in Sobolev spaces of sufficiently high order [9]. The "regularity gap" between these spaces and the weaker norms in which there is a natural energy estimate for the linearization is closed by the application of a generalized chain rule for the nonlinear operator which expresses certain invariance properties of the underlying boundary value problem. Well-posedness as well as stability of the equilibria follow then from a standard procedure via Galerkin approximations.
- Introduction of Lagrangian coordinates, derivation of a fixed point problem and proving its solvability by a contraction argument in Hölder spaces [15].
- Proof of a maximal regularity property for the linearized problem in little Hölder spaces [6]. Both well-posedness of the problem and the smoothing property of the evolution (even up to analyticity) follows then rather straightforwardly from abstract arguments.
- Explicit series expansion with respect to a parameter representing the smallness of the perturbation. [7, 8]

All these approaches use parabolic estimates for the linearized evolution operator. However, these estimates degenerate in the thin film limit, when the the first order evolution operator degenerates to an operator of order 4.

It is the aim of the present paper to show that the first approach is flexible enough to be applied in the treatment of this limit. Thereby, a strict justification of the thin film equation as an approximation of Stokes flow with surface tension will be obtained.

To be more precise, we consider the Stokes flow problem in a layer geometry over a fixed plane horizontal bottom and assume periodicity with respect to all horizontal coordinates. Being interested in thin films, we assume that the height h of the layer is in the order of $\varepsilon \ll 1$. By rescaling, we obtain an ε-dependent problem, given in (2.3), (2.6) below, on a domain whose height $\widetilde{h} = \varepsilon^{-1}h$ is in the order of 1 (see Sect. 2). It is well known that formal calculations yield the so-called thin film equation (2.8) as limit evolution for $\varepsilon \to 0$. This approach is also known as lubrication approximation (see e.g. [12], Ch. 4)

Roughly speaking, our task here is to control the difference between the solution of

(2.3),(2.6) and the solution of (2.8) with equal initial conditions and to show that this difference vanishes in the limit $\varepsilon \to 0$. (In a certain sense, this is comparable to the approach in [16].) The precise main result is given in Theorem 2.2. Accordingly, the present paper is rather an investigation of Stokes flow than of the thin film equation. In particular, we assume strict positivity of h, hence degeneration of (2.8) is excluded.

This paper is organized as follows: In Section 2, we state the moving boundary problem precisely and perform the rescaling. We sketch a formal derivation of the thin film equation and formulate the main result. Section 3 is devoted to preliminary estimates concerning a scaled Laplacian and a scaled version of Korn's inequality. The scaled Stokes equations and the dependence of their solution on the domain perturbations are discussed in Section 4. In both sections, ε-dependent norms are used. Moreover, a guiding idea here is the use of two types of estimates: those in which the regularity is optimal but the constants blow up as $\varepsilon \to 0$, and those with uniform constants (or slower blowup) but with a loss of regularity. The proof depends critically on the interplay between these.

In Section 5 we introduce series expansions with respect to ε and justify them by giving estimates for the remainder terms. Finally, using the previous results, Section 6 provides the energy estimates and the proof of the main result.

We remark that we have not strived for optimality with respect to the smoothness demands as this is not a core issue in our context and we did not want to obscure the presentation by additional technicalities.

Furthermore, we want to point out that the related problem of thin film motion driven by gravity instead of surface tension appears to be more complicated, although the formal calculations are analogue and the resulting formal limit evolution is very similar. This is due to the lack of parabolicity of the nonlocal evolution for $\varepsilon > 0$ in the gravity driven case.

2. Statement of the problem and main result

We consider the problem of Stokes flow driven by surface tension in a (thin) layer of incompressible fluid over a fixed m-dimensional plane horizontal bottom. As usual, we adopt the no-slip boundary condition at the bottom. To keep the technicalities simple we assume spatial periodicity in all horizontal coordinates. Moreover, we normalize the surface tension coefficient to 1. The time dependent fluid domain $\Omega(t) \subset \mathbb{T}^m \times \mathbb{R}$ is then given as

$$\Omega(t) := \big\{(x, y) \mid x \in \mathbb{T}^m, 0 < y < h(x, t)\big\},$$

with lower fixed and upper free surface part

$$\Gamma_0 := \mathbb{T}^m \times \{0\} \quad \text{and} \quad \Gamma(t) := \big\{\big(x, h(x, t)\big) \mid x \in \mathbb{T}^m\big\},$$

where $h = h(x, t)$ is a positive function describing the height of the layer and $\mathbb{T}^m := \mathbb{R}^m/\mathbb{Z}^m$ is the m-dimensional torus identified with the unit cube of $\mathbb{R}^m$ in the usual manner. If we denote the velocity and pressure fields inside $\Omega(t)$ by u and p, the problem we consider can be written in the following form:

$$\left.\begin{aligned} -\Delta u + \nabla p &= 0 && \text{in } \Omega(t), \\ \operatorname{div} u &= 0 && \text{in } \Omega(t), \\ T(u,p)n &= \kappa n && \text{on } \Gamma(t), \\ u &= 0 && \text{on } \Gamma_0, \\ \partial_t h - u \cdot n\sqrt{1+|\nabla_x h|^2} &= 0 && \text{on } \Gamma(t). \end{aligned}\right\} \tag{2.1}$$

Here

$$T(u,p) = [T_{ij}(u,p)] := [(\partial_i u_j + \partial_j u_i) - p\delta_{ij}], \quad i,j = 1,\dots,m+1$$

is the stress tensor,

$$\kappa = \operatorname{div}_x \left(\nabla_x h \,/\, \sqrt{1+|\nabla_x h|^2} \right),$$

denotes the (m-fold) mean curvature of $\Gamma(t)$ (negative if $\Omega(t)$ is convex) and

$$n = (-\nabla_x h,\, 1)^\top / \sqrt{1+|\nabla_x h|^2}$$

is the outer unit normal to $\Omega(t)$. Here and in the sequel, the index x on differential operators is used to indicate that the operator is to be applied with respect to the coordinates $x \in \mathbb{T}^m$ only. Eq. $(2.1)_5$ is the usual kinematic boundary condition expressing the fact that while moving with velocity $u\big(x(t), h(x(t),t)\big)$, a particle at $\Gamma(t)$ will remain at the moving boundary for all time.

For the mathematical treatment of this evolution problem it is a convenient feature that by solving fixed-time boundary value problems $(2.1)_1$-$(2.1)_4$ in dependence of known $h(t)$, i.e. known domains $\Omega(t)$, the kinematic boundary condition $(2.1)_5$ describes the evolution of the free boundary $\Gamma(t)$ by a scalar nonlocal evolution equation

$$\dot{h} = \mathscr{F}(h). \tag{2.2}$$

It turns out that $\mathscr{F}$ acts as a smooth, nonlinear operator of first order

$$\mathscr{F} \in C^\infty\big(H^s_+(\mathbb{T}^m), H^{s-1}(\mathbb{T}^m)\big), \quad s \geq s_0,$$

where H^s denote the usual L^2-based Sobolev spaces, H^s_+ the cone of strictly positive functions in H^s and $s_0 = s_0(m)$ is a sufficiently large number. Up to terms with better regularity the linearization $\mathscr{F}'(h)$ is given by a first-order elliptic operator plus a first-order differential operator, hence the evolution equation (2.2) is a parabolic one. Concerning the solvability of the initial value problem for (2.2) we have the following result which can be proved along the lines given (for a different geometry) in [9] for (i) and (ii) and [6] for (iii):

Theorem 2.1. *Let $s \geq s_0$ and $h_0 \in H^s_+(\mathbb{T}^m)$. Then we have:*

(i) (Existence and uniqueness)
There exists a unique solution

$$h \in C\big([0,T_*), H^s_+(\mathbb{T}^m)\big) \cap C^1\big([0,T_*), H^{s-1}_+(\mathbb{T}^m)\big)$$

of (2.2) with $h(0) = h_0$ on a maximal time intervall $[0,T_*)$ with $T_* > 0$ or $T_* = \infty$.

(ii) (Blow up)

If $T_* < +\infty$, then

$$\liminf_{t\to T_*} \min_{x\in\mathbb{T}^m} h(x,t) = 0 \quad \text{or} \quad \limsup_{t\to T_*} \|h\|_{s_0} = +\infty.$$

(iii) (Spatial smoothing)
$h = h(\cdot,t)$ is a (real) analytic function on $\mathbb{T}^m$ for $t \in (0,T_*)$.

In further considerations, being interested in the limit behavior for small h, we rescale our problem by setting

$$y = \varepsilon\widetilde{y}, \quad h = \varepsilon\widetilde{h},$$

where $0 < \varepsilon \ll 1$. Moreover, we write

$$u(x,y) = \widetilde{u}(x,\widetilde{y}) = \big(\widetilde{v}(x,\widetilde{y}), \widetilde{w}(x,\widetilde{y})\big)^\top, \quad p(x,y) = \widetilde{p}(x,\widetilde{y}),$$

where $\widetilde{v}$ and $\widetilde{w}$ have values in $\mathbb{R}^m$ and $\mathbb{R}$, respectively.

This rescaling brings $(2.1)_1$-$(2.1)_4$ in the following form:

$$\left.\begin{aligned}
-\varepsilon^2\Delta_x\widetilde{v} - \partial_{\widetilde{y}\widetilde{y}}\widetilde{v} + \varepsilon^2\nabla_x\widetilde{p} &= 0 && \text{in } \widetilde{\Omega}(t),\\
-\varepsilon^2\Delta_x\widetilde{w} - \partial_{\widetilde{y}\widetilde{y}}\widetilde{w} + \varepsilon\partial_{\widetilde{y}}\widetilde{p} &= 0 && \text{in } \widetilde{\Omega}(t),\\
\varepsilon\mathrm{div}_x\widetilde{v} + \partial_{\widetilde{y}}\widetilde{w} &= 0 && \text{in } \widetilde{\Omega}(t),\\
-\varepsilon^2(\nabla_x\widetilde{v} + (\nabla_x\widetilde{v})^\top)\nabla_x\widetilde{h} + \varepsilon\nabla_x\widetilde{w} + \varepsilon^2\widetilde{p}\nabla_x\widetilde{h} + \partial_{\widetilde{y}}\widetilde{v} &= -\varepsilon^3\widetilde{\kappa}_\varepsilon\nabla_x\widetilde{h} && \text{on } \widetilde{\Gamma}(t),\\
-\varepsilon(\varepsilon\nabla_x\widetilde{w} + \partial_{\widetilde{y}}\widetilde{v})\cdot\nabla_x\widetilde{h} + 2\partial_{\widetilde{y}}\widetilde{w} - \varepsilon\widetilde{p} &= \varepsilon^2\widetilde{\kappa}_\varepsilon && \text{on } \widetilde{\Gamma}(t),\\
\widetilde{v} &= 0 && \text{on } \Gamma_0,\\
\widetilde{w} &= 0 && \text{on } \Gamma_0
\end{aligned}\right\} \tag{2.3}$$

with

$$\widetilde{\Omega}(t) := \big\{(x,\widetilde{y}) \mid 0 < \widetilde{y} < \widetilde{h}(x,t)\big\}, \quad \widetilde{\Gamma}(t) := \big\{(x,\widetilde{h}(x,t)) \mid x \in \mathbb{T}^m\big\},$$
$$\widetilde{\kappa}_\varepsilon := \mathrm{div}_x\left(\nabla_x\widetilde{h}\Big/\sqrt{1+\varepsilon^2|\nabla_x\widetilde{h}|^2}\right).$$

The following introductory calculations are purely formal. Consider $(2.3)_1$–$(2.3)_7$ for given fixed t and $\widetilde{h}$. If one inserts power series expansions

$$(\widetilde{v},\widetilde{w},\widetilde{p}) = \sum_{k\geq 0}(v_k,w_k,p_k)\varepsilon^k,$$

one obtains by comparison of coefficients

$$\widetilde{v} = \varepsilon^3 v_3 + O(\varepsilon^4), \quad \widetilde{w} = \varepsilon^4 w_4 + O(\varepsilon^5), \quad \widetilde{p} = \varepsilon p_1 + O(\varepsilon^2),$$

where

$$v_3(x,\widetilde{y}) = -\nabla_x\big(\Delta_x\widetilde{h}\big)(x)\left(\frac{y^2}{2} - \widetilde{h}(x)\widetilde{y}\right), \tag{2.4}$$

$$w_4(x,\widetilde{y}) = \Delta_x^2\widetilde{h}(x)\left(\frac{y^3}{6} - \widetilde{h}(x)\frac{y^2}{2}\right) - \nabla_x\big(\Delta_x h\big)(x)\cdot\nabla_x\widetilde{h}(x)\frac{y^2}{2}, \tag{2.5}$$

$$p_1(x,\widetilde{y}) = -\Delta_x\widetilde{h}(x)$$

with the time argument suppressed for the sake of brevity. Thus, according to $(2.1)_5$,

$$\partial_t\widetilde{h} = -\widetilde{v}\cdot\nabla_x\widetilde{h} + \varepsilon^{-1}\widetilde{w} = O(\varepsilon^3),$$

which suggests a rescaling of time by $t = \varepsilon^{-3}\widetilde{t}$. This yields

$$\partial_{\widetilde{t}}\widetilde{h} = \varepsilon^{-4}(-\varepsilon\widetilde{v}\cdot\nabla_x\widetilde{h} + \widetilde{w}) = \varepsilon^{-4}\widetilde{n}_\varepsilon\cdot\widetilde{u} \tag{2.6}$$

where $\widetilde{n}_\varepsilon := (-\varepsilon\nabla_x\widetilde{h}, 1)^\top$.

In what follows, all tildes will be suppressed in the notation. Note that due to (2.4)-(2.6)

$$\partial_t h = -\frac{1}{3}\mathrm{div}_x(h^3\nabla_x\Delta_x h) + O(\varepsilon). \tag{2.7}$$

Setting $\varepsilon = 0$ in the last equation we obtain the well-known "thin film equation" solved by $h = h_0$:

$$\partial_t h_0 = -\frac{1}{3}\mathrm{div}_x(h_0^3\nabla_x\Delta_x h_0) \tag{2.8}$$

In this manner, starting from h_0 and considering higher order powers of ε, we obtain an expansion

$$h(t) = h_0(t) + \varepsilon h_1(t) + \varepsilon^2 h_2(t) + \dots$$

for the solution of (2.7); details of this construction, which justify these formal calculations, are given in Section 5. Now we are in position to formulate our main result precisely; a somewhat more elaborate version together with a complete proof is contained in Proposition 6.6.

Theorem 2.2. *Let $n \in \mathbb{N}$, $T > 0$ and $s \geq s_{min}(m)$ be given, assume $\sigma = \sigma(n, m, s)$ sufficiently large and let*

$$h_0 \in C\big([0,T], H_+^\sigma(\mathbb{T}^m)\big) \cap C^1\big([0,T], H^{\sigma-4}(\mathbb{T}^m)\big)$$

be any solution of the thin film equation (2.8)*. Then for $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$ sufficiently small, there exists*

$$h_\varepsilon \in C\big([0,T], H_+^s(\mathbb{T}^m)\big) \cap C^1\big([0,T], H^{s-1}(\mathbb{T}^m)\big)$$

satisfying (together with appropriate $u_\varepsilon, v_\varepsilon, p_\varepsilon$) the system (2.3), (2.6) *and* $h_\varepsilon(0) = h_0(0)$*. Moreover,*

$$\|h_\varepsilon(t) - h_0(t) - \varepsilon h_1(t) - \ldots - \varepsilon^{n-1}h_{n-1}(t)\|_s \leq C\varepsilon^n, \quad t \in [0,T].$$

The constants C, ε_0 depend on n, s, m, T and h_0.

Remark 2.3. (i) One remarkable point here (apart from the convergence result) is the fact that the theorem provides a lower bound for the existence time of the solutions to (2.3), (2.6) (independent of ε). Recall, moreover, that a solution of (2.3), (2.6) in a time scale $O(1)$ corresponds to a solution of the unscaled problem (2.1) in a time scale $O(\varepsilon^{-3})$.
(ii) Usually, the formal derivation of (2.8) is done using the different scaling

$$u = (\varepsilon^3\widehat{v}, \varepsilon^4\widehat{w})^\top, \quad p = \varepsilon\widehat{p},$$

which is suggested by the asymptotics given above. In that scaling, (2.1) gets a form in which $\varepsilon = 0$ can be filled in directly and (2.8) results. However, we prefer the form (2.3) because of its closer resemblance to the original Stokes system. This resemblance will be useful in our analysis.

3. Scaled estimates

Define the domain

$$\Omega := \mathbb{T}^m \times (0,1) = \big\{(x,y) \in \mathbb{R}^{m+1} \mid x \in \mathbb{T}^m, y \in (0,1)\big\}$$

with upper and lower boundary

$$\Gamma := \mathbb{T}^m \times \{1\}, \quad \Gamma_0 := \mathbb{T}^m \times \{0\}.$$

For $M = \Omega$ or $M = \Gamma, \Gamma_0$ let $H^s(M,\mathbb{R})$, $H^s(M,\mathbb{R}^n)$ be the usual L^2-based Sobolev spaces of order s with values in $\mathbb{R}$ or $\mathbb{R}^n$; the corresponding norms will be denoted by $\|\cdot\|_s^M$; the upper index M is dropped when no confusion seems likely. Depending on the context, we will use the notations x_{m+1} an y as well as ∂_{m+1} and ∂_y synonymously. Summation over indices occurring twice in a product has to be carried out from 1 to $m+1$, unless indicated otherwise.

Considering functions $f \in H^s(\Gamma)$ as periodic functions on $\mathbb{T}^m$, we can write these functions in terms of their Fourier series

$$f(x,1) = \sum_{n\in\mathbb{Z}^m} f_n e^{2\pi i\, n\cdot x}, \quad x \in \mathbb{T}^m \tag{3.1}$$

and have in the sense of norm equivalence

$$\|f\|_s^\Gamma \sim \Big(\sum_{n\in\mathbb{Z}^m} \big(1+|n|^{2s}\big)|f_n|^2\Big)^{1/2}. \tag{3.2}$$

Correspondingly, we write functions $u \in H^s(\Omega)$ in terms of Fourier series with respect to the first m arguments, i.e.

$$u(x,y) = \sum_{n\in\mathbb{Z}^m} u_n(y) e^{2\pi i n\cdot x}, \quad (x,y) \in \Omega \tag{3.3}$$

and note

$$\|u\|_s^\Omega \sim \Big(\sum_{n\in\mathbb{Z}^m} \big(\|u_n\|_s^2 + |n|^{2s}\|u_n\|_0^2\big)\Big)^{1/2} \tag{3.4}$$

With a small parameter $\varepsilon > 0$ we define the norms

$$\|u\|_{s,\varepsilon} := \|u\|_{s-1} + \|\partial_y u\|_{s-1} + \varepsilon\|u\|_s$$

for $s \geq 1$. In terms of these norms, a simple scaled version of the usual trace and extension theorem for H^s-functions reads

Lemma 3.1. *(Scaled trace and extension theorems)*

(i) Let T denote the trace operator from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$. Then for any $u \in H^1(\Omega)$ and any $\varepsilon \in (0,1)$ we have

$$\|Tu\|_0^\Gamma + \sqrt{\varepsilon}\|Tu\|_{1/2}^\Gamma \leq C\|u\|_{1,\varepsilon}^\Omega \tag{3.5}$$

with C independent of u and ε.

(ii) For any fixed $\varepsilon \in (0,1]$ there exist operators

$$E_1 \in \mathscr{L}\big(H^{1/2}(\Gamma), H^1(\Omega)\big), \quad E_2 \in \mathscr{L}\big(H^{1/2}(\Gamma), H^2(\Omega)\big)$$

such that $E_1u|_\Gamma = u$ *and* $E_2u|_\Gamma = 0$, $\partial_y E_2 u|_\Gamma = u$, *and*

$$\|E_1u\|_{1,\varepsilon}^{\Omega}, \varepsilon\|E_2u\|_{2,\varepsilon}^{\Omega}, \|\partial_y E_2 u\|_{1,\varepsilon}^{\Omega} \le C\big(\|u\|_0^\Gamma + \sqrt{\varepsilon}\|u\|_{1/2}^\Gamma\big) \tag{3.6}$$

for all $u \in H^{1/2}(\Gamma)$ *with* C *independent of* u *and* ε.

Proof. (i) It is sufficient to show the assertion for smooth functions u. Write u in terms of its Fourier series according to (3.3). Let f be any (complex) C^1-function on $[0,1]$ and choose $\xi \in [0,1]$ with $|f(y)| \ge |f(\xi)|$ for all $y \in [0,1]$. Then

$$|f(1)|^2 = \int_\xi^1 \big(f(y)\bar{f}(y)\big)'\,dy + |f(\xi)|^2 \le 2\|f\|_{L^2(0,1)}\|f'\|_{L^2(0,1)} + \|f\|_{L^2(0,1)}^2,$$

and consequently for any $\delta \ge 0$

$$(1+\delta)|f(1)|^2 \le 3\big((1+\delta^2)\|f\|_{L^2(0,1)}^2 + \|f'\|_{L^2(0,1)}^2\big).$$

Setting now $f = u_n$, $\delta = \varepsilon n$ and summing over $n \in \mathbb{Z}^m$, we obtain the estimate (3.5) in view of (3.2), (3.4).
(ii) Defining the extension operators E_1, E_2 by

$$(E_1u)(x,y) := \sum_{n\in\mathbb{Z}^m} u_n e^{\varepsilon|n|(y-1)} e^{2\pi i n\cdot x}, \quad (x,y) \in \Omega,$$

and

$$(E_2u)(x,y) := \sum_{n\in\mathbb{Z}^m} (y-1) u_n e^{\varepsilon|n|(y-1)} e^{2\pi i n\cdot x},$$

where u_n, $n \in \mathbb{Z}^m$ are the coefficients of the Fourier series of any $u \in H^{1/2}(\Gamma)$, it is easy to check that E_1, E_2 satisfy the estimates (3.6). $\square$

As a consequence, we note for later use the boundary integral estimate

$$\left|\int_\Gamma u\phi\,d\Gamma\right| \le C\|u\|_\delta^\Gamma \|\phi\|_{-\delta}^\Gamma \le C\varepsilon^{-\delta}\|u\|_{1,\varepsilon}^\Omega \|\phi\|_{-\delta}^\Gamma \tag{3.7}$$

holding for $\delta \in [0,1/2]$, $u \in H^1(\Omega)$, $\phi \in L^2(\Gamma)$. For coefficients $a = (a_1, \dots, a_{m+1}) \in C^2(\Omega, \mathbb{R}^{m+1})$ such that

$$\|a\|_{C^2} \le M, \quad a_{m+1}(x,y) \ge \lambda \text{ for all } (x,y) \in \Omega \tag{3.8}$$

with given positive constants M, λ and

$$\partial_i(a_{m+1}^{-1}) + \partial_{m+1}(a_i/a_{m+1}) = 0, \qquad i = 1, \dots, m, \tag{3.9}$$

we define differential operators D_i by

$$D_i := D_{i,\varepsilon,a} := \begin{cases} \varepsilon(\partial_i + a_i\partial_{m+1}) & \text{for } i = 1, \dots, m, \\ a_{m+1}\partial_{m+1} & \text{for } i = m+1. \end{cases} \tag{3.10}$$

The demand (3.9) expresses the fact that in our application the operators D_i are transformed versions of operators with constant coefficients. As a consequence, we get from integration by parts

$$\int_\Omega a_{m+1}^{-1} D_i w\, v\,dx = -\int_\Omega a_{m+1}^{-1} w D_i v\,dx + \int_\Gamma \tfrac{a_{i,\varepsilon}}{a_{m+1}} vw\,d\Gamma - \textstyle\int_{\Gamma_0} \tfrac{a_{i,\varepsilon}}{a_{m+1}} vw\,d\Gamma_0. \tag{3.11}$$

Moreover, to economize our notation we define

$$\begin{aligned}
\partial_{i,\varepsilon} &:= \begin{cases} \varepsilon\partial_i & i = 1\dots,m \\ \partial_{m+1} & i = m+1, \end{cases} \\
\nabla_\varepsilon &:= (\partial_{1,\varepsilon},\dots,\partial_{m+1,\varepsilon})^\top \\
a_{i,\varepsilon} &:= \begin{cases} \varepsilon a_i & i = 1\dots,m \\ a_{m+1} & i = m+1. \end{cases}
\end{aligned} \tag{3.12}$$

For later use we note the commutator

$$[D_i,\partial_l] = -\partial_l a_{i,\varepsilon}\partial_{m+1}. \tag{3.13}$$

We will need two version of Poincaré's inequality under different conditions, namely,

$$\|u\|_0 \le C\|\nabla_\varepsilon u\|_0 \quad \text{for } u \in H^1(\Omega),\ u|_{\Gamma_0} = 0, \tag{3.14}$$

$$\|u\|_0 \le C\varepsilon^{-1}\|\nabla_\varepsilon u\|_0 \quad \text{for } u \in H^1(\Omega),\ \int_\Omega u\,dx = 0. \tag{3.15}$$

The second one is standard while the first one follows from the fact that due to the boundary condition, u can be recovered from $\partial_y u$ by integration in the unscaled direction only.

The different behavior of the constants in (3.14) and (3.15) for $\varepsilon \downarrow 0$ suggests a corresponding difference in the a priori estimates for the generalized Laplacian D_iD_i. This is verified in the next lemmas.

Lemma 3.2. *(H^1 -a priori estimates for D_iD_i)*
There are constants $\varepsilon_0 > 0$ and $C > 0$ depending only on M and λ such that for all $\varepsilon \in (0,\varepsilon_0)$, $u \in H^2(\Omega)$, $f \in L^2(\Omega)$, $f_i \in H^1(\Omega)$, $i = 1,\dots,m+1$, with

$$\left.\begin{aligned} -D_iD_iu &= f + \partial_{i,\varepsilon}f_i \ \text{ in } \Omega, \\ -a_{i,\varepsilon}D_iu &= f_{m+1} \qquad \text{on } \Gamma \end{aligned}\right\}$$

and

(i) *(mixed boundary conditions)*

$$u = 0 \quad \text{on } \Gamma_0$$

or

(ii) *(Neumann boundary conditions)*

$$\begin{aligned} -a_{i,\varepsilon}D_iu &= f_{m+1} \ \text{ on } \Gamma_0, \\ \textstyle\int_\Omega u\,dx &= 0 \end{aligned}$$

we have

(i) $$\|u\|_{1,\varepsilon}^\Omega \le C\left(\|f\|_0^\Omega + \textstyle\sum_{i=1}^{m+1}\|f_i\|_0^\Omega\right)$$

or

(ii) $$\varepsilon\|u\|_0^\Omega + \|\nabla_\varepsilon u\|_0^\Omega \le C\left(\varepsilon^{-1}\|f\|_0^\Omega + \textstyle\sum_{i=1}^{m}\|f_i\|_0^\Omega + \varepsilon^{-1}\|f_{m+1}\|_0^\Omega\right),$$

respectively.

Proof. Applying (3.11), integration by parts, and the boundary conditions we get

$$\begin{aligned}
&\int_\Omega a_{m+1}^{-1} D_i u D_i u\,dx = \int_\Omega a_{m+1}^{-1}(fu - f_i \partial_{i,\varepsilon} u)\,dx - \int_\Omega \partial_{i,\varepsilon}(a_{m+1}^{-1}) f_i u\,dx \\
&\le\ C \begin{cases} \left(\|f\|_0 + \sum_{i=1}^{m+1} \|f_i\|_0\right) \|u\|_{1,\varepsilon} & \text{in case (i),} \\ \left(\varepsilon^{-1}\|f\|_0 + \sum_{i=1}^{m} \|f_i\|_0 + \varepsilon^{-1}\|f_{m+1}\|_0\right)(\varepsilon\|u\|_0 + \|\nabla_\varepsilon u\|_0) & \text{in case (ii).} \end{cases}
\end{aligned} \tag{3.16}$$

Furthermore, straightforward calculation yields

$$D_i u D_i u \ge c\partial_{i,\varepsilon} u \partial_{i,\varepsilon} u - C\varepsilon^2(\partial_{m+1} u)^2 \ge c\partial_{i,\varepsilon} u \partial_{i,\varepsilon} u$$

with $c = c(\lambda, M) > 0$ and ε sufficiently small. By (3.14) and (3.15), respectively, we get from this

$$\begin{aligned}
&\int_\Omega a_{m+1}^{-1} D_i u D_i u\,dx \ge c \int_\Omega D_i u D_i u\,dx \ge c{\|\nabla_\varepsilon u\|_0}^2 \ge c\|u\|_{1,\varepsilon}^2 \\
&\ge\ c \begin{cases} \|u\|_{1,\varepsilon}^2 & \text{in case (i),} \\ (\varepsilon\|u\|_0 + \|\nabla_\varepsilon u\|_0)^2 & \text{in case (ii).} \end{cases}
\end{aligned}$$

Together with (3.16), this implies the result. □

Lemma 3.3. *(H^2-a priori estimate for $D_i D_i$)*

There are constants $\varepsilon_0 > 0$ and $C > 0$ depending only on M and λ such that if $\varepsilon \in (0, \varepsilon_0)$, $u \in H^2(\Omega)$,

$$\left.\begin{aligned} -D_i D_i u &= f \text{ in } \Omega, \\ -a_{i,\varepsilon} D_i u &= 0 \text{ on } \Gamma \end{aligned}\right\} \tag{3.17}$$

and

(i) *(mixed boundary conditions)*

$$u = 0 \qquad \textit{on } \Gamma_0$$

or

(ii) *(Neumann boundary conditions)*

$$\begin{aligned} -a_{i,\varepsilon} D_i u &= 0 \text{ on } \Gamma_0, \\ \textstyle\int_\Omega u\,dx &= 0 \end{aligned}$$

we have

(i) $$\|u\|_{1,\varepsilon}^\Omega + \textstyle\sum_{i=1}^{m+1} \|\partial_{i,\varepsilon} u\|_{1,\varepsilon}^\Omega \le C\|f\|_0^\Omega$$

or

(ii) $$\varepsilon\|u\|_0^\Omega + \textstyle\sum_{i=1}^{m+1} \|\partial_{i,\varepsilon} u\|_{1,\varepsilon}^\Omega \le C\varepsilon^{-1}\|f\|_0^\Omega,$$

respectively.

Proof. The estimates for $\|u\|_0$ and $\|\partial_{i,\varepsilon} u\|_0$ are immediate from Lemma 3.2. It remains to estimate the second derivatives. For $j = 1, \dots, m$ we apply $\partial_{j,\varepsilon}$ to (3.17) to obtain (using (3.13))

$$\left.\begin{array}{rcll} -D_iD_i\partial_{j,\varepsilon}u & = & \partial_{j,\varepsilon}f-\varepsilon([D_i,\partial_j]u+D_i[D_i,\partial_j]u) & \\ & = & \partial_{j,\varepsilon}f+\varepsilon\big(\partial_j a_{i,\varepsilon}\partial_{m+1}D_iu & \\ & & +\sum_{i=1}^m\varepsilon\partial_i(\partial_j a_{i,\varepsilon}\partial_{m+1}u)+a_{i,\varepsilon}\partial_{m+1}(\partial_j a_{i,\varepsilon}\partial_{m+1}u)\big) & \\ & = & \widetilde{f}+\partial_{i,\varepsilon}\widetilde{f}_i & \text{in } \Omega, \\ -a_{i,\varepsilon}D_i\partial_{j,\varepsilon}u & = & -\varepsilon[a_{i,\varepsilon}D_i,\partial_{j,\varepsilon}]u=\widetilde{f}_{m+1} & \text{on } \Gamma, \end{array}\right\}$$

where

$$\begin{aligned} \widetilde{f} &:= \varepsilon\big(-\partial_{j,m+1}a_{i,\varepsilon}D_iu-\partial_{m+1}a_{i,\varepsilon}\partial_j a_{i,\varepsilon}\partial_{m+1}u\big), \\ \widetilde{f}_i &:= \varepsilon\partial_j a_{i,\varepsilon}\partial_{m+1}u+\delta_{ij}f, \quad i=1,\dots,m, \\ \widetilde{f}_{m+1} &:= \varepsilon\big(\partial_j a_{i,\varepsilon}D_iu+a_{i,\varepsilon}\partial_j a_{i,\varepsilon}\partial_{m+1}u\big). \end{aligned}$$

Furthermore, $\int_\Omega \partial_{j,\varepsilon}u\,dx=0$, in case (i)

$$\partial_{j,\varepsilon}u=0 \quad \text{on } \Gamma_0,$$

and in case (ii)

$$-a_{i,\varepsilon}D_i\partial_{j,\varepsilon}u=\widetilde{f}_{m+1} \quad \text{on } \Gamma_0.$$

By Lemma 3.2, we get in case (i)

$$\|\partial_{j,\varepsilon}u\|_{1,\varepsilon}\le C\big(\|\widetilde{f}\|_0+\textstyle\sum_{i=1}^{m+1}\|\widetilde{f}_i\|_0\big)\le C\big(\|f\|_0+\varepsilon\|u\|_{1,\varepsilon}\big)\le C\|f\|_0$$

and in case (ii)

$$\begin{aligned} \|\nabla_\varepsilon\partial_{j,\varepsilon}u\|_0 &\le C\left(\varepsilon^{-1}\|\widetilde{f}\|_0+\textstyle\sum_{i=0}^m\|\widetilde{f}_i\|_0+\varepsilon^{-1}\|\widetilde{f}_{m+1}\|_0\right) \\ &\le C\left(\|f\|_0+\|\nabla_\varepsilon u\|_0\right)\le C\varepsilon^{-1}\|f\|_0. \end{aligned}$$

This yields the estimates for all second derivatives except ∂_{m+1}^2u.

Finally, we write $(3.17)_1$ in the form

$$\begin{aligned} \partial_{m+1}^2u &= \left(a_{m+1}^2-\varepsilon^2\textstyle\sum_{i=1}^m\right)^{-1}\Big(-f-\varepsilon^2\textstyle\sum_{i=1}^m(\partial_i^2u+\partial_i(a_i\partial_{m+1}u)+a_i\partial_{i,m+1}u) \\ &\quad -a_{i,\varepsilon}\partial_{m+1}a_{i,\varepsilon}\partial_{m+1}u\Big) \end{aligned}$$

and estimate from this and the previous results

$$\|\partial_{m+1}^2u\|_0\le C\left(\|f\|_0+\|u\|_{1,\varepsilon}+\textstyle\sum_{j=1}^m\|\partial_{j,\varepsilon}u\|_{1,\varepsilon}\right).$$

The results follow now from the previous estimates. □

The natural basic space for the velocity fields in our problem is

$$V:=\big\{v\in H^1(\Omega,\mathbb{R}^{m+1}) \mid v|_{\mathbb{T}^m\times\{0\}}=0\big\}.$$

For such velocity fields we have the following scaled version of Korn's inequality:

Proposition 3.4. *Let $M,\lambda>0$ be given. There are positive constants $C=C(M,\lambda)$, $\varepsilon_0=\varepsilon_0(M,\lambda)$ such that*

$$\|u\|_{1,\varepsilon}^{\Omega\;2}\le C\sum_{i,j=1}^{m+1}\|D_iu_j+D_ju_i\|_0^{\Omega\;2} \tag{3.18}$$

for $\varepsilon \in (0, \varepsilon_0)$ *and* $u \in V$.

Proof. In a first step we consider the special case

$$a_1 = \ldots = a_m \equiv 0, \quad a_{m+1} \equiv 1,$$

i.e. $D_i = \partial_{i,\varepsilon}$ in above notation. This is done by coupling Korn's inequality in the strip

$$\Omega_L := \mathbb{R}^m \times (0,1) = \big\{(x,y) \mid x \in \mathbb{R}^m, y \in (0,1)\big\}$$

with an obvious extension and cut-off technique to handle the periodic case. Then, in a second step, the general case follows by a simple perturbation argument based on the special structure (3.10) of the differential operators.

Step 1: Let

$$V_L := \big\{u \in H^1(\Omega_L, \mathbb{R}^{m+1}) \mid u|_{\mathbb{R}^m \times \{0\}} = 0\big\}.$$

By Theorem 3.3 in [5], Ch. 3, there is a C_0 such that

$$\sum_{i,j=1}^{m+1} \|\partial_i u_j\|^2_{L^2(\Omega_L)} \le C_0 \sum_{i,j=1}^{m+1} \|\partial_i u_j + \partial_j u_i\|^2_{L^2(\Omega_L)} \ \text{ for } \ u \in V_L. \tag{3.19}$$

By ε-scaling w.r.t. the first m variables, this clearly implies

$$\sum_{i,j=1}^{m+1} \|\partial_{i,\varepsilon} u_j\|^2_{L^2(\Omega_L)} \le C_0 \sum_{i,j=1}^{m+1} \|\partial_{i,\varepsilon} u_j + \partial_{j,\varepsilon} u_i\|^2_{L^2(\Omega_L)} \ \text{ for } \ u \in V_L, \ \varepsilon > 0. \tag{3.20}$$

Now let $u \in V$ with components considered as 1-periodic functions w.r.t. the first m arguments x. Moreover, let $\eta \in C^1(\mathbb{R})$ be a cutoff function such that $\eta \equiv 1$ on $(-\infty, 0]$ and $\eta \equiv 0$ on $[1, \infty)$ and define for $n \in \mathbb{N}$

$$\begin{aligned} u^{(n)}(x,y) &:= u(x,y)\,\eta(|x_1| - n) \cdot \ldots \cdot \eta(|x_m| - n), \\ \Omega_n &:= \big\{(x,y) \in \Omega_L \ \big| \ |x_i| < n, i = 1, \ldots, m\big\}. \end{aligned}$$

Then $u^{(n)} \in V_L$, $\operatorname{supp} u^{(n)} \subset \bar{\Omega}_{n+1}$, and (3.20) applied to $u^{(n)}$ yields

$$\begin{aligned} &\sum_{i,j=1}^{m+1} \Big((2n)^m \|\partial_{i,\varepsilon} u_j\|^2_{L^2(\Omega)} + \|\partial_{i,\varepsilon} u_j^{(n)}\|^2_{L^2(\Omega_{n+1} \setminus \Omega_n)} \Big) \\ &\qquad \le C_0 \sum_{i,j=1}^{m+1} \Big((2n)^m \|\partial_{i,\varepsilon} u_j + \partial_{j,\varepsilon} u_i\|^2_{L^2(\Omega)} + \|\partial_{i,\varepsilon} u_j^{(n)} + \partial_{j,\varepsilon} u_i^{(n)}\|^2_{L^2(\Omega_{n+1} \setminus \Omega_n)} \Big) \end{aligned}$$

Note that due to the periodicity of u we have

$$\|\partial_{i,\varepsilon} u_j^{(n)}\|_{L^2(\Omega_{n+1} \setminus \Omega_n)} \le C \|u\|_{H^1(\Omega_{n+1} \setminus \Omega_n)} \le C' n^{m-1} \|u\|^2_{H^1(\Omega)},$$

and consequently, after division by $(2n)^m$

$$\sum_{i,j=1}^{m+1} \|\partial_{i,\varepsilon} u_j\|^2_{L^2(\Omega)} \le C_0 \sum_{i,j=1}^{m+1} \|\partial_{i,\varepsilon} u_j + \partial_{j,\varepsilon} u_i\|^2_{L^2(\Omega)} + C n^{-1} \|u\|^2_{H^1(\Omega)}.$$

Thus the assertion (3.18) follows from taking $n \to \infty$ together with Poincaré's inequality.

Step 2: First observe that for any $b \in C^1(\overline{\Omega})$ with $\|b\|_{C^1} \le B$, $b \ge b_0 > 0$ and any

$w \in H^1(\Omega)$ with $w(x,0) = 0$ we have

$$\begin{aligned}\|\partial_y w\|_0 &\le C\|b\partial_y w\|_0 \le C(\|\partial_y(bw)\|_0 + \|\partial_y b\, w\|)\\ &\le C(\|\partial_y(bw)\|_0 + \|w\|_0) \le C(\|\partial_y(bw)\|_0 + \|bw\|_0) \le C\|\partial_y(bw)\|_0, \quad (3.21)\end{aligned}$$

where Poincaré's inequality has been used in the last step and $C = C(B, b_0)$.

For $u \in V$ define $v := (u_1, \dots, u_m, a_{m+1}^{-1}u_{m+1}) \in V$. Then by (3.8), (3.10) we have

$$\|\partial_{i,\varepsilon}v_j + \partial_{j,\varepsilon}v_i\|_0^2 \le C\big(\|D_iu_j + D_ju_i\|_0^2 + \varepsilon^2\|u\|_0^2 + \varepsilon^2\|\partial_y u\|_0^2\big) \qquad (3.22)$$

for $i, j = 1, \dots, m+1$ with a constant $C = C(M, \lambda)$. For $i, j = 1, \dots, m$ this follows simply from the identity

$$\partial_{i,\varepsilon}v_j + \partial_{j,\varepsilon}v_i = D_iu_j + D_ju_i - \varepsilon(a_i\partial_y u_j + a_j\partial_y u_i),$$

for $i = 1, \dots, m$ and $j = m+1$ (or the other way round) we get (3.22) from

$$\partial_{i,\varepsilon}v_j + \partial_{j,\varepsilon}v_i = a_{m+1}^{-1}(D_iu_j + D_ju_i) + \varepsilon(\partial_i a_{m+1}^{-1}u_{m+1} - a_{m+1}^{-1}a_i\partial_y u_{m+1}),$$

and, finally, for $i = j = m+1$ we use (3.21) with $b = a_{m+1}$, $w = v_{m+1}$. Further

$$\sum_{i,j=1}^{m+1}\|\partial_{i,\varepsilon}u_j\|_0^2 \le C\sum_{i,j=1}^{m+1}\|\partial_{i,\varepsilon}v_j\|^2 + C\varepsilon^2\|u\|_0^2$$

by (3.21) with $b = a_{m+1}^{-1}$, $w = u_{m+1}$. Thus, applying the estimate from the first step to v and Poincaré's inequality again we obtain

$$\begin{aligned}\sum_{i,j=1}^{m+1}\|\partial_{i,\varepsilon}u_j\|_0^2 &\le C\sum_{i,j=1}^{m+1}\|D_iu_j + D_ju_i\|_0^2 + C\varepsilon^2(\|u\|_0^2 + \|\partial_y u\|_0^2)\\ &\le C\sum_{i,j=1}^{m+1}\|D_iu_j + D_ju_i\|_0^2 + C\varepsilon^2\|\partial_y u\|_0^2,\end{aligned}$$

which implies the assertion (3.18) provided ε suffiently small. □

4. The transformed Stokes system

Fix $s_0 > m/2 + 1$ such that $s_0 + 1/2$ is integer. For $M, \alpha > 0$ and $s \ge s_0$ we define

$$\mathscr{U}(s, M, \alpha) := \big\{h \in H^s(\mathbb{T}^m) \;\big|\; \|h\|_s < M \text{ and } h(x) > \alpha \text{ for } x \in \mathbb{T}^m\big\}. \qquad (4.1)$$

We will fix M, α and abbreviate $\mathscr{U} := \mathscr{U}(s_0, M, \alpha)$.

The spaces $H^{s_0+1/2}(\Omega)$ and $H^{s_0}(\Gamma)$ form Banach algebras, and we have continuous embeddings

$$H^{s_0+1/2}(\Omega) \hookrightarrow C_b^1(\Omega), \quad H^{s_0}(\Gamma) \hookrightarrow C^1(\Gamma).$$

Here and in what follows, $C_b^k(O)$ will denote the usual Banach space of all k times continuously differentiable functions on the open set O whose derivatives op to order k are bounded. We will use the corresponding estimates without explicit mentioning.

For $h \in \mathscr{U}$ we set

$$\Omega_h := \big\{(x, y) \;\big|\; x \in \mathbb{T}^m, y \in (0, h(x))\big\}.$$

It will be necessary to transform our boundary value problem to a fixed domain for which we choose Ω. For this we need a corresponding diffeomorphism. The natural choice $[(x,y)\mapsto(x,h(x)y)]$ would produce a loss of "1/2 order of differentiability" in our Sobolev scale. To avoid this, we use the following construction.

Lemma 4.1. *(Extension of h)*
There is a map

$$[h\mapsto \widetilde{h}]\in\mathcal{L}\left(H^\sigma(\Gamma),H^{\sigma+1/2}(\Omega)\right),\qquad \sigma>0$$

with the following properties:

(i) $\widetilde{h}|_\Gamma=h$, $(\partial_y\widetilde{h})|_\Gamma=0$,

(ii) *If* $h\in\mathscr{U}$ *then* $[(x,y)\mapsto(x,\widetilde{h}y)]\in\mathrm{Diff}(\Omega,\Omega_h)$.

Proof. There is an extension operator

$$\mathcal{T}\in\mathcal{L}\left(H^\sigma(\mathbb{T}^m),H^{\sigma+1/2}(\Omega)\right)$$

such that $(\partial_y\mathcal{T}h)|_\Gamma=0$. As $H^{s_0+1/2}(\Omega)\hookrightarrow C^1(\Omega)$, there is an $\eta\in(0,1)$ such that

$$\mathcal{T}h(x,y)>\frac{3}{4}\alpha,\quad |\partial_y\mathcal{T}h(x,y)|<\frac{\alpha}{4}\qquad\text{as } y\ge\eta \tag{4.2}$$

for $h\in\mathscr{U}$. Pick now $\chi\in C^\infty(0,1)$ with $\chi'\ge0$, $\operatorname{supp}\chi\in(\eta,1)$, and $\chi\equiv1$ near 1. Define

$$\widetilde{h}:=(1-\chi)\delta h_0+\chi\mathcal{T}h,$$

where $h_0:=\int_\Gamma h\,dx$ denotes the average value of h. and $\delta>0$ chosen small enough that

$$\delta h_0<\mathcal{T}h(x,y)\qquad\text{as } y\ge\eta. \tag{4.3}$$

It is clear that $\widetilde{h}$ satisfies (i).

Fix $h\in\mathscr{U}$. For $y\in(0,\eta)$ we have $\widetilde{h}=\delta h_0$, therefore

$$\partial_y(\widetilde{h}y)=\widetilde{h}>0.$$

For $y\in[\eta,1)$, (4.2), (4.3) imply

$$\begin{aligned}\partial_y(\widetilde{h}y)&=\widetilde{h}+(-\chi'\delta h_0+\chi\partial_y\mathcal{T}h+\chi'\mathcal{T}h)y\\&=(1-\chi)\delta h_0+\chi(\mathcal{T}h+y\partial_y\mathcal{T}h)+y\chi'(\mathcal{T}h-\delta h_0)\\&\ge(1-\chi)\delta h_0+\chi\frac{\alpha}{2}\ge\lambda>0.\end{aligned}$$

This implies (ii). □

Set now

$$\begin{aligned}a_i&:=-y\partial_i\widetilde{h}/\partial_y(\widetilde{h}y),\qquad i=1,\dots,m,\\a_{m+1}&:=1/\partial_y(\widetilde{h}y).\end{aligned}\tag{4.4}$$

Note that with this choice, (3.8) is satisfied uniformly for $h\in\mathscr{U}$; this follows by the same arguments used in the proof of Lemma 4.1. Recall the definitions (3.10) and (3.12) of $D_i=D_{i,\varepsilon,a}$ and $\partial_{i,\varepsilon}$. Note that

$$D_iD_j=D_jD_i,\quad i,j=1,\dots,m+1 \tag{4.5}$$

and

$$D_i'\{h_1\}\varphi = -D_i(\tilde{h}_1 y)D_{m+1}\varphi, \quad i = 1, \dots, m+1, \tag{4.6}$$

where the prime indicates the Fréchet derivative with respect to h. Moreover, define

$$\begin{aligned} g &:= (\partial_y(\widetilde{h}y))^2, \\ \nu &:= h^{-1}(-\varepsilon\nabla h, 1), \\ T_{ij}(u,p) &:= D_i u_j + D_j u_i - \delta_{ij} p, \quad i, j = 1, \dots, m+1. \end{aligned}$$

Using the diffeomorphism from Lemma 4.1 we can transform the boundary value problem (2.3) from the domain $\Omega(t) = \Omega_{h(t)}$ to the fixed domain Ω. We abuse notation slightly and continue to use u_j for the pull-back of the components of the velocity field. Moreover, for convenience we rescale the pressure by replacing p by $\varepsilon^{-1}p$.

This results in the transformed Stokes-type system

$$\left.\begin{aligned} -D_i D_i u_j + D_j p &= 0 && \text{in } \Omega, \\ D_i u_i &= 0 && \text{in } \Omega, \\ T_{ij}(u,p)\nu_i &= \varepsilon^2 \kappa_\varepsilon \nu_j && \text{on } \Gamma, \\ u &= 0 && \text{on } \Gamma_0, \end{aligned}\right\} \tag{4.7}$$

which will be investigated in this section. Unless stated otherwise, constants that occur in estimates are independent of $\varepsilon \in (0,1)$ and of $\mathscr{U}$ but may depend on $\mathscr{U}$ itself.

We start with a preliminary result concerning two estimates for the solution of the transformed divergence equation. Similarly to the situation for the Laplacian $D_i D_i$ discussed before, the problem behaves differently for $\varepsilon \downarrow 0$, in dependence on the boundary conditions.

Lemma 4.2. *(Divergence equation)*

(i) *(One-sided boundary condition)*
There is a $C > 0$ such that for all $G \in L^2(\Omega)$ there is a $u \in H^1(\Omega, \mathbb{R}^{m+1})$ satisfying

$$\left.\begin{aligned} D_i u_i &= G \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \Gamma_0, \end{aligned}\right\} \tag{4.8}$$

and

$$\|u\|_{1,\varepsilon} \le C\|G\|_0. \tag{4.9}$$

(ii) *(Two-sided boundary conditions)*
There is a $C > 0$ such that for all $G \in L^2(\Omega)$ with

$$\int_\Omega \sqrt{g} G \, dx = 0 \tag{4.10}$$

there is a $u \in H^1(\Omega, \mathbb{R}^{m+1})$ satisfying

$$\left.\begin{aligned} D_i u_i &= G \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega, \end{aligned}\right\} \tag{4.11}$$

and

$$\|u\|_{1,\varepsilon} \le C\varepsilon^{-1}\|G\|_0. \tag{4.12}$$

Proof. We construct $u = (u_1, \dots, u_{m+1})$ by the ansatz

$$u_i = \begin{cases} D_i\Phi + D_{m+1}\Phi_i, & i = 1, \dots, m \\ D_{m+1}\Phi - \sum_{j=1}^m D_j\Phi_j, & i = m+1 \end{cases}$$

where Φ solves the boundary value problem

$$\left.\begin{aligned} D_iD_i\Phi &= G \quad \text{in } \Omega, \\ \partial_{m+1}\Phi &= 0 \quad \text{on } \Gamma_0, \\ \Phi &= 0 \quad \text{on } \Gamma \text{ in case (i)}, \\ \nu_i D_i\Phi &= 0 \quad \text{on } \Gamma \text{ in case (ii)}. \end{aligned}\right\} \tag{4.13}$$

The solvability of this problem in case (ii) in ensured by (4.10). Note that with (4.4), Lemma 3.3 is applicable to (4.13), and we have

$$\|D_j\Phi\|_{1,\varepsilon} \le C \begin{cases} \|G\|_0 & \text{in case (i)}, \\ \varepsilon^{-1}\|G\|_0 & \text{in case (ii)}, \end{cases} \qquad j = 1 \dots, m+1.$$

Define

$$\varphi_i := D_i\Phi|_{\partial\Omega}, \quad i = 1, \dots, m,$$

and observe that from Lemma 3.1 (i) we get

$$\|\varphi_i\|_0^{\partial\Omega} + \sqrt{\varepsilon}\|\varphi_i\|_{1/2}^{\partial\Omega} \le C \begin{cases} \|G\|_0 & \text{in case (i)}, \\ \varepsilon^{-1}\|G\|_0 & \text{in case (ii)}. \end{cases}$$

Now Lemma 3.1 (ii) implies the existence of $\Phi_i \in H^1(\Omega)$ such that

$$D_{m+1}\Phi_i = -\varphi_i, \quad \Phi_i = 0 \qquad \text{on } \partial\Omega$$

and

$$\|D_j\Phi_i\|_{1,\varepsilon} \le C \begin{cases} \|G\|_0 & \text{in case (i)}, \\ \varepsilon^{-1}\|G\|_0 & \text{in case (ii)}, \end{cases} \qquad j = 1 \dots, m+1.$$

With that choice of Φ and Φ_i, we find that u satisfies (4.8) and (4.9) in case (i) and (4.11) and (4.12) in case (ii), respectively. □

Next we prove a regularity result for a Stokes type system generalizing (4.7).

Lemma 4.3. *(Sobolev regularity for the scaled Stokes system)*
Let $\varepsilon \in (0,1)$, $F \in H^{s_0-3/2}(\Omega, \mathbb{R}^{(m+1)\times(m+1)})$, $G \in H^{s_0-3/2}(\Omega)$, $H \in H^{s_0-3/2}(\Omega, \mathbb{R}^{m+1})$, $E \in H^{s_0-3/2}(\Gamma, \mathbb{R}^{m+1})$. *Let* (u,p) *be the solution of the transformed Stokes system*

$$\begin{aligned} D_iT_{ij}(u,p) &= \partial_{i,\varepsilon}F_{ij} + H_j \quad \text{in } \Omega, \\ D_iu_i &= G \quad \text{in } \Omega, \\ T_{ij}(u,p)\nu_i &= E_j - F_{m+1,j} \quad \text{on } \Gamma, \\ u &= 0 \quad \text{on } \Gamma_0. \end{aligned} \tag{4.14}$$

Then

$$\|u\|_{t,\varepsilon}^\Omega + \|p\|_{t-1}^\Omega \le C(\|F\|_{t-1}^\Omega + \|G\|_{t-1}^\Omega + \|H\|_{t-1}^\Omega + \varepsilon^{-\delta}\|E\|_{t-1-\delta}^\Gamma)$$

for $t \in [1, s_0 - 1/2]$, $\delta \in [0, 1/2]$.

Proof. It is sufficient to prove the result for integer t; the general case follows by interpolation. This will be done by induction over t.

Assume $t = 1$. Defining for $u, v \in H^1(\Omega, \mathbb{R}^{m+1})$ with $u|_{\Gamma_0} = v|_{\Gamma_0} = 0$

$$a(u,v) := \frac{1}{2}\int_\Omega (D_i u_j + D_j u_i)(D_i v_j + D_j v_i)\sqrt{g}\,dx$$

and using a transformed version of the Green identity for the Stokes operator, namely,

$$a(u,v) + \int_\Omega \big(D_i(T_{ij}(u,p))v_j - pD_i v_i\big)\sqrt{g}\,dx = \int_\Gamma T_{ij}(u,p)\nu_i v_j \sqrt{g}\,d\Gamma, \tag{4.15}$$

we get

$$a(u,v) - \int_\Omega \sqrt{g} p D_i v_i\,dx = f(v) \tag{4.16}$$

with

$$\begin{aligned} f(v) &:= \int_\Omega \sqrt{g}(\partial_{i,\varepsilon}F_{ij} + H_j)v_j\,dx + \int_\Gamma \sqrt{g}(E_j - F_{m+1,j})v_j\,d\Gamma \\ &= \int_\Omega (-F_{ij}\partial_{i,\varepsilon}(\sqrt{g}v_j) + \sqrt{g}H_j v_j)\,dx + \int_\Gamma \sqrt{g}E_j v_j\,d\Gamma. \end{aligned}$$

We obviously have $f \in \big(H^1(\Omega, \mathbb{R}^{m+1})\big)'$ with (cf. (3.7))

$$|f(v)| \le C\left(\|F\|_0 + \|H\|_0 + \varepsilon^{-\delta}\|E\|^\Gamma_{-\delta}\right)\|v\|_{1,\varepsilon}. \tag{4.17}$$

According to Lemma 4.2 (i) there is an u_G such that $D_i(u_G)_i = G$, $u_G|_{\Gamma_0} = 0$ and

$$\|u_G\|_{1,\varepsilon} \le C\|G\|_0. \tag{4.18}$$

Setting $u = \widetilde{u} + u_G$ we get $D_i\widetilde{u}_i = 0$ and therefore from (4.16)

$$a(\widetilde{u},\widetilde{u}) = f(\widetilde{u}) - a(u_G,\widetilde{u}).$$

Consequently, we get from Korn's inequality (3.18) and (4.17)

$$\|\widetilde{u}\|_{1,\varepsilon} \le C\left(\|F\|_0 + \|H\|_0 + \varepsilon^{-\delta}\|E\|^\Gamma_{-\delta} + \|u_G\|_{1,\varepsilon}\right),$$

and thus with (4.18)

$$\|u\|_{1,\varepsilon} \le C\left(\|F\|_0 + \|H\|_0 + \varepsilon^{-\delta}\|E\|^\Gamma_{-\delta} + \|G\|_0\right). \tag{4.19}$$

To estimate p we apply Lemma 4.2 (i) to find a $v \in H^1(\Omega, \mathbb{R}^{m+1})$ such that $D_i v_i = p$, $v|_{\Gamma_0} = 0$, and

$$\|v\|_{1,\varepsilon} \le C\|p\|_0. \tag{4.20}$$

Then

$$(\|p\|_0)^2 \le C\int_\Omega \sqrt{g}pD_i v_i\,dx = a(u,v) - f(v). \tag{4.21}$$

Consequently, using (4.17), we get

$$(\|p\|_0)^2 \le C\left(\|u\|_{1,\varepsilon} + \|F\|_0 + \|H\|_0 + \varepsilon^{-\delta}\|E\|^\Gamma_{-\delta}\right)\|v\|_{1,\varepsilon}$$

and further by (4.19), (4.20)

$$\|p\|_0 \le C\left(\|F\|_0 + \|H\|_0 + \varepsilon^{-\delta}\|E\|^\Gamma_{-\delta} + \|G\|_0\right).$$

This completes the proof for $t = 1$.

Assume now that the result holds for some integer $t = k \in [1, s_0 - 3/2]$. Differentiating (4.14) with respect to horizontal directions and using the fact that $\nu_i = a_{i,\varepsilon}$ at Γ and (3.13) yields

$$\begin{aligned} D_i T_{ij}(\partial_l u, \partial_l p) &= \partial_l \partial_{i,\varepsilon} F_{ij} + \partial_l H_j + [D_i, \partial_l] T_{ij}(u,p) + D_i [T_{ij}, \partial_l](u,p) \\ &= \partial_{i,\varepsilon} \widetilde{F}_{ij}^{(l)} + \widetilde{H}_j^{(l)} && \text{in } \Omega, \\ D_i \partial_l u_i &= \partial_l G + [D_i, \partial_l] u_i =: \widetilde{G}^{(l)} && \text{in } \Omega, \\ T_{ij}(\partial_l u, \partial_l p)\nu_i &= \partial_l E_j - \partial_l F_{m+1,j} + [T_{kj}, \partial_l](u,p)\nu_i - T_{kj}(u,p)\partial_l \nu_i \\ &= \partial_l E_j - \widetilde{F}_{m+1,j}^{(l)} && \text{on } \Gamma, \\ \partial_l u &= 0 && \text{on } \Gamma_0, \end{aligned}$$

where

$$\begin{aligned} \widetilde{F}_{ij}^{(l)} &:= \partial_l F_{ij} + [T_{ij}, \partial_l](u,p) \quad (i = 1, \dots, m), \\ \widetilde{F}_{m+1,j}^{(l)} &:= \partial_l F_{m+1,j} - a_{r,\varepsilon}[T_{rj}, \partial_l](u,p) + \partial_l a_{r,\varepsilon} T_{rj}(u,p), \\ \widetilde{H}_j^{(l)} &:= \partial_l H_j + \partial_{l,m+1} a_{r,\varepsilon} T_{rj}(u,p) - \partial_{m+1} a_{r,\varepsilon}[T_{rj}, \partial_l](u,p). \end{aligned}$$

Application of the induction assumption to this system yields

$$\begin{aligned} \|\partial_l u\|_{k,\varepsilon} + \|\partial_l p\|_{k-1} &\leq C\left(\|\widetilde{F}\|_{k-1} + \|\widetilde{G}\|_{k-1} + \|\widetilde{H}\|_{k-1} + \varepsilon^{-\delta}\|\partial_l E\|_{k-1-\delta}^{\Gamma}\right) \\ &\leq C\left(\|F\|_k + \|G\|_k + \|H\|_k + \varepsilon^{-\delta}\|E\|_{k-\delta}^{\Gamma} + \|u\|_{k,\varepsilon} + \|p\|_{k-1}\right) \\ &\leq C\left(\|F\|_k + \|G\|_k + \|H\|_k + \varepsilon^{-\delta}\|E\|_{k-\delta}^{\Gamma}\right), \end{aligned} \tag{4.22}$$

where the induction assumption for the original system has been used in the last step.

To estimate derivatives in the vertical direction we write the transformed Stokes equations in the form

$$\begin{aligned} &-(a_{i,\varepsilon}a_{i,\varepsilon})\partial_{m+1}^2 u_j + a_{j,\varepsilon}\partial_{m+1} p \\ =\ &\partial_{i,\varepsilon} F_{ij} + H_j + D_j G - (1 - \delta_{j,m+1})\varepsilon \partial_j p \\ &+ \sum_{i=1}^m (\varepsilon \partial_i D_i + \varepsilon^2 a_i \partial_i \partial_{m+1}) u_j + a_{m+1}\partial_{m+1} a_{m+1} \partial_{m+1} u_j =: R_j, \end{aligned} \tag{4.23}$$

$j = 1, \dots, m+1$. By differentiation of the divergence equation with respect to y we get

$$a_{i,\varepsilon}\partial_{m+1}^2 u_i = \partial_{m+1} G + [D_i, \partial_{m+1}] u_i - \sum_{i=1}^m \partial_i \partial_{m+1} u_i =: R_{m+2} \tag{4.24}$$

Using the induction assumption and (4.22) we get for $j = 1, \dots, m+2$:

$$\begin{aligned} \|R_j\|_{k-1} &\leq C\left(\|F\|_k + \|G\|_k + \|H\|_k + \|u\|_{k,\varepsilon} + \sum_{l=1}^m (\|\partial_l u\|_{k\varepsilon} + \|\partial_l p\|_{k-1})\right) \\ &\leq C\left(\|F\|_k + \|G\|_k + \|H\|_k + \|e\|_k^{\Gamma}\right). \end{aligned} \tag{4.25}$$

Observe that (4.23), (4.24) form a linear system in $\partial_{m+1}^2 u_j$ and $\partial_{m+1} p$. Its coefficient matrix has determinant $(-\sum_{i=1}^{m+1} |a_{i,\varepsilon}|^2)^{m+1}$ which is bounded away from zero uniformly with respect to $\varepsilon \in (0,1)$ and $h \in \mathcal{U}$. Inverting this system and taking H^{k-1}-norms we conclude from (4.25)

$$\|\partial_{m+1}^2 u\|_{k-1} + \|\partial_{m+1} p\|_{k-1} \le C\left(\|F\|_k + \|G\|_k + \|H\|_k + \|e\|_k^\Gamma\right).$$

Together with (4.22), this implies the statement of the lemma for $t = k+1$. □

Remark 4.4. It is clear that the estimate is not optimal with respect to H, this is due to the fact that we avoid to work with norms with negative index. For our purposes, however, the given estimate will be sufficient.

We will consider now the solution (u, p) of the transformed Stokes system

$$\left.\begin{array}{rcll} -D_i T_{ij}(u,p) & = & 0 & \text{in } \Omega, \\ D_i u_i & = & 0 & \text{in } \Omega, \\ T_{ij}(u,p)\nu_j & = & E_j & \text{on } \Gamma, \\ u & = & 0 & \text{on } \Gamma_0 \end{array}\right\} \tag{4.26}$$

as functions $(u,p) = (u(\varepsilon,h), p(\varepsilon,h))E$ of $h \in \mathscr{U}$, $\varepsilon \in (0,1)$, $E \in H^{s_0-3/2}(\Gamma, \mathbb{R}^{m+1})$. The dependence on h is smooth as (4.26) forms a regular elliptic system for $h \in \mathscr{U}$ and the dependence of all occurring operators on h is smooth. More precisely, we have the following estimates for the k-th order Fréchet derivatives:

Lemma 4.5. *(Estimates for Fréchet derivatives)*
Let $u = u(\varepsilon,h)E$ be the solution of (4.26) *and let $u^{(k)}(\varepsilon,h)\{h_1,\dots,h_k\}E$ denote its k-th order Fréchet derivative with respect to h. There is a constant $C = C(\mathscr{U},k)$ such that*

$$\begin{aligned} \|u^{(k)}(\varepsilon,h)\{h_1,\dots,h_k\}E\|_{1,\varepsilon}^\Omega &\le C\|h_1\|_{s_0}^\Gamma \cdots \|h_k\|_{s_0}^\Gamma \varepsilon^{-\delta}\|E\|_{-\delta}^\Gamma, \\ \|u^{(k)}(\varepsilon,h)\{h_1,\dots,h_k\}E\|_{1,\varepsilon}^\Omega &\le C\|h_1\|_{1/2}^\Gamma \|h_2\|_{s_0}^\Gamma \cdots \|h_k\|_{s_0}^\Gamma \|E\|_{s_0-3/2}^\Gamma, \end{aligned}$$

$\delta \in [0, 1/2]$.

Proof. We will show the more general estimates

$$\begin{aligned} &\|u^{(k)}(\varepsilon,h)\{h_1,\dots,h_k\}E\|_{t,\varepsilon} + \|p^{(k)}(\varepsilon,h)\{h_1,\dots,h_k\}E\|_{t-1} \\ &\qquad \le C\|h_1\|_{s_0}^\Gamma \cdots \|h_k\|_{s_0}^\Gamma \varepsilon^{-\delta}\|E\|_{t-1-\delta}^\Gamma, \quad t \in [1, s_0 - 1/2],\ \delta \in [0, 1/2], \end{aligned} \tag{4.27}$$

$$\begin{aligned} &\|u^{(k)}(\varepsilon,h)\{h_1,\dots,h_k\}E\|_{t,\varepsilon} + \|p^{(k)}(\varepsilon,h)\{h_1,\dots,h_k\}E\|_{t-1} \\ &\qquad \le C\|h_1\|_{t-1/2}^\Gamma \|h_2\|_{s_0}^\Gamma \cdots \|h_k\|_{s_0}^\Gamma \|E\|_{s_0-3/2}^\Gamma, \quad t \in [1, s_0 - 3/2], \end{aligned} \tag{4.28}$$

by induction over k.

For $k = 0$, the result is given in Lemma 4.3. Assume now that (4.27), (4.28) are true for $0 \le k \le l-1$. Fréchet differentiation of (4.26) leads to

$$\left.\begin{array}{rcll} -D_i T_{ij}(u^{(l)}, p^{(l)}) & = & \sum D_i^{(\lambda)} T_{ij}^{(\mu)}(u^{(\sigma)}, p^{(\sigma)}) & \text{in } \Omega, \\ D_i u_i^{(l)} & = & -\sum D_i^{(\lambda)} u_i^{(\sigma)} & \text{in } \Omega, \\ T_{ij}(u^{(l)}, p^{(l)})\nu_j & = & -\sum T_{ij}^{(\mu)}(u^{(\sigma)}, p^{(\sigma)})\nu_i^{(\lambda)} & \text{on } \Gamma, \\ u^{(l)} & = & 0 & \text{on } \Gamma_0. \end{array}\right\}$$

The sums are to be taken over all partitions $l = \lambda + \mu + \sigma$ with $\sigma < l$ and all permutations $\pi \in \mathsf{S}_l$. In the second sum, we demand $\mu = 0$. Here and in the sequel, we suppress the arguments of the Fréchet derivatives, for example,

$$D_i^{(\lambda)}T_{ij}^{(\mu)}\big(u^{(\sigma)},p^{(\sigma)}\big) = D_i^{(\lambda)}\{h_{\pi(1)},\dots,h_{\pi(\lambda)}\}T_{ij}^{(\mu)}\{h_{\pi(\lambda+1)},\dots,h_{\pi(\lambda+\mu)}\}$$
$$\big(u^{(\sigma)}\{h_{\pi(\lambda+\mu+1)},\dots,h_{\pi(l)}\},p^{(\sigma)}\{h_{\pi(\lambda+\mu+1)},\dots,h_{\pi(l)}\}\big).$$

Note that $D_i^{(\lambda)} = a_{i,\varepsilon}^{(\lambda)}\partial_{m+1}$ for $\lambda > 0$ and $\nu_i^{(\lambda)} = a_{i,\varepsilon}^{(\lambda)}$ on Γ.

1. We start by showing (4.27) for $k = l$. Fix λ, μ, σ, and π and define

$$\begin{aligned}
F_{ij}^{\lambda,\mu,\sigma} &:= \begin{cases} T_{ij}^{(\mu)}(u^{(\sigma)},p^{(\sigma)}), & \lambda = 0,\\ 0, & \lambda > 0,\end{cases} \qquad i = 1,\dots,m,\\
F_{m+1,j}^{\lambda,\mu,\sigma} &:= a_{r,\varepsilon}^{(\lambda)}T_{ij}^{(\mu)}(u^{(\sigma)},p^{(\sigma)}),\\
H_j^{\lambda,\mu,\sigma} &:= -\partial_{m+1}a_{r,\varepsilon}^{(\lambda)}T_{ij}^{(\mu)}(u^{(\sigma)},p^{(\sigma)}),\\
G^{\lambda,\mu,\sigma} &:= \begin{cases} -D_i^{(\lambda)}u_i^{(\sigma)}, & \mu = 0,\\ 0, & \mu > 0,\end{cases}\\
\widehat{E}_j^{\lambda,\mu,\sigma} &:= 0.
\end{aligned}$$

Then

$$\left.\begin{aligned}
-D_iT_{ij}(u^{(l)},p^{(l)}) &= \textstyle\sum \partial_{i,\varepsilon}F_{ij}^{\lambda,\mu,\sigma} + H_j^{\lambda,\mu,\sigma} && \text{in } \Omega,\\
D_iu_i^{(l)} &= \textstyle\sum G^{\lambda,\mu,\sigma} && \text{in } \Omega,\\
T_{ij}(u^{(l)},p^{(l)})\nu_j &= -\textstyle\sum F_{m+1,j}^{\lambda,\mu,\sigma} + \widehat{E}_j^{\lambda,\mu,\sigma} && \text{on } \Gamma,\\
u^{(l)} &= 0 && \text{on } \Gamma_0,
\end{aligned}\right\} \tag{4.29}$$

and, using standard product estimates in Sobolev spaces and the induction assumption,

$$\begin{aligned}
&\|F^{\lambda,\mu,\sigma}\|_{t-1} + \|G^{\lambda,\mu,\sigma}\|_{t-1} + \|H^{\lambda,\mu,\sigma}\|_{t-1}\\
\le\ & C\|h_{\pi(1)}\|_{s_0}^{\Gamma}\dots\|h_{\pi(\lambda+\mu)}\|_{s_0}^{\Gamma}\Big(\|u^{(\sigma)}\|_{t,\varepsilon} + \|p^{(\sigma)}\|_{t-1}\Big)\\
\le\ & C\|h_1\|_{s_0}^{\Gamma}\dots\|h_l\|_{s_0}^{\Gamma}\varepsilon^{-\delta}\|E\|_{t-1-\delta}^{\Gamma}.
\end{aligned}$$

The estimate (4.27) for $k = l$ follows now from Lemma 4.3.

2. We show (4.28) for $k = l$.

2.1. Assume $1 \in \pi(\{1,\dots,l\})$. Define

$$\begin{aligned}
F_{ij}^{\lambda,\mu,\sigma} &:= 0,\\
H_j^{\lambda,\mu,\sigma} &:= D_i^{(\lambda)}T_{ij}^{(\mu)}(u^{(\sigma)},p^{(\sigma)})\\
G^{\lambda,\mu,\sigma} &:= \begin{cases} -D_i^{(\lambda)}u_i^{(\sigma)}, & \mu = 0,\\ 0, & \mu > 0,\end{cases}\\
\widehat{E}_j^{\lambda,\mu,\sigma} &:= \widehat{E}_{j,1}^{\lambda,\mu,\sigma} + \widehat{E}_{j,2}^{\lambda,\mu,\sigma},\\
\widehat{E}_{j,1}^{\lambda,\mu,\sigma} &:= \sum_{r=1}^{m} T_{rj}^{(\mu)}(u^{(\sigma)},p^{(\sigma)})\nu_r^{(\lambda)} = \varepsilon\sum_{r=1}^{m} a_r^{(\lambda)}T_{rj}^{(\mu)}(u^{(\sigma)},p^{(\sigma)}),\\
\widehat{E}_{j,2}^{\lambda,\mu,\sigma} &:= T_{m+1,j}^{(\mu)}(u^{(\sigma)},p^{(\sigma)})\nu_{m+1}^{(\lambda)} = (h^{-1})^{(\lambda)}T_{m+1,j}^{(\mu)}(u^{(\sigma)},p^{(\sigma)}).
\end{aligned}$$

Then (4.29) is satisfied again. Here we estimate, using the induction assumption (4.27),

$$\begin{aligned}
&\varepsilon^{-1/2}\|\widehat{E}^{\lambda,\mu,\sigma}_{j,1}\|^{\Gamma}_{t-3/2} \\
&\le C\varepsilon^{1/2}\|h_1\|^{\Gamma}_{t-1/2}\prod_{\substack{r=1\ldots\lambda\\ \pi(r)\neq 1}}\|h_{\pi(r)}\|^{\Gamma}_{s_0}\Big(\|u^{(\sigma)}\|_{s_0-3/2,\varepsilon}+\|p^{(\sigma)}\|_{s_0-5/2}\Big) \\
&\le C\|h_1\|^{\Gamma}_{t-1/2}\|h_2\|^{\Gamma}_{s_0}\ldots\|h_k\|^{\Gamma}_{s_0}\|E\|_{s_0-3/2},
\end{aligned}$$

$$\begin{aligned}
\|\widehat{E}^{\lambda,\mu,\sigma}_{j,2}\|^{\Gamma}_{t-1} &\le C\|h_1\|^{\Gamma}_{t-1}\prod_{\substack{r=1\ldots\lambda\\ \pi(r)\neq 1}}\|h_{\pi(r)}\|^{\Gamma}_{s_0}\Big(\|u^{(\sigma)}\|_{s_0-1,\varepsilon}+\|p^{(\sigma)}\|_{s_0-2}\Big) \\
&\le C\|h_1\|^{\Gamma}_{t-1/2}\|h_2\|^{\Gamma}_{s_0}\ldots\|h_k\|^{\Gamma}_{s_0}\|E\|_{s_0-3/2},
\end{aligned}$$

$$\begin{aligned}
&\|G^{\lambda,\mu,\sigma}\|_{t-1}+\|H^{\lambda,\mu,\sigma}\|_{t-1} \\
&\le C\|h_1\|^{\Gamma}_{t-1/2}\prod_{\substack{r=1\ldots\lambda\\ \pi(r)\neq 1}}\|h_{\pi(r)}\|^{\Gamma}_{s_0}\Big(\|u^{(\sigma)}\|_{s_0-1/2,\varepsilon}+\|p^{(\sigma)}\|_{s_0-3/2}\Big) \\
&\le C\|h_1\|^{\Gamma}_{t-1/2}\|h_2\|^{\Gamma}_{s_0}\ldots\|h_k\|^{\Gamma}_{s_0}\|E\|_{s_0-3/2},
\end{aligned}$$

2.2. Assume $1\in\pi(\{\lambda+1,\ldots,\lambda+\mu\})$. We proceed as in 1. and estimate

$$\begin{aligned}
&\|F^{\lambda,\mu,\sigma}\|_{t-1}+\|G^{\lambda,\mu,\sigma}\|_{t-1}+\|H^{\lambda,\mu,\sigma}\|_{t-1} \\
&\le C\|h_1\|^{\Gamma}_{t+1/2}\prod_{\substack{r=1\ldots\lambda+\mu\\ \pi(r)\neq 1}}\|h_{\pi(r)}\|^{\Gamma}_{s_0}\Big(\|u^{(\sigma)}\|_{s_0-3/2,\varepsilon}+\|p^{(\sigma)}\|_{s_0-5/2}\Big) \\
&\le C\|h_1\|^{\Gamma}_{t-1/2}\|h_2\|^{\Gamma}_{s_0}\ldots\|h_k\|^{\Gamma}_{s_0}\|E\|_{s_0-3/2}.
\end{aligned}$$

2.3. Assume $1\in\pi(\{\lambda+\mu+1,\ldots,l\})$. We proceed as in 1. and estimate, using the induction assumption (4.28),

$$\begin{aligned}
&\|F^{\lambda,\mu,\sigma}\|_{t-1}+\|G^{\lambda,\mu,\sigma}\|_{t-1}+\|H^{\lambda,\mu,\sigma}\|_{t-1} \\
&\le C\|h_{\pi(1)}\|^{\Gamma}_{s_0}\ldots\|h_{\pi(\lambda+\mu)}\|^{\Gamma}_{s_0}\Big(\|u^{(\sigma)}\|_{t,\varepsilon}+\|p^{(\sigma)}\|_{t-1}\Big) \\
&\le C\|h_1\|^{\Gamma}_{t-1/2}\|h_2\|^{\Gamma}_{s_0}\ldots\|h_k\|^{\Gamma}_{s_0}\|E\|_{s_0-3/2}.
\end{aligned}$$

In all three cases, (4.28) for $k=l$ follows from applying Lemma 4.3 to (4.29). □

Next, we give a first preliminary result about the behavior of the mapping $(\varepsilon,h)\mapsto u(\varepsilon,h)E$ as $\varepsilon\downarrow 0$. In these considerations we specialize the right hand side of (4.26) to

$$E := f\nu,\quad f\in H^{s_0-3/2}(\Gamma). \tag{4.30}$$

which is sensible in view of (4.7). Note that the estimates here are in weaker norms, i.e. there is a "loss of smoothness" which compensates for the degeneracy of the operator. Moreover, note the different behavior of the tangential and normal components. Series expansions in powers of ε are postponed to Section 5.

Lemma 4.6. *Let $2 \le s \le s_0 - 3/2$, $u = u(\varepsilon, h)[f\nu]$, $p = p(\varepsilon, h)[f\nu]$ and set $p^{[0]}(x, y) := f(x)$ for $(x, y) \in \Omega$. Then we have*

$$\|p - p^{[0]}\|_s,\ \|u\|^{\Omega}_{s-1},\ \|u\|^{\Gamma}_{s-1} \le C\varepsilon\|f\|^{\Gamma}_s, \tag{4.31}$$

and moreover

$$\|u_{m+1}\|^{\Omega}_{s-2},\ \|u_{m+1}\|^{\Gamma}_{s-2} \le C\varepsilon^2\|f\|^{\Gamma}_s. \tag{4.32}$$

Proof. Clearly $(\tilde u, \tilde p) := (u, p - p^{[0]})$ is the solution of the Stokes system (4.14) with data $F_{ij} \equiv 0$, $G_i \equiv 0$, $E \equiv 0$ and $H_j = \varepsilon\partial_j f$ for $i = 1, \dots, m$, $H_{m+1} = 0$. Hence Lemma 4.3 gives

$$\|\tilde p\|^{\Omega}_{s-1} + \|\tilde u\|_{s,\varepsilon} \le C\|H\|^{\Omega}_{s-1} \le C\varepsilon\|f\|^{\Gamma}_s.$$

Together with the scaled trace inequality this implies (4.31). Using the divergence relation $D_i u_i = 0$ and $u|_{\Gamma_0} = 0$, we find

$$\|u_{m+1}\|^{\Omega}_{s-2}, \|u_{m+1}\|^{\Gamma}_{s-2} \le C\|D_{m+1}u_{m+1}\|^{\Omega}_{s-2} \le C\sum_{i=1}^{m}\|D_i u_i\|^{\Omega}_{s-2} \le C\varepsilon\|u\|^{\Omega}_{s-1},$$

hence (4.32) follows from (4.31). □

The next lemma provides a similar result for the first derivative with respect to h. From now on, we will assume $s_0 > m/2 + 8$.

Lemma 4.7. *Setting*

$$v(\varepsilon, h)\{h_1\}f := u'(\varepsilon, h)\{h_1\}[f\nu] + u(\varepsilon, h)[T(u, p)\nu'\{h_1\}],$$

we have the estimate

$$\|v(\varepsilon, h)\{h_1\}f\|^{\Gamma}_{1/2} \le C\varepsilon\|h_1\|^{\Gamma}_{1/2}\|f\|^{\Gamma}_{s_0}, \tag{4.33}$$

and moreover for the $(m+1)$-th component of v

$$\|v_{m+1}(\varepsilon, h)\{h_1\}f\|^{\Gamma}_{1/2} \le C\varepsilon^2\|h_1\|^{\Gamma}_{3/2}\|f\|^{\Gamma}_{s_0}. \tag{4.34}$$

Proof. First we verify that

$$v(\varepsilon, h)\{h_1\}f = h_1 D_{m+1}u + u(\varepsilon, h)[h_1 D_{m+1}T(u, p)\nu]. \tag{4.35}$$

To see this we differentiate the Stokes system (4.26) as in the proof of Lemma 4.5 with respect to h to obtain

$$\begin{aligned} D_i T_{ij}(u', p') + D'_i T_{ij}(u, p) + D_i T'_{ij}(u, p) &= 0 \text{ in } \Omega,\\ D_i u'_i + D'_i u_i &= 0 \text{ in } \Omega,\\ T_{ij}(u', p')\nu_i + T'_{ij}(u, p)\nu_i + T_{ij}(u, p)\nu'_i &= 0 \text{ on } \Gamma,\\ u' &= 0 \text{ on } \Gamma_0. \end{aligned}$$

Recalling (4.5), (4.6) and using that $D_i u_i = 0$ and $D_i T_{ij}(u, p) = 0$ we find easily

$$\begin{aligned} D'_i u_i &= -D_i\big(\tilde h_1 y D_{m+1}u_i\big),\\ D'_i T_{ij}(u, p) &= -D_i\big(\tilde h_1 y D_{m+1}T_{ij}(u, p)\big),\\ T'_{ij}(u, p) &= -\tfrac12\big(D_j(\tilde h_1 y)D_{m+1}u_i + D_i(\tilde h_1 y)D_{m+1}u_j\big), \end{aligned}$$

and therefore

$$D_i' T_{ij}(u,p) + D_i T_{ij}'(u,p) = -D_i T_{ij}\big(\tilde h_1 y D_{m+1} u, \tilde h_1 y D_{m+1} p\big).$$

Thus (w,q) given by

$$\begin{aligned} w &:= u'(\varepsilon,h)\{h_1\}[f\nu] + u(\varepsilon,h)[T(u,p)\nu'\{h_1\}] - \tilde h_1 y D_{m+1} u \\ q &:= p'(\varepsilon,h)\{h_1\}[f\nu] + p(\varepsilon,h)[T(u,p)\nu'\{h_1\}] - \tilde h_1 y D_{m+1} p \end{aligned}$$

satisfies the system

$$\begin{aligned} D_i T_{ij}(w,q) &= 0 \ \text{ in } \ \Omega, \\ D_i w_i &= 0 \ \text{ in } \ \Omega, \\ T_{ij}(w,q)\nu_i &= -h_1 D_{m+1} T_{ij}(u,p)\nu_i \ \text{ on } \ \Gamma, \\ w &= 0 \ \text{ on } \ \Gamma_0. \end{aligned}$$

Consequently, we have

$$w = -u(\varepsilon,h)[h_1 D_{m+1} T(u,p)\nu],$$

which gives (4.35).

Now, to prove (4.33), (4.34) we use the estimates

$$\|u_i\|_{C_b^4(\Omega)} \le C\|u_i\|^{\Omega}_{s_0-7/2} \le \begin{cases} C\varepsilon\|f\|_{s_0}, & i = 1,\dots,m, \\ C\varepsilon^2\|f\|_{s_0}, & i = m+1 \end{cases} \tag{4.36}$$

which follow from (4.31), (4.32), and $s_0 > m/2 + 8$. Moreover, we will apply

$$\|\partial_y p\|_{C_b^2(\Omega)} \le C\|D_i D_i u_{m+1}\|_{C_b^2(\Omega)} \le C\|u_{m+1}\|_{C_b^4(\Omega)} \le C\varepsilon^2\|f\|_{s_0}. \tag{4.37}$$

Using these estimates we get

$$\|h_1 D_{m+1} u_i\|^{\Gamma}_{1/2} \le C\|h_1\|_{1/2}\|u_i\|_{C_b^2(\Omega)} \le \begin{cases} C\varepsilon\|h_1\|_{1/2}\|f\|_{s_0}, & i = 1,\dots,m, \\ C\varepsilon^2\|h_1\|_{1/2}\|f\|_{s_0}, & i = m+1. \end{cases} \tag{4.38}$$

In the same manner we obtain for $s = 1/2$ and $s = 3/2$

$$\|h_1 D_{m+1}(T(u,p))\,\nu\|^{\Gamma}_s \le C\big(\|u\|_{C_b^4(\Omega)} + \|\partial_y p\|_{C_b^2(\Omega)}\big)\|h_1\|_s \le C\varepsilon\|h_1\|_s\|f\|_{s_0},$$

hence by the scaled trace inequality (3.5) and Lemma 4.3

$$\|w\|^{\Gamma}_s \le C\|w\|^{\Omega}_{s+1,\varepsilon} \le C\varepsilon\|h_1\|_s\|f\|_{s_0}. \tag{4.39}$$

Finally, using this estimate, the divergence relation $D_i w_i = 0$, and $w|_{\Gamma_0} = 0$ we find

$$\|w_{m+1}\|^{\Gamma}_{1/2} \le C\|\partial_y w_{m+1}\|^{\Omega}_{1/2} \le C\varepsilon\|w\|^{\Omega}_{5/2,\varepsilon} \le C\varepsilon^2\|h_1\|_{3/2}\|f\|_{s_0}. \tag{4.40}$$

In view of (4.35), the estimates (4.38)-(4.40) imply (4.33), (4.34) and the proof is complete. □

Remark 4.8. As an by-product of the preceding proof we notice the following relation for the first derivative of u with respect to h:

$$u'(\varepsilon,h)\{h_1\}E = -h_1 D_{m+1} u - u(\varepsilon,h)\big[h_1(D_{m+1}T(u,p))\nu + T(u,p)\nu'\{h_1\}\big].$$

Formally, this relation can be straightforwardly obtained by "variation of the domain" and taking into account the additional terms arising from the variation of the corresponding diffeomorphism.

Interpreting $u = u(\varepsilon, h)E$ as solution operator to (4.26) we can formulate our moving boundary problem now as a single nonlocal Cauchy problem. For given $h \in \mathscr{U}$ and $f \in H^{s_0-3/2}(\Gamma)$ let $u = u(\varepsilon, h)[f\nu]$, $p = p(\varepsilon, h)[f\nu]$ be the solution of the Stokes system (4.26) with boundary data $E = f\nu$ and define

$$F(\varepsilon, h)f := \varepsilon^{-2}(-\varepsilon\nabla_x h, 1) \cdot u(\varepsilon, h)[f\nu], \quad \text{and} \quad \mathscr{F}(\varepsilon, h) := F(\varepsilon, h)\kappa(\varepsilon, h)$$

where

$$\kappa(\varepsilon, h) := \operatorname{div}_x \left(\nabla_x h / \sqrt{1 + \varepsilon^2|\nabla_x h|^2}\right)$$

denotes the rescaled curvature expression as introduced in Sect. 2. With this notation, the evolution equation (2.6) gets the form

$$\dot{h} = \mathscr{F}(\varepsilon, h). \tag{4.41}$$

Note that $F(\varepsilon, h)f$ has zero mean value over Γ, and consequently

$$\int_\Gamma h(t)\, d\Gamma = \int_\Gamma h(0)\, d\Gamma \tag{4.42}$$

for any solution $h = h(t)$ of (4.41) expressing the conservation of volume of $\Omega_{h(t)}$.

5. Approximation by series expansion

We continue the investigation of (4.26), (4.30) and take care also of the different behavior of the components of the velocity vector u as $\varepsilon \downarrow 0$. Thus we write $u = (v, w)$ with $v = (u_1, \dots, u_m)$, $w = u_{m+1}$ and write the transformed Stokes system (4.26),(4.30) accordingly componentwise in the form

$$\begin{aligned} S(\varepsilon)(v, w, p)^\top &= (0, 0, 0, \varepsilon a f, a_{m+1} f)^\top, \\ (v, w) &= 0 \ \text{ on } \ \Gamma_0 \end{aligned}$$

where $a := -y\nabla_x \widetilde{h} / \partial_y (y\widetilde{h})$ and

$$S(\varepsilon) := S_0 + \varepsilon S_1 + \varepsilon^2 S_2, \qquad S_0 := \begin{bmatrix} -D_{m+1}^2 I & 0 & 0 \\ 0 & -D_{m+1}^2 & D_{m+1} \\ 0 & D_{m+1} & 0 \\ \frac{1}{h} D_{m+1} I & 0 & 0 \\ 0 & \frac{2}{h} D_{m+1} & -\frac{1}{h} \end{bmatrix},$$

$$S_1 := \begin{bmatrix} 0 & 0 & \widetilde{\nabla} \\ 0 & 0 & 0 \\ \widetilde{\nabla}^\top & 0 & 0 \\ 0 & \frac{1}{h}\widetilde{\nabla} & -a \\ a^\top D_{m+1} & 0 & 0 \end{bmatrix}, \qquad S_2 := \begin{bmatrix} -\widetilde{\nabla}^\top \widetilde{\nabla} I & 0 & 0 \\ 0 & -\widetilde{\nabla}^\top \widetilde{\nabla} & 0 \\ 0 & 0 & 0 \\ (a^\top \widetilde{\nabla}) I + a \widetilde{\nabla}^\top & 0 & 0 \\ 0 & (a^\top \widetilde{\nabla}) & 0 \end{bmatrix},$$

and $\widetilde{\nabla} := \nabla_x + a\partial_{m+1}$. Clearly, the last two components represent the boundary conditions at Γ; the trace operator has been suppressed in the notation for the sake of simplicity.

In all further considerations of this section we assume

$$h \in \mathscr{U}_1 := \mathscr{U}(s_1, M_1, \alpha)$$

with some sufficiently large $s_1 \geq s_0$ specified later on and some fixed $M_1, \alpha > 0$. This set will be considered as open subset of $H^{s_1}(\Gamma)$. If not stated explicitly, constants that occur in estimates will always depend on $\mathscr{U}_1$, i.e. on s_1 and on α, M_1.

First observe that the problem

$$S_0(h)(v,w,p)^\top = (R_1, \ldots, R_5)^\top$$

is straightforwardly solved by

$$v(x,y) = h(x)R_4(x)\widetilde{h}(x,y)y + \int_0^y \partial_\eta\big(\widetilde{h}(x,\eta)\eta\big) \int_\eta^1 \partial_\zeta\big(\widetilde{h}(x,\zeta)\zeta\big) R_1(x,\zeta)\, d\zeta d\eta, \tag{5.1}$$

$$w(x,y) = \int_0^y \partial_\eta\big(\widetilde{h}(x,\eta)\eta\big) R_3(x,\eta)\, d\eta, \tag{5.2}$$

$$p(x,y) = R_3(x,1) - h(x)R_5(x) + R_3(x,y) - \int_y^1 \partial_\eta\big(\widetilde{h}(x,\eta)\eta\big) R_2(x,\eta)\, d\eta. \tag{5.3}$$

Further, as $h \in \mathscr{U}_1$ implies that $(x,y) \mapsto \partial_y(\widetilde{h}(x,y)y)$ belongs to $H^{s_1-1/2}(\Omega)$, we easily obtain from this explicit representation formulas the estimates

$$\begin{aligned} \|v\|_s^\Gamma + \|v\|_s^\Omega &\leq C\big(\|R_1\|_s + \|R_4\|_s^\Gamma\big), \\ \|w\|_s^\Gamma + \|w\|_s^\Omega &\leq C\|R_3\|_s, \\ \|p\|_{s-1}^\Gamma + \|p\|_{s-1}^\Omega &\leq C\big(\|R_2\|_{s-1} + \|R_3\|_{s-1/2} + \|R_5\|_{s-1}^\Gamma\big) \end{aligned}$$

provided $1 \leq s \leq s_1 - 1/2$. Moreover h enters smoothly into (5.1)-(5.3). Thus, introducing the spaces

$$\begin{aligned} X^s &:= H^s(\Omega, \mathbb{R}^{m+1}) \times H^s(\Omega, \mathbb{R}) \times H^s(\Gamma, \mathbb{R}^{m+1}), \\ Y^s &:= H^s(\Omega, \mathbb{R}^{m+1}) \times H^{s-1}(\Omega, \mathbb{R}) \end{aligned}$$

with norms defined correspondingly, we have

$$[h \mapsto S_0(h)^{-1}] \in C_b^\infty\big(\mathscr{U}_1, \mathscr{L}(X^s, Y^s)\big). \tag{5.4}$$

By reinspecting the structure of the operators $S_1(h)$, $S_2(h)$ we find similarly

$$[h \mapsto S_1(h)], [h \mapsto S_2(h)] \in C_b^\infty\big(\mathscr{U}_1, \mathscr{L}(Y^s, X^{s-2})\big) \tag{5.5}$$

for $2 \leq s \leq s_1 - 1/2$.

Lemma 5.1. *Let $n \in \mathbb{N}$, $s \geq 0$ be given and assume $s_1 \geq s + 2n + 5/2$. Then for $f \in H^{s_1}(\Gamma)$ and $h \in \mathscr{U}_1$ there exists an expansion of the form*

$$u(\varepsilon, h)E = \sum_{k=0}^n \varepsilon^k u^{[k]}(h)f + r_n(\varepsilon, h)f \tag{5.6}$$

where $[h \mapsto u^{[k]}(h))] \in C_b^\infty\big(\mathscr{U}_1, \mathscr{L}(H^{s_1}(\Gamma), H^s(\Omega, \mathbb{R}^{m+1}))\big)$ and the remainder term satisfies

$$\|r_n(\varepsilon, h)f\|_s \leq C\varepsilon^{n+1}\|f\|_{s_1} \tag{5.7}$$

with $C = C(\mathscr{U}_1)$. In particular, we have $u^{[0]} \equiv 0$, $u^{[1]} = (v^{[1]}, 0)$ and $u^{[2]} = (v^{[2]}, w^{[2]})$ with $v^{[1]}$, $w^{[2]}$ given by

$$v^{[1]} = \frac{1}{2}h^2 \nabla_x f,\ w^{[2]} = -\frac{1}{2}h^2 \nabla_x h \nabla_x f - h^3 \frac{1}{3}\Delta_x f \quad on\ \Gamma. \tag{5.8}$$

Proof. Inserting the expansion (5.6) and its analogue

$$p(\varepsilon, h)E = \sum_{k=0}^{n} \varepsilon^k p^{[k]}(h) f + q_n(\varepsilon, h) f \tag{5.9}$$

for p into the left of the equation

$$S(\varepsilon, h)(u, p)^\top = (0, 0, 0, \varepsilon a f, a_{m+1} f)^\top, \tag{5.10}$$

the terms $u^{[k]}$, $p^{[k]}$, $k = 0, \dots, n$ are determined successively by comparison of coefficients with respect to equal powers of ε (up to order n). This leads to

$$S_0(h)\big(u^{[0]}, p^{[0]}\big)^\top = (0, 0, 0, 0, a_{m+1} f)^\top,$$
$$S_0(h)\big(u^{[1]}, p^{[1]}\big)^\top = (0, 0, 0, af, 0)^\top - S_1(h)\big(u^{[0]}, p^{[0]}\big),$$

and

$$S_0(h)\big(u^{[k]}, p^{[k]}\big)^\top = -S_1(h)\big(u^{[k-1]}, p^{[k-1]}\big)^\top - S_2(h)\big(u^{[k-2]}, p^{[k-2]}\big)^\top$$

for $2 \le k \le n$. Hence, based on (5.4), (5.5) we successively obtain

$$\big[h \mapsto \big(u^{[k]}, p^{[k]}\big)\big] \in C_b^\infty\big(\mathscr{U}_1, \mathscr{L}(H^{s_1 - \frac{1}{2}}(\Gamma), Y^{s_1 - \frac{1}{2} - 2k})\big) \tag{5.11}$$

for $k = 0, \dots, n$. Now, writing the equation (5.10) as

$$S(\varepsilon, h)(r_n, q_n) = \varepsilon^{n+1} R \tag{5.12}$$

with the right hand side given by

$$R := S_1\big(u^{[n-1]}, p^{[n-1]}\big)^\top + \varepsilon S_2\big(u^{[n]}, p^{[n]}\big)^\top,$$

we find from (5.5) and (5.11)

$$\|R\|_{X^s} \le C\big(\|(u^{[n-1]}, p^{[n-1]})\|_{Y^{s+2}} + \|(u^{[n]}, p^{[n]})\|_{Y^{s+2}}\big) \le C\|f\|_{s_1 - 1/2}$$

because of $s_1 \ge s + 2n + 5/2$. Hence, after applying Lemma 4.3 to (5.12), we get

$$\|r_n\|_s^\Gamma \le C\|r_n\|_{s+1,\varepsilon} \le C\varepsilon^{n+1}\|R\|_{X^s} \le C\varepsilon^{n+1}\|f\|_{s_1 - \frac{1}{2}}$$

This proves the estimate (5.7). The further statements are consequences of explicit calculations. ☐

From Lemma 5.1 we obtain easily

Lemma 5.2. *Let $k \in \mathbb{N}$ and $s \ge 0$ be given, set $s_1 = s + \tau(k)$, $\tau = \tau(k)$ sufficiently large. Then there exists a mapping*

$$\big[(\varepsilon, h) \mapsto \mathscr{F}_k(\varepsilon, h)\big] \in C_b^\infty\big([0, 1) \times (\mathscr{U}_1), H^s(\Gamma)\big)$$

such that for all $(\varepsilon, h) \in (0, 1) \times \mathscr{U}_1$

$$\|\mathscr{F}(\varepsilon, h) - \mathscr{F}_k(\varepsilon, h)\|_s \le C\varepsilon^{k+1}$$

with C independent of ε and h. ☐

As the following lemma shows, we can always, starting with a sufficiently smooth solution of the thin film equation, construct an approximative solution of (4.41) up to remainder terms of any order $O(\varepsilon^k)$.

Lemma 5.3. *Let $k \in \mathbb{N}$, $s \geq 0$ and $T > 0$ be given, assume $s_2 = s_2(s,k) > s$ sufficiently large ,define $\mathscr{U}_2 := \mathscr{U}(s_2, M_2, \alpha)$ and let*

$$h_0 \in C^0\big([0,T], \mathscr{U}_2\big) \cap C^1\big([0,T], H^{s_2-4}(\Gamma)\big)$$

be a solution of the thin film equation (2.8)*. Then there exists*

$$h_{\varepsilon,k} \in C^1\big([0,T], H^s(\Gamma)\big) \quad \textit{with} \quad h_{\varepsilon,k}(0) = h_0(0)$$

such that

$$h_{\varepsilon,k}(t) > \alpha/2, \quad \|\dot{h}_{\varepsilon,k}(t)\|_s, \|h_{\varepsilon,k}(t)\|_s \leq C, \tag{5.13}$$

$$\|\dot{h}_{\varepsilon,k}(t) - \mathscr{F}(\varepsilon, h_{\varepsilon,k}(t))\|_s \leq C\varepsilon^{k+1} \tag{5.14}$$

for all $t \in [0,T]$ and $\varepsilon \in (0, \varepsilon_0)$ with constants $\varepsilon_0 > 0$, $C = C(s,k,T,\mathscr{U}_1)$. Furthermore,

$$\int_\Gamma h_{\varepsilon,k}(t)\, d\Gamma = \int_\Gamma h_0(0)\, d\Gamma. \tag{5.15}$$

Proof. Choose $\mathscr{F}_k$ and $\tau = \tau(k) > 0$ according to Lemma 5.2, define $s_2 := s + (k+1)\tau$ and let h_0 be as presupposed. Then it is sufficient to construct $h_{\varepsilon,k} = h_{\varepsilon,k}(t)$ with

$$\|\dot{h}_{\varepsilon,k}(t)\|_s, \|h_{\varepsilon,k}(t)\|_{s+\tau} \leq C, \quad \|\dot{h}_{\varepsilon,k}(t) - \mathscr{F}_k(\varepsilon, h_{\varepsilon,k}(t))\|_s \leq C\varepsilon^{k+1}$$

for $\varepsilon \in (0, \varepsilon_0)$, $t \in [0,T]$ and $\varepsilon_0 > 0$ sufficiently small. Because of the k-times differentiability of $\mathscr{F}_k$ with respect to both variables ε and h this can be done straightforwardly by inserting the ansatz

$$h_{\varepsilon,k}(t) = h_0(t) + \varepsilon h_1(t) + \ldots + \varepsilon^k h_k(t)$$

with

$$h_i(0) = 0, \quad \|h_i(t)\|_{s_2 - i\tau}, \|\dot{h}_i(t)\|_{s_2-(i+1)\tau} \leq C, \quad i = 1, \ldots, k,\ t \in [0,T] \tag{5.16}$$

into the evolution equation $\dot{h}_{\varepsilon,k} = \mathscr{F}_k(\varepsilon, h_{\varepsilon,k})$ and comparing coefficients with equal powers of ε up to order k. To determine h_{l+1} in this way from already known $h_0, \ldots, h_l$ we have to solve a linear evolution equation of the form

$$\dot{h}_{l+1}(t) = \mathscr{F}_k'\big(0, h_0(t)\big)h_{l+1}(t) + R_l(t), \quad h_{l+1}(0) = 0,$$

where $R_l(t)$ is a finite sum of terms

$$D_\varepsilon^i D_h^j \mathscr{F}_k\big(0, h_0(t)\big)\big\{h_{i_1}(t), \ldots, h_{i_j}(t)\big\}$$

with $1 \leq i + j \leq l+1$, $i_1, \ldots i_j \in \{0, \ldots, l\}$. Thus, assuming (5.16) for $i = 0, \ldots, l$, we obtain from Lemma 5.2 (with s replaced by $s_2 - (l+1)\tau$), $s_1 = s_2 - l\tau$)

$$\|R_l(t)\|_{s_2-(l+1)\tau} \leq C, \quad t \in [0,T]. \tag{5.17}$$

Further, in view of (4.31) we find $\mathscr{F}_k'(0, h_0)$ to be the linear fourth order differential operator

$$\mathscr{F}_k'(0, h_0)h = \operatorname{div}_x\big(h_0^2 h \nabla_x(\Delta_x h_0)\big) + \frac{1}{3}\operatorname{div}_x\big(h_0^3 \nabla_x(\Delta_x h)\big)$$

which is uniformly elliptic as long as $h_0(t)$ varies in $\mathscr{U}_2$. Thus by standard L^2-based energy estimates we obtain that any solution $h = h(t)$ of the inhomogeneous evolution equation

$$\dot{h} = \mathscr{F}_k'(0, h_0)h + f \quad \text{in} \quad [0, T].$$

satisfies

$$\|h(t)\|_\sigma \le C\big(\|h(0)\|_\sigma + \max_{t\in[0,T]} \|f(t)\|_\sigma\big)$$

Applying this estimate to h_{l+1} with $\sigma = s_2 - (l+1)\tau$ and using (5.17) together with $\tau > 4$ we get (5.16) also for $i = l+1$. The estimate (5.14) is now a consequence of Lemma 5.2 and Taylor's theorem

$$\Big\|\mathscr{F}_k(\varepsilon, h_{\varepsilon,k}(t)) - \sum_{l=0}^{k} \frac{\varepsilon^l}{l!}\frac{d^l}{d\varepsilon^l}\mathscr{F}_k(\varepsilon, h_{\varepsilon,k}(t))|_{\varepsilon=0}\Big\|_s \le C\varepsilon^{k+1} \max_{\eta\in[0,\varepsilon]}\Big\|\frac{d^{k+1}}{d\eta^{k+1}}\mathscr{F}_k(\eta, h_{\eta,k}(t))\Big\|_s.$$

Finally, as $\mathscr{F}(\varepsilon, h)$ has always zero mean value over Γ, we find

$$\int_\Gamma \dot{h}_{\varepsilon,k}(t)\, d\Gamma = O(\varepsilon^{k+1}),$$

by (5.14), and consequently $\int_\Gamma h_k(t)\, d\Gamma = 0$, $i = 1, \dots, k$ as $h_i(0) = 0$. This completes the proof. □

6. Energy estimates and proof of the main result

Define $\mathscr{U}$ as in Section 4. Again, if not stated otherwise, constants occurring in estimates are independent of $\varepsilon \in (0,1)$ and $h \in \mathscr{U}$.

From Lemma 4.3 together with the scaled trace inequality we obtain

$$\|F(\varepsilon, h)f\|_{s-1} + \sqrt{\varepsilon}\|F(\varepsilon, h)f\|_{s-1/2} \le C\varepsilon^{-5/2}\min\big\{\|f\|^\Gamma_{s-3/2}, \sqrt{\varepsilon}\|f\|^\Gamma_{s-1}\big\} \tag{6.1}$$

for $1 \le s \le s_0 - 1/2$. On the other hand, Lemma 4.6 yields

$$\|F(\varepsilon, h)f\|_{s-2} \le C\|f\|_s$$

for $2 \le s \le s_0 - 3/2$. In particular this implies the boundedness

$$\|\mathscr{F}(\varepsilon, h)\|_{s_0-4} \le C \tag{6.2}$$

independent of $\varepsilon \in [0,1]$ and $h \in \mathscr{U}$. Moreover, the estimate of Fréchet derivatives in Lemma 4.5 together with the scaled trace inequality imply

$$\|F^{(l)}(\varepsilon, h)\{h_1, \dots, h_l\}f\|_{1/2} \le C\varepsilon^{-3}\|h_1\|_{s_0}\cdots\|h_l\|_{s_0}\|f\|_{-1/2} \tag{6.3}$$

and

$$\|F^{(l)}(\varepsilon, h)\{h_1, \dots, h_l\}f\|_{1/2} \le C\varepsilon^{-3}\|h_1\|_{3/2}\|h_2\|_{s_0}\cdots\|h_l\|_{s_0}\|f\|_{s_0} \tag{6.4}$$

Actually, by exploiting the structure of F more precisely, (6.4) can be improved. For $l = 1$ this will be done in the following lemma.

Lemma 6.1. *There holds*

$$\|F'(\varepsilon, h)\{h_1\}f\|_{1/2} \le C\|h_1\|_{3/2}\|f\|_{s_0}. \tag{6.5}$$

Proof. By definition of F we have

$$F'(\varepsilon,h)\{h_1\}f = \varepsilon^{-2}\big((-\varepsilon\nabla_x h_1,0)\cdot u(\varepsilon,h)[f\nu] \\ + (-\varepsilon\nabla_x h,1)\cdot u'(\varepsilon,h)\{h_1\}[f\nu] + (-\varepsilon\nabla_x h,1)\cdot u(\varepsilon,h)[\nu'\{h_1\}f]\big).$$

Hence, in view of the estimates (4.31)-(4.34), it is sufficient to check the estimate

$$\big\|(T(u,p)-f)\nu'\{h_1\}\big\|_{1/2} \le C\varepsilon^2\|h_1\|_{3/2}\|f\|_{s_0}. \tag{6.6}$$

As

$$\nu'\{h_1\} = -h_1h^{-1}\nu - \varepsilon h^{-1}(\nabla_x h_1,0)$$

and $T(u,p)\nu = f\nu$, we have

$$(T(u,p)-f)\nu'\{h_1\} = -\varepsilon h^{-1}(T(u,p)-f)(\nabla_x h_1,0).$$

As a consequence of (4.31) we obtain

$$\|T_{ij}(u,p)-f\delta_{ij}\|_{C^1(\Gamma)} \le C\varepsilon\|f\|_{s_0},$$

which implies (6.6). □

To find estimates in higher Sobolev norms we will use the invariance of our problem with respect to horizontal translations. For this purpose, we first derive a chain rule expressing this invariance.

For $\mu\in\mathbb{T}^m$ let $\mathcal{S}_\mu \in \mathscr{L}\big(L^2(\mathbb{T}^m)\big)$ denote the translation operator given by

$$\mathcal{S}_\mu u(x) := u(x+\mu).$$

We clearly have

$$\mathcal{S}_\mu F(\varepsilon,h)f = F(\mathcal{S}_\mu h,\varepsilon)\mathcal{S}_\mu f$$

and therefore, after differentiation with respect to μ at $\mu=0$,

$$\partial_i(F(\varepsilon,h)f) = F'(\varepsilon,h)\{\partial_i h\}f + F(\varepsilon,h)\partial_i f$$

for $i=1,\dots,m$, $h\in H^{s_0+1}(\Gamma)$ and $f\in H^{s_0-1/2}(\Gamma)$. Taking the k-th order Fréchet derivative of this equation yields

$$\partial_i F^{(k)}(\varepsilon,h)\{h_1,\dots,h_k\}f = F^{(k+1)}(\varepsilon,h)\{h_1,\dots,h_k,\partial_i h\}f \\ + \sum_{l=1}^{k} F^{(k)}(\varepsilon,h)\{h_1,\dots,h_{l-1},\partial_i h_l,h_{l+1}\dots,h_k\}f + F^{(k)}(\varepsilon,h)\{h_1,\dots,h_k\}\partial_i f. \tag{6.7}$$

For higher derivatives in horizontal directions, we get under appropriate smoothness assumptions on h and f

$$\partial^\gamma F(\varepsilon,h)f = \sum C_{\gamma_1,\dots,\gamma_{k+1}}F^{(k)}(\varepsilon,h)\{\partial^{\gamma_1}h,\dots\partial^{\gamma_k}h\}\partial^{\gamma_{k+1}}f, \tag{6.8}$$

where the sum is to be taken over all $(k+1)$-tuples of multiindices $\gamma_i\in\mathbb{N}^m$ such that $\gamma_1+\ldots+\gamma_{k+1}=\gamma$, $\gamma_1,\dots\gamma_k>0$. In particular, one has $C_\gamma = C_{\gamma,0}=1$. This follows from (6.7) by induction over $|\gamma|$.

Lemma 6.2. *Let $\gamma\in\mathbb{N}^m$ be a multiindex and $s := |\gamma|+1/2 \ge 2s_0+7/2$. Then*

$$\|\partial^\gamma F(\varepsilon,h)f - F(\varepsilon,h)\partial^\gamma f\|_{1/2} \le C\varepsilon^{-3}\|f\|_{s-2}\big(1+\varepsilon^3\|h\|_{s+1}\big). \tag{6.9}$$

with $C = C(s, \|h\|_s)$. Moreover we have

$$\partial^\gamma\big(F(\varepsilon,h)f - F(\varepsilon,\bar h)f\big) = \big(F(\varepsilon,h) - F(\varepsilon,\bar h)\big)\partial^\gamma f + Q_\gamma(\varepsilon,h,\bar h,f),$$

where Q_γ allows for the estimate

$$\|Q_\gamma(\varepsilon,h,\bar h,f)\|_{1/2} \le C\|f\|_{s-2}\big(\|h-\bar h\|_{s+1} + \varepsilon^{-3}\|h-\bar h\|_s\|\bar h\|_{s+1}\big) \tag{6.10}$$

with a constant $C = C\big(s, \|h\|_s, \|\bar h\|_s\big)$.

Proof. By the chain rule (6.8) we have

$$\partial^\gamma F(\varepsilon,h)f = F(\varepsilon,h)\partial^\gamma f + F'(\varepsilon,h)\{\partial^\gamma h\}f + R_\gamma(\varepsilon,h)$$

with

$$R_\gamma(\varepsilon,h) = \sum F^{(l)}(\varepsilon,h)\{\partial^{\gamma_1}h,\dots,\partial^{\gamma_l}h\}\partial^\beta f$$

where the sum is to be taken over all multiindices $\gamma_1,\dots,\gamma_l$, β and $l = 1,2,\dots$ with

$$\gamma = \gamma_1 + \ldots + \gamma_l + \beta, \quad |\gamma_1|,\dots,|\gamma_l|,|\beta| \le |\gamma| - 1. \tag{6.11}$$

To estimate these terms, we first consider the case $|\beta| \ge \max_{i=1,\dots,l}|\gamma_i|$, in which we have

$$|\beta| \le s - \tfrac{3}{2}, \quad |\gamma_1|,\dots,|\gamma_l| \le \tfrac12(s-\tfrac12) \le s - 2 - s_0,$$

and we get by (6.3)

$$\begin{aligned}\|F^{(l)}(\varepsilon,h)\{\partial^{\gamma_1}h,\dots,\partial^{\gamma_l}h\}\partial^\beta f\|_{1/2} &\le C\varepsilon^{-3}\|\partial^{\gamma_1}h\|_{s_0}\dots\|\partial^{\gamma_l}h\|_{s_0}\|\partial^\beta f\|_{-1/2}\\ &\le C\varepsilon^{-3}\|f\|_{s-2}\|h\|_{s-2}^l.\end{aligned} \tag{6.12}$$

On the other hand, if $|\beta| \le |\gamma_j| = \max_{i=1,\dots,l}|\gamma_i|$ with some $j \in \{1,\dots,l\}$ then

$$|\gamma_j| \le s - \tfrac32, \quad |\beta|,|\gamma_i| \le \tfrac12(s-\tfrac12) \le s-2-s_0 \ \text{ for } \ i = 1,\dots,l,\ i \ne j,$$

we assume without loss of generality $j = 1$ and find from (6.4)

$$\begin{aligned}\|F^{(l)}(\varepsilon,h)\{\partial^{\gamma_1}h,\dots,\partial^{\gamma_l}h\}\partial^\beta f\|_{1/2} &\le C\varepsilon^{-3}\|\partial^{\gamma_1}h\|_{3/2}\|\partial^{\gamma_s}h\|_{s_0}\dots\|\partial^\beta f\|_{s_0}\\ &\le C\varepsilon^{-3}\|f\|_{s-2}\|h\|_s^l\end{aligned} \tag{6.13}$$

From Lemma 6.1 we get

$$\|F'(\varepsilon,h)\{\partial^\gamma h\}f\|_{1/2} \le C\|f\|_{s-2}\|h\|_{s+1}.$$

Together with (6.12) and (6.13), this proves (6.9).

To show (6.10) note first that $R_\gamma(\varepsilon,h) - R_\gamma(\varepsilon,\bar h)$ is a finite sum of terms of the form

$$\begin{aligned} I &:= F^{(l)}(\varepsilon,h)\{\partial^{\gamma_1}(h-\bar h),\partial^{\gamma_2}\hat h_2,\dots\partial^{\gamma_l}\hat h_l\}\partial^\beta f,\\ J &:= \big(F^{(l)}(\varepsilon,h) - F^{(l)}(\varepsilon,\bar h)\big)\{\partial^{\gamma_1}\bar h,\dots\partial^{\gamma_l}\bar h\}\partial^\beta f,\end{aligned}$$

with multiindices satisfying (6.11) and $\hat h_i = h$ or $\hat h_i = \bar h$, $i = 2,\dots,l-1$. Writing

$$J = \int_0^1 F^{(l+1)}\big(\varepsilon,\bar h + \theta(h-\bar h)\big)\{h-\bar h,\partial^{\gamma_1}\bar h,\dots\partial^{\gamma_l}\bar h\}\partial^\beta f\,d\theta$$

and proceeding as in the estimation of $R_\gamma(\varepsilon,h)$ above, we get

$$\begin{aligned}\|I\|_{1/2} &\le C\varepsilon^{-3}\|f\|_{s-2}\|h-\bar h\|_s\left(\|h\|_s + \|\bar h\|_s\right)^{l-1},\\ \|J\|_{1/2} &\le C\varepsilon^{-3}\|f\|_{s-2}\|h-\bar h\|_s\left(\|h\|_s + \|\bar h\|_s\right)^{l}.\end{aligned}$$

Finally, as

$$
\begin{aligned}
&F'(\varepsilon,h)\{\partial^\gamma h\}f - F'(\varepsilon,\bar h)\{\partial^\gamma \bar h\}f\\
&\qquad = F'(\varepsilon,h)\{\partial^\gamma(h-\bar h)\}f + \int_0^1 F''(\varepsilon,\bar h+\theta(h-\bar h))\{h-\bar h,\partial^\gamma \bar h\}f\,d\theta,
\end{aligned}
$$

(6.10) follows from

$$\|F'(\varepsilon,h)\{\partial^\gamma(h-\bar h)\|_{1/2} \le C\|\partial^\gamma(h-\bar h)\|_{3/2}\|f\|_{s_0} \le C\|h-\bar h\|_{s+1}\|f\|_{s-2}$$

which holds because of Lemma 6.1, and

$$
\begin{aligned}
&\|F''(\varepsilon,\bar h+\theta(h-\bar h))\{h-\bar h,\partial^\gamma \bar h\}f\|_{1/2}\\
&\qquad \le C\varepsilon^{-3}\|h-\bar h\|_{s_0}\|\partial^\gamma \bar h\|_{3/2}\|f\|_{s_0} \le C\varepsilon^{-3}\|h-\bar h\|_{s-2}\|\bar h\|_{s+1}\|f\|_{s-2}
\end{aligned}
$$

because of (6.4). This completes the proof. □

Lemma 6.3. *Under the assumptions of Lemma* 6.2*,*

$$P_\gamma(\varepsilon,h,\bar h) := \partial^\gamma\big(\mathscr{F}(\varepsilon,h)-\mathscr{F}(\varepsilon,\bar h)\big) - F(\varepsilon,h)\big[\kappa'(\varepsilon,h)\{\partial^\gamma(h-\bar h)\}\big]$$

satisfies

$$\|P_\gamma(\varepsilon,h,\bar h)\|_{1/2} \le C\big(\|h-\bar h\|_{s+1} + \varepsilon^{-3}\|h-\bar h\|_s\|\bar h\|_{s+1}\big)$$

with $C = C(s,\|h\|_s,\|\bar h\|_s)$*.*

Proof. Because of the structure of $h \mapsto \kappa(\varepsilon,h)$ as a quasi-linear second order differential operator we find easily

$$\|\kappa(\varepsilon,h)\|_{s-2} \le C\|h\|_s, \quad \|\kappa(\varepsilon,h)-\kappa(\varepsilon,\bar h)\|_{s-2} \le C\|h-\bar h\|_s,$$

as well as

$$\|\partial^\gamma(\kappa(\varepsilon,h)-\kappa(\varepsilon,\bar h)) - \kappa'(\varepsilon,h)\{\partial^\gamma(h-\bar h)\}\|_{-1/2} \le C(\|h\|_s,\|\bar h\|_s)\|\bar h\|_{s+1}\|h-\bar h\|_s.$$

and by (6.1) with $s=1$

$$\|F(\varepsilon,h)[\partial^\gamma(\kappa(\varepsilon,h)-\kappa(\varepsilon,\bar h)) - \kappa'(\varepsilon,h)\{\partial^\gamma(h-\bar h)\}]\|_{1/2} \le C\varepsilon^{-3}\|h-\bar h\|_s\|\bar h\|_{s+1}. \tag{6.14}$$

Furthermore, from

$$(F(\varepsilon,h)-F(\varepsilon,\bar h))\partial^\gamma\kappa(\varepsilon,\bar h) = \int_0^1 F'(\varepsilon,\bar h+\theta(h-\bar h))\{h-\bar h\}[\partial^\gamma\kappa(\varepsilon,\bar h)]\,d\theta$$

and (6.3)) we get

$$\|(F(\varepsilon,h)-F(\varepsilon,\bar h))\partial^\gamma\kappa(\varepsilon,\bar h)\|_{1/2} \le C\varepsilon^{-3}\|h-\bar h\|_{s_0}\|\partial^\gamma\kappa(\varepsilon,\bar h)\|_{-1/2} \le C\varepsilon^{-3}\|h-\bar h\|_s\|\bar h\|_{s+1}. \tag{6.15}$$

Using the notation of Lemma 6.2, we split

$$
\begin{aligned}
P_\gamma(\varepsilon,h,\bar h) \;=\;& (F(\varepsilon,h)-F(\varepsilon,\bar h))\partial^\gamma\kappa(\varepsilon,\bar h) + Q_\gamma(\varepsilon,h,\bar h,\kappa(\varepsilon,\bar h)),\\
&+\partial^\gamma F(\varepsilon,h)(\kappa(\varepsilon,h)-\kappa(\varepsilon,\bar h)) - F(\varepsilon,h)\partial^\gamma(\kappa(\varepsilon,h)-\kappa(\varepsilon,\bar h))\\
&+F(\varepsilon,h)\left(\partial^\gamma(\kappa(\varepsilon,h)-\kappa(\varepsilon,\bar h)) - \kappa'(\varepsilon,h)\{\partial^\gamma(h-\bar h)\}\right).
\end{aligned}
$$

The estimate follows now from (6.9), (6.10), (6.14), and (6.15). □

The following lemma states a coercivity property of the linear operator $\varphi \mapsto F(\varepsilon, h)\varphi$ independent of ε, which is crucial for our further considerations. To formulate this result we define

$$\|\varphi\|_{-1/2,\varepsilon} := \sup_{\psi \in H^{1/2}(\Gamma),\, \psi \neq 0} \frac{\langle \varphi, \psi \rangle_{L^2(\Gamma)}}{\|\psi\|_0 + \sqrt{\varepsilon}\|\psi\|_{1/2}}.$$

Clearly there holds

$$C_1\|\varphi\|_{-1/2} \leq \|\varphi\|_{-1/2,\varepsilon} \leq C_2 \min\big\{\|\varphi\|_0, \varepsilon^{-1/2}\|\varphi\|_{-1/2}\big\}$$

with some positive C_1, C_2. Moreover, for any $\mu > 0$ there exist $C = C(\mu)$ and $\varepsilon_0 = \varepsilon_0(\mu) > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$

$$\|\varphi\|_{-1/2} \leq \mu\|\varphi\|_{-1/2,\varepsilon} + C(\mu)\|\varphi\|_{-1}. \tag{6.16}$$

This is seen as follows. Expand φ in a Fourier series $\varphi(x) = \sum_{n \in \mathbb{Z}^m} \hat{\varphi}_n e^{2\pi i n \cdot x}$ and define

$$\varphi_k(x) := \sum_{|n| \geq k} \hat{\varphi}_n e^{2\pi i n \cdot x}, \quad \psi_k(x) := \sum_{|n| \geq k} |n|^{-1}\hat{\varphi}_n e^{2\pi i n \cdot x}, \qquad k = 1, 2, \ldots.$$

Then

$$\begin{aligned} \|\varphi - \varphi_k\|_{-1/2} &\leq Ck^{1/2}\|\varphi\|_{-1}, \\ \|\psi_k\|_0 &\leq Ck^{-1/2}\|\varphi_k\|_{-1/2}, \\ \|\psi_k\|_{1/2} &\leq C\|\varphi_k\|_{-1/2}, \end{aligned} \tag{6.17}$$

and

$$\langle \varphi, \psi_k \rangle_{L^2(\Gamma)} \sim \|\varphi_k\|_{-1/2}^2.$$

Consequently, we have

$$\|\varphi\|_{-1/2,\varepsilon} \geq \frac{\langle \varphi, \psi_k \rangle_{L^2(\Gamma)}}{\|\psi_k\|_0 + \sqrt{\varepsilon}\|\psi_k\|_{1/2}} \geq \frac{C}{k^{-1/2} + \varepsilon^{1/2}}\|\varphi_k\|_{-1/2}.$$

Thus, given $\mu > 0$, we choose k sufficiently large, $\varepsilon_0 > 0$ sufficiently small and obtain

$$\|\varphi_k\|_{-1/2} \leq \mu\|\varphi\|_{-1/2,\varepsilon}$$

for $\varepsilon \in (0, \varepsilon_0)$. Hence together with (6.17) we obtain (6.16).

Lemma 6.4. *There exists a positive constant c independent of $\varepsilon \in (0,1)$, such that for all $h \in \mathscr{U}$ and all $\varphi \in H^{1/2}(\Gamma)$ that satisfy*

$$\int_\Gamma \varphi \, d\Gamma = 0 \tag{6.18}$$

we have

$$\big\langle \varphi, F(\varepsilon, h)\varphi \big\rangle_{L^2(\Gamma)} \geq c\|\varphi\|_{-1/2,\varepsilon}^2. \tag{6.19}$$

Proof. Let (u, p) be the solution to (4.26) with $E := \varphi\nu$ and observe

$$F(\varepsilon, h)\varphi = \varepsilon^{-2}\sqrt{g}\nu \cdot u,$$

thus

$$\langle \varphi, F(\varepsilon, h)\varphi\rangle_{L^2(\Gamma)} = \varepsilon^{-2}\int_\Gamma \sqrt{g}\varphi\nu\cdot u\,d\Gamma = \varepsilon^{-2}\int_\Gamma \sqrt{g}T_{ij}(u,p)\nu_j u_i\,d\Gamma$$
$$= \varepsilon^{-2}a(u,u) \geq c\varepsilon^{-2}\|u\|^2_{1,\varepsilon}$$

with some $c>0$ by the Green identity (4.15) and Korn's inequality, hence it suffices to prove

$$\|\varphi\|_{-1/2,\varepsilon} \leq C\varepsilon^{-1}\|u\|_{1,\varepsilon}. \tag{6.20}$$

It follows from Lemmas 3.1, (ii) and 4.2, (ii) that for any $\psi \in H^{1/2}(\Gamma)$ satisfying

$$\int_\Gamma \psi\,d\Gamma = 0,$$

there is a $v \in H^1(\Omega, \mathbb{R}^{m+1})$ such that

$$D_i v_i = 0 \ \text{ on } \ \Omega, \quad v_i\nu_i = \psi/\sqrt{g} \ \text{ on } \ \Gamma, \quad v = 0 \ \text{ on } \ \Gamma_0$$

and

$$\|v\|_{1,\varepsilon} \leq C\varepsilon^{-1}\big(\|\psi\|^\Gamma_0 + \sqrt{\varepsilon}\|\psi\|^\Gamma_{1/2}\big).$$

Thus, applying (4.15) again,

$$\int_\Gamma \varphi\psi\,d\Gamma = \int_\Gamma \sqrt{g}\varphi\nu_j v_j\,d\Gamma = \int_\Gamma \sqrt{g}T_{ij}(u,p)\nu_i v_j\,d\Gamma$$
$$= a(u,v) \leq C\|u\|_{1,\varepsilon}\|v\|_{1,\varepsilon} \leq C\varepsilon^{-1}\|u\|_{1,\varepsilon}\big(\|\psi\|^\Gamma_0 + \sqrt{\varepsilon}\|\psi\|^\Gamma_{1/2}\big).$$

Due to (6.18), the last inequality is also valid for arbitrary $\psi \in H^{1/2}(\Gamma)$. This implies (6.20). $\square$

Remark 6.5. From the facts that $F(\varepsilon, h)\varphi$ has zero mean value and vanishes on constants it is easily seen that the condition (6.18) is natural.

The following proposition is the main result of this section. In particular, together with the existence and blow up statement of Theorem 2.1 it implies Theorem 2.2.

Proposition 6.6. *Let $n \in \mathbb{N}$, $T > 0$ and let $s \geq 2s_0$ integer. Assume $k = k(s,n) \in \mathbb{N}$ sufficiently large and then let $s_2 = s_2(k,s)$, $h_0 = h_0(t)$ and $h_{\varepsilon,k} = h_{\varepsilon,k}(t)$ as in Lemma* 5.3*. Then, if $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$ sufficiently small and $h_\varepsilon = h_\varepsilon(t) > 0$ is the solution of* (4.41) *of class*

$$h_\varepsilon \in C^0\big([0,T_\varepsilon], H^s_+(\Gamma)\big) \cap C^1\big([0,T_\varepsilon], H^{s-1}(\Gamma)\big)$$

in some time interval $[0,T_\varepsilon]$, $T_\varepsilon \in (0,T]$ with $h_\varepsilon(0) = h_0(0)$, we have

$$\|h_\varepsilon(t) - h_{\varepsilon,k}(t)\|_s \leq C\varepsilon^n \ \text{ for } \ t \in [0,T_\varepsilon]. \tag{6.21}$$

The constants C and ε_0 depend on s,k,n,M_2,α.

Proof. By the construction of Lemma 5.3 we have

$$\dot{h}_{\varepsilon,k}(t) = F\big(\varepsilon, h_{\varepsilon,k}(t)\big)\kappa\big(\varepsilon, h_{\varepsilon,k}(t)\big) + R_1(t), \quad t \in [0,T]$$

where the remainder term satisfies

$$\|R_1(t)\|_s \leq C\varepsilon^k. \tag{6.22}$$

Let $h_\varepsilon = h_\varepsilon(t)$, $\varepsilon \in (0,\varepsilon_0)$ be a solution of (4.41) as presupposed and choose $\tilde{T}_\varepsilon \in (0, T_\varepsilon]$ maximal such that

$$h_\varepsilon(t) \geq \alpha/4, \quad \|h_\varepsilon(t) - h_{\varepsilon,k}(t)\|_s \leq 1 \text{ for } t \in [0, \tilde{T}_\varepsilon].$$

It suffices to prove

$$\|h_\varepsilon(t) - h_{\varepsilon,k}(t)\|_s \leq C\varepsilon^n \text{ for } t \in [0, \tilde{T}_\varepsilon] \tag{6.23}$$

with some constant C depending on s, k, n, M_2, α.

Actually, we have $\tilde{T}_\varepsilon = T_\varepsilon$ for $\varepsilon_0 > 0$ sufficiently small, as (6.23) then implies

$$h_\varepsilon(t) \geq \alpha/2, \quad \|h_\varepsilon(t) - h_{\varepsilon,k}(t)\|_s \leq 1/2 \text{ for } t \in [0, \tilde{T}_\varepsilon],$$

which contradicts the maximality of $\tilde{T}_\varepsilon$ if $\tilde{T}_\varepsilon < T_\varepsilon$.

As a first step we prove the estimate (6.23) for $s = 1$ in the sharper form

$$\|h_\varepsilon(t) - h_{\varepsilon,k}(t)\|_1 \leq C\varepsilon^k, \tag{6.24}$$

and then in a second step, assuming additionally $k \geq n + 6s - 3$, the full claim (6.23).

Step 1. Introducing the differences

$$d(t) := h_\varepsilon(t) - h_{\varepsilon,k}(t), \quad d_1(t) := \kappa\big(\varepsilon, h_\varepsilon(t)\big) - \kappa\big(\varepsilon, h_{\varepsilon,k}(t)\big),$$

we obtain for $d = d(t)$ the evolution equation

$$\dot{d}(t) = F\big(\varepsilon, h_\varepsilon(t)\big)d_1(t) + R_2(t) - R_1(t), \tag{6.25}$$

where

$$R_2(t) := \int_0^1 F'\big(\varepsilon, h_{\varepsilon,k}(t) + \theta d(t)\big)\{d(t)\}\kappa\big(\varepsilon, h_{\varepsilon,k}(t)\big)\, d\theta.$$

Note that $d(t)$ and $d_1(t)$ have zero mean value over Γ as a consequence of (4.42), (5.15) and the divergence structure of κ, respectively. Writing

$$d_1(t) = \int_0^1 \kappa'\big(\varepsilon, h_{\varepsilon,k}(t) + \theta d(t)\big)\{d(t)\}\, d\theta$$

and using

$$\kappa'(\varepsilon, h)\{d\} = \operatorname{div}_x\left(\frac{\nabla_x d}{(1+\varepsilon^2|\nabla_x h|^2)^{1/2}} - \frac{\varepsilon^2\nabla_x h}{(1+\varepsilon^2|\nabla_x h|^2)^{3/2}}\nabla_x h \nabla_x d\right),$$

we find for $\sigma \in [1,2]$

$$c_1\|d(t)\|_\sigma \leq \|d_1(t)\|_{\sigma-2} \leq c_2\|d(t)\|_\sigma$$

with positive constants $c_1, c_2 > 0$. Thus, by Lemma 6.1, the term R_2 satisfies

$$\|R_2(t)\|_{1/2} \leq C\|d(t)\|_{3/2}\|\kappa\big(\varepsilon, h_{\varepsilon,k}(t)\big)\|_{s_0} \leq C'\|d_1(t)\|_{-1/2}. \tag{6.26}$$

Moreover, an integration by parts yields

$$-\langle \dot{d}, d_1\rangle_{L^2(\Gamma)} = B(\varepsilon, h_\varepsilon, h_{\varepsilon,k})(\dot{d}, d)$$

where $B(\varepsilon, h, \bar{h}), B(\varepsilon, h) : H^1(\Gamma) \times H^1(\Gamma) \to \mathbb{R}$ are the bilinear forms defined by

$$B(\varepsilon, h, \bar{h})(e, f) := \int_0^1 B\big(\varepsilon, \bar{h} + \theta(h - \bar{h})\big)(e, f)\, d\theta$$

and

$$B(\varepsilon,v)(e,f) := \int_\Gamma \left(\frac{\nabla_x e \nabla_x f}{(1+\varepsilon^2|\nabla_x v|^2)^{1/2}} - \varepsilon^2 \frac{(\nabla_x v \nabla_x e)(\nabla_x v \nabla_x f)}{(1+\varepsilon^2|\nabla_x v|^2)^{3/2}} \right) dx.$$

Hence, using

$$\|h_\varepsilon\|_{C^1(\Gamma)}, \|\dot{h}_\varepsilon\|_{C^1(\Gamma)}, \|h_{\varepsilon,k}\|_{C^1(\Gamma)}, \|\dot{h}_{\varepsilon,k}\|_{C^1(\Gamma)} \le C,$$

we find

$$-\langle \dot{d}, d_1\rangle_{L^2(\Gamma)} \ge \frac{1}{2}\frac{d}{dt}\big(B(\varepsilon,h_\varepsilon,h_{\varepsilon,k})d^2\big) - C\|d\|_1^2.$$

Note that

$$c_1\|d\|_1^2 \le B(\varepsilon,h_\varepsilon,h_{\varepsilon,k})d^2 \le c_2\|d\|_1^2 \tag{6.27}$$

with some constants $c_1,c_2>0$. On the other hand, by scalar multiplication of (6.25) with $-d_1(t)$ we have

$$-\langle \dot{d}, d_1\rangle_{L^2(\Gamma)} = -\langle F(\varepsilon,h_\varepsilon)d_1, d_1\rangle_{L^2(\Gamma)} + \langle R_1, d_1\rangle_{L^2(\Gamma)} - \langle R_2, d_1\rangle_{L^2(\Gamma)},$$

thus by Lemma 6.4 together with (6.16) and (6.22), (6.26)

$$\begin{aligned} -\langle \dot{d}, d_1\rangle_{L^2(\Gamma)} &\le -\mu\|d_1\|_{-1/2}^2 + C(\mu)\|d_1\|_{-1}^2 + C\|R_1\|_1\|d_1\|_{-1} + C\|R_2\|_{1/2}\|d_1\|_{-1/2} \\ &\le C\|d_1\|_{-1}^2 + C\varepsilon^{2k} \end{aligned}$$

for μ sufficiently large and $\varepsilon\in[0,\varepsilon_0)$ with $\varepsilon_0>0$ sufficiently small. Therefore we find

$$\frac{d}{dt}\big(B\big(\varepsilon,h_\varepsilon(t),h_{\varepsilon,k}(t)\big)d(t)^2\big) \le C\Big(\varepsilon^{2k} + B\big(\varepsilon,h_\varepsilon(t),h_{\varepsilon,k}(t)\big)d(t)^2\Big)$$

for $t\in[0,\tilde{T}_\varepsilon]$ and $\varepsilon\in(0,\varepsilon_0)$. Due to $d(0)=0$ and (6.27), this implies (6.24).

Step 2. Let γ be a multiindex with $|\gamma|=s-1$. Then we have by (5.14) and Lemma 6.3

$$\partial^\gamma \dot{d}(t) = F\big(\varepsilon,h_\varepsilon(t)\big)d_\gamma(t) + R_\gamma(t) \tag{6.28}$$

where $d_\gamma(t) := \kappa'(\varepsilon,h_{\varepsilon,k}(t))\{\partial^\gamma d(t)\}$ and the remainder R_γ satisfies

$$\|R_\gamma(t)\|_{1/2} \le C\big(\varepsilon^k + \|d(t)\|_{s+1/2} + \varepsilon^{-3}\|d(t)\|_s\big).$$

Hence, similarly as in Step 1 above, we have

$$-\langle \partial^\gamma\dot{d}, d_\gamma\rangle_{L^2(\Gamma)} \ge \frac{1}{2}\frac{d}{dt}\big(B(\varepsilon,h_\varepsilon)(\partial^\gamma d)^2\big) - C\varepsilon^2\|d\|_s^2$$

and

$$-\langle \partial^\gamma\dot{d}, d_\gamma\rangle_{L^2(\Gamma)} \le -\mu\|d_\gamma\|_{-1/2}^2 + C(\mu)\|d_\gamma\|_{-1}^2 + \|R_\gamma\|_{1/2}\|d_\gamma\|_{-1/2},$$

and consequently

$$\frac{d}{dt}\Big(\sum_{|\gamma|=s-1} B\big(\varepsilon,h_\varepsilon(t)\big)\big(\partial^\gamma d(t)\big)^2\Big) \le \varepsilon^{2k} - \|d(t)\|_{s+1/2}^2 + c_1\varepsilon^{-6}\|d(t)\|_s^2$$

for $t\in[0,\tilde{T}_\varepsilon]$ and $\varepsilon\in(0,\varepsilon_0)$, $\varepsilon_0>0$ sufficiently small. Further, by interpolation and Young's inequality we obtain

$$c_1\varepsilon^{-6}\|d(t)\|_s^2 \le \|d(t)\|_{s+1/2}^2 + C\varepsilon^{-12(s-1/2)}\|d(t)\|_1^2,$$

hence together with (6.24)

$$\frac{d}{dt}\Big(\sum_{|\gamma|=s-1} B(\varepsilon,h(t))(\partial^\gamma d(t))^2\Big) \le C\big(\varepsilon^{2k}+\varepsilon^{2k-12(s-1/2)}\big) \le C\varepsilon^{2n}$$

because of our choice of k. Thus the proof is complete. □

References

[1] Antanovskii, L.K.: The bi-analytic stress-stream function in plane quasistationary problems of capillary hydrodynamics (in Russian), *Sibirsk. Mat. Zh.* **33** (1992) 1 3–15, English translation: *Siberian Math. J.* **33** (1992) 1 1–11

[2] Antanovskii, L.K.: Creeping thermocapillary motion of a two-dimensional deformable bubble: existence theorem and numerical simulation, *Eur. J. Mech., B/Fluids* **11** (1992) 6 741–758

[3] Beale, J.T.: Large-time regularity of viscous surface waves, *Arch. Rat. Mech. Anal.* **84** (1983/84), 307–352

[4] Cummings, L.J., Howison, S.D., King, J.R.: Conserved quantities in Stokes flow with free surfaces, *Phys. Fluids* **9** (1997) 3 477–480

[5] Duvaut, G., Lions, J.-L.: *Les inéquations en mécanique et en physique*, Dunod, Paris 1972

[6] Escher, J., Prokert, G.: Analyticity of solutions to nonlinear parabolic equations on manifolds and an application to Stokes flow, *J. Math. Fluid Mech.* **8** (2006) 1–35

[7] Friedman, A., Reitich, F.: Quasistatic motion of a capillary drop. I. The two-dimensional case, *J. Diff. Eq.* **178** (2002) 212–263

[8] Friedman, A., Reitich, F.: Nonlinear stability of a quasi-static Stefan problem with surface tension: a continuation approach. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4)* **30** (2001) 341–403

[9] Günther, M., Prokert, G.: Existence results for the quasistationary motion of a capillary liquid drop, *Zeitschrift für Analysis und ihre Anwendungen* **16** (1997) 2 311–348

[10] Hopper, R.W.: Plane Stokes flow driven by capillarity on a free surface, *J. Fluid Mech.* **213** (1990) 349–375

[11] Howison, S.D., Richardson, S.: Cusp development in free boundaries, and two-dimensional viscous flows. European Journ. Appl. Math. **6** (1995) 441–454

[12] Ockendon, H., Ockendon, J.R.: *Viscous flows*, Cambridge University Press, Cambridge 1995

[13] Pukhnachev, V.V.: The quasistationary approximation in the problem of the motion of an isolated volume of a viscous incompressible capillary liquid, *J. Appl. Maths. Mechs,* **62** (1998) 927–932

[14] Solonnikov, V.A.: On a non-steady motion of an isolated volume of a finite liquid mass bounded by a free surface (in Russian), *Zap. nauch. sem. LOMI* **152** (1986) 137–157

[15] Solonnikov, V.A.: On quasistationary approximation in the problem of motion of a capillary drop, in: Escher, J., Simonett, G. (Eds.):*Topics in nonlinear analysis*, Birkhäuser 1999, 643–671

[16] Solonnikov, V.A.: On the justification of the quasistationary approximation in the problem of motion of a viscous capillary drop. *Interfaces and Free Boundaries* **1** (1999), 125–173