Spectral stability of small-amplitude viscous shock waves in several space dimensions

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Abstract

A planar viscous shock profile of a hyperbolic-parabolic system of conservation laws is a steady solution in a moving coordinate frame. The asymptotic stability of viscous profiles and the related vanishing-viscosity limit are delicate questions already in the well understood case of one space dimension and even more so in the case of several space dimensions.

It is a natural idea to study the stability of viscous profiles by analysing the spectrum of the linearization about the profile. The Evans function method provides a geometric dynamical-systems framework to study the eigenvalue problem. In this approach eigenvalues correspond to zeros of an essentially analytic function $\mathcal{E}(\rho\lambda, \rho\omega)$ which detects nontrivial intersections of the so-called stable and unstable spaces, i. e., spaces of solutions that decay on one ("$-\infty$") or the other side ("$+\infty$") of the shock wave, respectively.

In a series of pioneering papers, Kevin Zumbrun and collaborators have established in various contexts that spectral stability, i. e., the non-vanishing of $\mathcal{E}(\rho\lambda, \rho\omega)$ and the non-vanishing of the Lopatinski-Kreiss-Majda function $\Delta(\lambda, \omega)$, imply nonlinear stability of viscous shock profiles in several space dimensions. In this paper we show that these conditions hold true for small amplitude extreme shocks under natural assumptions.

This is done by exploiting the slow-fast nature of the small-amplitude limit, which was used in a previous paper by the authors to prove spectral stability of small-amplitude shock waves in one space dimension. Geometric singular perturbation methods are applied to decompose the stable and unstable spaces into subbundles with good control over their limiting behaviour.

Three qualitatively different regimes are distinguished that relate the small strength ϵ of the shock wave to appropriate ranges of values of the spectral parameters $(\rho\lambda, \rho\omega)$. Various rescalings are used to overcome apparent degeneracies in the problem caused by loss of hyperbolicity or lack of transversality.

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0 Introduction

In this paper, we study the stability of planar viscous shock waves

$$v(x,t) = u(x \cdot N - st), \quad u(\pm\infty) = u^{\pm}, \tag{0.1}$$

in multidimensional systems of hyperbolic-parabolic viscous conservation laws

$$\frac{\partial}{\partial t} v + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}(f_j(v)) = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} v, \tag{0.2}$$

with

$$f_j : V \to \mathbb{R}^n, j = 1, \dots, d \text{ smooth}, V \subset \mathbb{R}^n \text{ convex and open} \tag{0.3}$$

and

$$\sum_{j=1}^{d} \xi_j Df_j(v) \ \mathbb{R}\text{-diagonalizable for all } v \in V \text{ and } \xi \in S^{d-1}. \tag{0.4}$$

Assuming w. l. o. g. that $N = (1, 0, \dots, 0)$ and Fourier transforming with respect to time t and transverse space variables $(x_2, \dots, x_d)$, one obtains the eigenvalue problem

$$(\check{\lambda} I + iB^{\check{\omega}}(u) + |\check{\omega}|^2 I)p + ((A(u) - sI)p)' = p'' \tag{0.5}$$

with frequencies

$$(\check{\lambda}, \check{\omega}) \in \mathbb{C} \times \mathbb{R}^{d-1}.$$

In Eqs. (0.5), we have written

$$A \equiv Df_1, \qquad B^{\check{\omega}} \equiv D(f^{\check{\omega}}), \ f^{\check{\omega}} \equiv \sum_{j=2}^{d} \check{\omega}_j f_j, \quad \check{\omega} = (\check{\omega}_2, \dots, \check{\omega}_d).$$

Any solution p of (0.5) that decays at both infinities, $p(\pm\infty) = 0$, is called an eigenfunction; e. g. and notably, for $(\check{\lambda}, \check{\omega}) = (0, 0)$, there is always the ("trivial") eigenfunction $p = u'$ corresponding to translation invariance. Roughly speaking, our goal will be to exclude eigenfunctions for

$$(\check{\lambda}, \check{\omega}) \neq (0, 0) \quad \text{with } \Re\lambda \geq 0.$$

Together with the equation for the profile u of the shock wave itself, Eq. (0.5) can be phrased as a first order autonomous system

$$\begin{aligned} u' &= f_1(u) - su - c \\ p' &= (A(u) - sI)p + q \\ q' &= (\check{\lambda} I + iB^{\check{\omega}}(u) + |\check{\omega}|^2 I)p. \end{aligned} \tag{0.6}$$

To prepare for a precise statemant of our goal, we fix $(\check{\lambda}, \check{\omega})$ and note that due to the linearity of the (p, q)-part of (0.6), this part can be viewed as a (non-autonomous) flow

$$X' = (\mathcal{A}_{\check{\lambda}, \check{\omega}}(u)(X) \tag{0.7}$$

on $\mathcal{G}_1^{2n}(\mathbb{C})$, the Grassmann manifold of all 1-dimensional subspaces of $\mathbb{C}^{2n}$. Let

$$S^-_{\check\lambda,\check\omega}, U^-_{\check\lambda,\check\omega}, S^+_{\check\lambda,\check\omega}, U^+_{\check\lambda,\check\omega}$$

be the (n-dimensional) stable and unstable spaces of the "frozen-end coefficient matrices" at $\mp\infty$, i. e., of

$$\begin{pmatrix} A(u^\mp) - sI & I \\ (\check\lambda I + iB^{\check\omega}(u^\mp) + |\check\omega|^2 I) & 0 \end{pmatrix}.$$

(More precisely, $S^-_{\check\lambda,\check\omega}, U^-_{\check\lambda,\check\omega}, S^+_{\check\lambda,\check\omega}, U^+_{\check\lambda,\check\omega}$ are continuous space-valued functions of $(\check\lambda,\check\omega)$ with the defining property that their values at $(\check\lambda,\check\omega)$ indeed are the stable resp. unstable spaces at least as long as $\Re\lambda > 0$. Cf. below.) Regarded as subsets of $\mathcal{G}_1^{2n}(\mathbb{C})$, $S^-_{\check\lambda,\check\omega}, U^-_{\check\lambda,\check\omega}$ and $S^+_{\check\lambda,\check\omega}, U^+_{\check\lambda,\check\omega}$ are invariant manifolds for the (autonomous) flows

$$X' = (\mathcal{A}_{\check\lambda,\check\omega}(u^\mp))(X), \tag{0.8}$$

respectively. We call any orbit $X : \mathbb{R} \to \mathcal{G}_1^{2n}(\mathbb{C})$ of (0.7) with limits

$$X(-\infty) \in U^-_{\check\lambda,\check\omega}, \quad X(+\infty) \in S^+_{\check\lambda,\check\omega}.$$

an *unstable-to-stable-bundle connection (USBC)* for (0.7).

Our findings concern shock waves of sufficiently small amplitude. We assume there exists a state $u_0 \in V$ such that

A.1. The matrices $Df_j(u_0), j = 1, \dots, d$ are symmetric.

A.2. For u near u_0 and $\check\omega$ near 0, the

smallest [resp. biggest] eigenvalue $\kappa(u,\check\omega)$ of $A(u) + B^{\check\omega}(u)$

is simple and satisfies
a. $D_u\kappa(u_0, 0) \notin$ left-ker$(A(u_0) - \kappa(u_0, 0)I)$ *(genuine nonlinearity in the sense of Lax),*
b. $D^2_{\check\omega}\kappa(u_0, 0) > 0$ *(strict convexity in the sense of Métivier [M1]).*

Assumption **A.1** implies that system the left-hand side of (0.2) is symmetric hyperbolic.[1]
Assumption **A.2.a** implies — cf. [L, MaPe] — that there are families

$$(u^-_\epsilon, u^+_\epsilon)_{0<\epsilon<\epsilon_0}, \quad (u_\epsilon)_{0<\epsilon<\epsilon_0} \qquad \text{with } \lim_{\epsilon\searrow 0} u_\epsilon = \lim_{\epsilon\searrow 0} u^\pm_\epsilon = u_0$$

of pairs of states and of profiles that solve (0.1) and (0.2) or, equivalently with the latter, $(0.6)_1$ with appropriate $c = c_\epsilon$ and

$$|u^+_\epsilon - u^-_\epsilon| \approx 2\epsilon,$$

covering locally all small-amplitude Lax 1-shocks [resp. n-shocks] of speeds

$$s = s_\epsilon \approx \kappa(u_0, 0).$$

[1] In fact, a more general context would also include a temporal component f_0 of the flux; for the minor adaptations needed to properly account for $f_0 \neq \mathrm{id}$, we refer the reader to the companion paper [FrSz3], which also covers state-dependent, and degenerate, viscosity and relaxation operators as opposed to the, though prototypical, Laplacian in (0.2).

Without loss of generality we will henceforth assume that we are dealing with 1-shocks (κ is the *smallest* eigenvalue). We note (from Lemma 6.2 of [M1]) that **A.2.b** implies

A.2.b′. $\sum_{j=2}^{n} B_{1j}^{\check\omega}(u_0)^2 > 0$ for any $\check\omega \in \mathbb{R}^{d-1} \setminus \{0\}$.

Finally, we assume w. l. o. g. that

$$A(u_0) = \mathrm{diag}(\kappa_1^0, \dots, \kappa_n^0) \text{ with } \kappa_1^0 = s_0 = 0 \quad \text{and} \quad B_{11}^{\check\omega}(u_0) \equiv 0 \text{ for any } \check\omega \in \mathbb{R}^{d-1},$$

which can be achieved by simple transformations. We will sometimes write $A^0 \equiv A(u_0)$, $B^{\omega,0} \equiv B^\omega(u_0)$.

Theorem 1. *Under the stated assumptions, if $\epsilon_0 > 0$ is sufficiently small, then for any $\epsilon \in (0, \epsilon_0]$ and any $(\lambda, \omega) \in$*

$$S \equiv \{(\lambda, \omega) \in \mathbb{C} \times \mathbb{R}^{d-1} : \Re\lambda \geq 0, |\lambda|^2 + |\omega|^2 = 1\},$$

(i) the ODE system

$$\begin{aligned} u' &= f(u) - s_\epsilon u - c_\epsilon \\ p' &= (A(u) - s_\epsilon I)p + q \\ q' &= \rho(\lambda I + iB^\omega(u) + \rho|\omega|^2 I)p. \end{aligned} \tag{0.9}$$

has no unstable-to-stable-bundle connection for any $\rho > 0$ and,
(ii) the quantity $\lambda[u_\epsilon] + i[f^\omega(u_\epsilon)]$ is transverse to the stable space of $(\lambda I + B^\omega(u_\epsilon^+))A(u_\epsilon^+)^{-1}$.

The motivation to prove Theorem 1 arose from the deep work [ZS, GMWZ1, GMWZ2, GMWZ3] of Kevin Zumbrun and collaborators in which it has been shown that any multidimensional planar Lax shock wave is nonlinearly stable in the viscous and the vanishing-viscosity context, if its so-called Evans and Lopatinski-Kreiss-Majda functions $\mathcal{E}, \Delta$ satisfy

$$\mathcal{E}(\rho\lambda, \rho\omega) \neq 0 \ \text{ for all } (\lambda, \omega) \in S \text{ and } \rho > 0 \tag{0.10}$$

and

$$\Delta(\lambda, \omega) \neq 0 \ \text{ for all } (\lambda, \omega) \in S. \tag{0.11}$$

As we will detail in Section 4, Theorem 1 readily means exactly that extreme shocks of small amplitude satisfy these two conditions and thus are nonlinearly stable, under the essentially sole assumptions **A.2** of genuine nonlinearity in the sense of Lax and of convexity in the sense of Métivier. We point out that part (ii) of Theorem 1 was already proved in [M1].

Instead of going into details on the, important, previous results, at this place we simply refer the reader to the fundamental papers [Go1, Li, SyX, ZH] on shock stability in one, and [Ma1, Ma2, M1, ZS, GMWZ1, GMWZ2, GMWZ3] in several space dimensions, as well as [AGaJ, GaJ, Sd, GaZ] on specific aspects of Evans functions, and the surveys [Z, M2]. The spectral stability of small-amplitude shock waves has been addressed in [FrSz1, PZ].

Part (i) of the Theorem 1 will be shown via the following three propositions:

Proposition 1. *(Inner regime.) For any $r_0 > 0$, the assertion of Theorem 1 holds under the restriction*

$$0 < \rho \leq \epsilon^2 r_0. \tag{0.12}$$

Proposition 2. *(Outer regime.) There exist $r_0, r_1 > 0$ such that the assertion of Theorem 1 holds under the restriction*

$$\epsilon^2 r_0 \leq \rho \leq r_1. \tag{0.13}$$

Proposition 3. *(Outmost regime.) For any $r_1 > 0$, the assertion of Theorem 1 holds under the restriction*

$$\rho \geq r_1. \tag{0.14}$$

Propositions 1,2,3 will be proved in Sections 1,2,3, respectively.

Part (ii) of Theorem 1 will be proved in Section 4 together with a brief discussion of the Evans and Lopatinski-Kreiss-Majda functions and a geometric-singular-pertubation re-derivation of the (general) fact — [ZS], here: Theorem 2 — that with a transversality coefficient Γ,

$$(\frac{\partial}{\partial\rho}\mathcal{E}(\rho\lambda, \rho\omega))|_{\rho=0} = \Gamma\Delta(\lambda, \omega), \quad (\lambda, \omega) \in S. \tag{0.15}$$

While we have to overcome a number of obstacles to obtain the exclusion of USBCs stated in Theorem 1, one prime difficulty consists in the fact that the abovementioned trivial eigenfunction manifests itself in various ways along portions of the boundaries of the inner and outer regime. To exclude USBCs at nearby interior points of these regimes, we will use a lemma on *transversal breaking of unstable-to-stable-bundle connections.* The rest of the present introductory section serves to just state this lemma, which will also be proved in Section 4.

Lemma 1. *Consider a family of autonomous systems*

$$\begin{aligned} \tau' &= g(\tau, \mu) \\ p' &= A(\tau, \mu)p + L(\tau, \mu)q \\ q' &= M(\tau, \mu)p \end{aligned} \tag{0.16}$$

on $\mathbb{R}^k \times \mathbb{C}^k \times \mathbb{C}^m$, parametrized by $\mu \in [0, \mu_0]$, with $g : \mathbb{R}^k \times [0, \mu_0] \to \mathbb{R}^k$,

$$A(\tau, \mu) \equiv D_\tau g(\tau, \mu) \in \mathbb{R}^{k\times k}, \quad L(\tau, \mu) \in \mathbb{C}^{k\times m}, \quad M(\tau, \mu) \in \mathbb{C}^{m\times k},$$

and

$$M(\tau, 0) \equiv 0.$$

Let τ_μ,

$$\tau'_\mu = g(\tau_\mu, \mu), \quad \tau_\mu(\pm\infty) = \tau_\mu^\pm,$$

be a corresponding family of transversal heteroclinic orbits and consider the naturally associated unstable-to-stable-bundle connection ("USBC") at $\mu = 0$ *spanned by*

$$(p, q) = (\tau'_0, 0).$$

Assume that the $A(\tau_0^\pm)$ *are hyperbolic and the dimensions of their unstable resp. stable spaces satisfy*

$$d_u^- + d_s^+ = k + 1.$$

For the (linear autonomous) left-end and right-end slow flows

$$\dot{q} = G_\mu^\pm q, \quad (G_0^\pm \equiv -D_\mu M(\tau^\pm, 0)(A(\tau^\pm, 0))^{-1} L(\tau^\pm, 0),)$$

assume that

$$\textit{the unstable space of } G_\mu^- \textit{ is} \subseteq E_\mu^-$$

and

$$\textit{the stable space of } G_\mu^+ \textit{ is} \subseteq E_\mu^+$$

with continuous bundles $\mu \mapsto E_\mu^-, E_\mu^+ \subset \mathbb{C}^n$*. If the Melnikov type vector quantity*

$$D_\mu[q]|_{\mu=0} \equiv \int_{-\infty}^{+\infty} D_\mu M(\tau_0, 0)\tau'_0 \quad \in \mathbb{C}^m$$

satisfies

$$E_0^- \oplus \mathbb{C} D_\mu[q]|_{\mu=0} \oplus E_0^+ = \mathbb{C}^m,$$

then the USBC breaks up transversely upon variation of μ *away from* 0.

1 Evans function: Inner regime

As in [FrSz1], we henceforth describe the profiles v_ϵ by the scalar center-manifold coordinate

$$\tau_\epsilon \equiv \epsilon^{-1}(u_\epsilon)_1,$$

which satisfies

$$\tau'_\epsilon = \epsilon(1 + O(\epsilon))(1 - \tau_\epsilon^2).$$

For concreteness and simplicity, we assume $\tau_\epsilon(0) = 0$ and henceforth write τ for τ_0.

In this section we study (0.9) in the case (0.12). Writing

$$\rho = \mu\epsilon^2,$$

we have to consider the range

$$0 < \mu \leq r_0. \tag{1.1}$$

Scaling q with ϵ, the (p,q)-part of Eqs. (0.9) reads

$$\begin{aligned}
p_1' &= -2\epsilon\tau p_1 + \epsilon q_1 + O(\epsilon p_j) + O(\epsilon^2 p) \\
p_j' &= \kappa_j^0 p_j + \epsilon q_j + O(\epsilon p), && j = 2, \dots, n \\
q_1' &= \epsilon\mu(\lambda p_1 + i{\textstyle\sum_m} B_{1m}^{\omega,0} p_m + O(\epsilon p)) \\
q_j' &= \epsilon\mu(\lambda p_j + i{\textstyle\sum_m} B_{jm}^{\omega,0} p_m + O(\epsilon p)), && j = 2, \dots, n.
\end{aligned} \tag{1.2}$$

In view of geometric singular perturbation theory [F, Sz], the next lemma is immediate.

Lemma 2. *For any $r > 0$, there exists an $\epsilon_r > 0$ such that the following holds uniformly for*

$$((\lambda,\omega),\mu,\epsilon) \in S^+ \times [0,r] \times [0,\epsilon_r] :$$

(i) The system that (1.2) induces on $\mathcal{G}_1^{2n}(\mathbb{C})$ possesses an n-dimensional normally hyperbolic attracting invariant

$$\textit{slow manifold} \quad \mathcal{M}_{\lambda,\omega}^{\epsilon,\mu} \ . \tag{1.3}$$

In the linear coordinates $(p,q) \in \mathbb{C}^{2n}$, $\mathcal{M}_{\lambda,\omega}^{\epsilon,\mu}$ is given by equations of the form

$$p_j = -\epsilon q_j / \kappa_j^0 + O(\epsilon^2), \quad j = 2, \dots, n$$

and the ("slow") flow on $\mathcal{M}_{\lambda,\omega}^{\epsilon,\mu}$ is governed, in linear coordinates $(p_1, q) \in \mathbb{C}^{n+1}$, by

$$\begin{aligned}
\dot p_1 &= -2\tau p_1 + q_1 + O(\epsilon) \\
\dot q_1 &= \mu(\lambda p_1 - i\epsilon{\textstyle\sum_{m\neq 1}}(B_{1m}^{\omega,0}/\kappa_m^0) q_m + O(\epsilon p_1) + O(\epsilon^2)) \\
\dot q_j &= \mu(iB_{j1}^{\omega,0} p_1 - i\epsilon{\textstyle\sum_{m\neq 1}}(B_{jm}^{\omega,0}/\kappa_m^0) q_m + O(\epsilon p_1) + O(\epsilon^2)), \quad j = 2, \dots, n
\end{aligned} \tag{1.4}$$

Since the manifold $\mathcal{M}_{\lambda,\omega}^{\epsilon,\mu}$ is attracting, any USBCs of (1.2) for $(\mu,\epsilon) \in (0,r_0] \times (0,\epsilon_0]$ must lie inside this slow manifold. Lemma 1 will thus be proved once we show

Lemma 3. *For any $r > 0$, there exists an $\epsilon_r > 0$ such that the slow manifold $\mathcal{M}_{\lambda,\omega}^{\epsilon,\mu}$ contains no USBC, for all values*

$$((\lambda,\omega),\mu,\epsilon) \in S \times (0,r] \times (0,\epsilon_r]$$

of the parameters.

The rest of this section serves to prove three further lemmata which together imply Lemma 3.

Lemma 4. *(Subregime I.) For any $c_I > 0$, the assertion of Lemma 3 holds under the restriction*

$$|\lambda| \geq c_I. \tag{1.5}$$

Proof. For $\epsilon = 0$, Eqs. (1.4) read

$$\begin{aligned} \dot{p}_1 &= -2\tau p_1 + q_1 \\ \dot{q}_1 &= \mu\lambda p_1 \\ \dot{q}_j &= i\mu B^{\omega,0}_{j1} p_1 \end{aligned} \quad . \tag{1.6}$$

We distinguish two cases.

Outer part of Subregime I: $r_I \le \mu \le r_0$ *for some* $0 < r_I < r_0$.
In this case $\tilde{\lambda} = \mu\lambda$ satisfies

$$0 < r_I c_I \le |\tilde{\lambda}| \le r_0,$$

and for the frozen-end systems at $\tau = \pm 1$, the eigenvalues are

$$\tau - \sqrt{1+\tilde{\lambda}} \ \text{(simple)}, \quad 0 \ \ ((n-1)\text{-fold}), \quad \tau + \sqrt{1+\tilde{\lambda}} \ \text{(simple)}.$$

The unstable bundles $\hat{U}^-, \hat{U}^+$ are hyperbolic attractors for the frozen-end flows. The (decoupled) (p_1, q_1) equations are just the eigenvalue problem for Burgers equation, which has no eigenvalues with $\Re\tilde{\lambda} \ge 0$ except $\tilde{\lambda} = 0$. Hence, the unique orbit with α-limit $\hat{U}^-$ has ω-limit $\hat{U}^+$. Because of hyperbolicity, this situation persists for small $\epsilon > 0$. In particular, no USBC can exist.

Inner part of Subregime I: $\mu \le r_I$ *for sufficiently small* $r_I > 0$. If also $\mu = 0$, there is a USBC, corresponding to

$$(p_1^*, q_1^*, \hat{q}^*) = (\tau', 0, 0). \tag{1.7}$$

We apply Lemma 1. In its terminology,

$$E_0^- = (0,0)^\top, \quad D_\mu[q]|_{\mu=0} = 2(\lambda, ib)^\top \notin E_0^+.$$

The latter holds as the right-end slow-flow ($p = 2q$) coefficient matrix,

$$\begin{pmatrix} \lambda/2 & 0 \\ ib/2 & 0 \end{pmatrix}, \tag{1.8}$$

has the stable space

$$E_0^+ = \{0\} \times \mathbb{C}^{n-1}.$$

The unstable and stable manifolds thus break away from each other transversely upon increasing μ away from 0. This transversality is robust in ϵ. Lemma 4 is proved. □

Lemma 5. *(Subregime II.) For sufficiently small* $c_{II} > 0$ *and sufficiently large* $C_{II} > 0$*, the assertion of Lemma 3 holds under the restriction*

$$C_{II}\sqrt{\epsilon} \le |\lambda| \le c_{II}. \tag{1.9}$$

Proof. Letting

$$\lambda = |\lambda|\tilde{\lambda}, \quad \epsilon = |\lambda|^2\tilde{\epsilon}, \quad \tilde{q}_j = |\lambda| q_j, j = 2, \dots, n, \quad \text{and} \quad \tilde{\mu} \equiv \mu|\lambda|$$

we write (1.4) as

$$\begin{aligned} \dot{p}_1 &= -2\tau p_1 + q_1 + O(\epsilon) \\ \dot{q}_1 &= \tilde{\mu}(\tilde{\lambda} p_1 + O(\tilde{\epsilon})) \\ \dot{\tilde{q}}_j &= \tilde{\mu}(iB_{j1}^{\omega,0} p_1 + O(\tilde{\epsilon})) \end{aligned} \tag{1.10}$$

It suffices to show that (1.10) has no USBC for sufficiently small $\tilde{\epsilon}, \mu > 0$. This, however, follows immediately, since we have recovered the situation of the inner part of Subregime I. □

Lemma 6. *(Subregime III.) For any $C_{III} > 0$ and any sufficiently small $c_{III} > 0$, the assertion of Lemma 3 holds under the restriction*

$$|\lambda| \leq \min\{c_{III}, C_{III}\sqrt{\epsilon}\}. \tag{1.11}$$

Proof. We introduce the scaling

$$\lambda = \sqrt{\epsilon}\hat{\lambda}, \quad \hat{q}_j = \sqrt{\epsilon} q_j, \ j = 2, \dots, n, \quad \text{and} \quad \hat{\mu} = \sqrt{\epsilon}\mu$$

and rewrite Eqs. (1.4) as

$$\begin{aligned} \dot{p}_1 &= -2\tau p_1 + q_1 + O(\epsilon) \\ \dot{q}_1 &= \hat{\mu}(\hat{\lambda} p_1 - i\textstyle\sum_{m\neq 1}(B_{1m}^{\omega,0}/\kappa_m^0)\hat{q}_m + O(\sqrt{\epsilon})) \\ \dot{\hat{q}}_j &= \hat{\mu}(iB_{j1}^{\omega,0} p_1 + O(\sqrt{\epsilon})). \end{aligned} \tag{1.12}$$

For $\hat{\mu} = 0$ and $\epsilon = 0$ there exists the USBC

$$(p_1^*, q_1^*, \hat{q}^*) = (\tau', 0, 0) \ .$$

We apply Lemma 1 with respect to $\hat{\mu}$. In its terminology,

$$E_0^- = (0,0)^\top, \quad D_{\hat{\mu}}[q]|_{\hat{\mu}=0} = 2(\hat{\lambda}, ib)^\top \notin E_0^+.$$

The latter holds as the right-end slow-flow ($p = 2q$) coefficient matrix,

$$\begin{pmatrix} \hat{\lambda}/2 & -i\tilde{b} \\ ib/2 & 0 \end{pmatrix}, \tag{1.13}$$

has the stable space

$$E_0^+ = (\{0\} \times \{\tilde{b}\}^-) + \mathbb{C}(1, i\beta b)^- \subset \mathbb{C} \times \mathbb{C}^{n-1}$$

with some β satisfying $\beta\hat{\lambda} \neq 1$. The unstable and stable manifolds thus break away from each other transversely upon increasing $\hat{\mu}$ away from 0. This transversality is robust in ϵ. Lemma 6 is proved. □

As Lemmata 4, 5, 6 imply Lemma 3, Proposition 1 is proved.

2 Evans function: Outer regime

In this section we study (0.9) in the case (0.13).

Letting

$$\epsilon = \alpha\sqrt{\rho} \tag{2.1}$$

and replacing q by $\sqrt{\rho}q$, Eqs. (0.9) read

$$\begin{aligned} p_1' &= -2\alpha\sqrt{\rho}\tau p_1 + \sqrt{\rho}q_1 + O(\alpha\sqrt{\rho}p_j) + O(\rho\alpha^2) \\ p_j' &= \kappa_j^0 p_j + \sqrt{\rho}q_j + O(\alpha\sqrt{\rho}p) \\ q' &= \sqrt{\rho}\,[\lambda I + iB^{\omega,0} + O(\alpha\sqrt{\rho}) + O(\rho)]p. \end{aligned} \tag{2.2}$$

By virtue of (2.1), the inequalities (0.13) defining the outer regime, $\epsilon^2 r_0 \leq \rho \leq r_1$, equivalently turn into

$$\alpha \leq 1/\sqrt{r_0}, \quad \rho \leq r_1. \tag{2.3}$$

Our task will thus be to understand (2.2) for small α and ρ.

We reduce system (2.2) further by (i) splitting off the $n-1$ unstable directions $p_j, j \geq 2$, via a slow-fast decomposition, (ii) rescaling the slow dynamics in the u_1, p_1, q variables, and (iii) decomposing $(q_2, \dots, q_n) \in \mathbb{C}^{n-1}$ into an "active" component $w \in \mathbb{C}$ and a "passive" component $z \in \mathbb{C}^{n-2}$.
For $\sqrt{\rho} = 0$, the equations $p_j = 0, j = 2, \dots, n$, define a normally hyperbolic critical manifold for (2.2). The corresponding slow manifold is given by

$$p_j = -\frac{\sqrt{\rho}q_j}{\kappa_j^0} + O(\alpha\sqrt{\rho}p_1) + O(\rho), \quad j = 2, \dots, n.$$

The slow dynamics is governed by the system

$$\begin{aligned} \dot{p}_1 &= -2\alpha\tau p_1 + q_1 + O(\sqrt{\rho}) \\ \dot{q}_1 &= \lambda p_1 - \sqrt{\rho}\sum_{j=2}^{n} i(B_{1j}^{\omega,0}/\kappa_j^0)q_j + O(\alpha\sqrt{\rho}p_1) + O(\rho) \\ \dot{q}_j &= iB_{j1}^{\omega,0}p_1 + O(\sqrt{\rho}). \end{aligned} \tag{2.4}$$

Letting

$$\rho = \delta^4 \quad \text{and} \quad q_j = -i\tilde{q}_j/\delta, \quad j = 2, \dots, n,$$

we obtain

$$\begin{aligned} \dot{p}_1 &= -2\alpha\tau p_1 + q_1 + O(\delta^2) \\ \dot{q}_1 &= \lambda p_1 + \delta\sum_{j=2}^{n}(B_{1j}^{\omega,0}/\kappa_j^0)\tilde{q}_j + O(\delta^2) \\ \dot{\tilde{q}}_j &= \delta B_{j1}^{\omega,0}p_1 + O(\delta^3). \end{aligned} \tag{2.5}$$

We define the active component

$$w := (\sum_{j=2}^{n} B_{1j}^{\omega,0} B_{j1}^{\omega,0}/\kappa_j^0)^{-1/2} \sum_{j=2}^{n} B_{1j}^{\omega,0} \tilde{q}_j$$

and the passive component $z = (z_3, \dots, z_n)$ by

$$z_l := \sum_{j=2}^{n} C_{lj} \tilde{q}_j \text{ with } C \text{ of full rank } n-2 \text{ and} \sum_{j=2}^{n} C_{lj} B_{j1}^{\omega,0} = 0$$

and replace δ with $(\sum_{j=2}^{n} B_{1j}^{\omega,0} B_{j1}^{\omega,0}/\kappa_j^0)^{-1/2}\delta$. In these variables, dropping subscripts and tildas, system (2.5) has the form

$$\begin{aligned} \dot{p} &= -2\alpha\tau p + q + O(\delta^2) \\ \dot{q} &= \lambda p + \delta w + O(\delta^2) \\ \dot{w} &= \delta p + O(\delta^3) \\ \dot{z} &= O(\delta^3). \end{aligned} \tag{2.6}$$

Momentarily neglecting the higher-order terms and the passive component z, we arrive at

$$\begin{aligned} \dot{p} &= -2\alpha\tau p + q \\ \dot{q} &= \lambda p + \delta w \\ \dot{w} &= \delta p \end{aligned} \tag{2.7}$$

on $\mathbb{C}^3$. System (2.7) is invariant under the scaling

$$\begin{aligned} (\alpha, \delta, \lambda) &\to (r^2\alpha, r^3\delta, r^4\lambda) \\ (p, q) &\to (r^{-1}p, rq). \end{aligned} \tag{2.8}$$

To understand (2.7), it hence suffices to study the system for parameter values on the sphere $\alpha^2 + \delta^2 + |\lambda|^2 = 1$.

Lemma 7. *The only parameter value in*

$$\mathcal{S} \equiv \{(\alpha, \delta, \lambda) \in [0, \infty) \times [0, \infty) \times \mathbb{C} \,:\, \Re\lambda \geq 0, \alpha^2 + \delta^2 + |\lambda|^2 = 1\}$$

for which (2.7) has a USBC is $(\alpha_*, \delta_*, \lambda_*) = (1, 0, 0)$. Upon variation of the parameter $(\alpha, \delta, \lambda) \in \mathcal{S}$ near $(\alpha_*, \delta_*, \lambda_*)$, the stable and unstable manifolds whose intersection that connection is move away from each other transversely.

Proof. *Subregime $\delta \geq \delta_0$ with some $\delta_0 > 0$.* The (constant-) coefficient matrices of (2.7) at both $\tau = -1$ and $\tau = +1$ are of the form

$$\begin{pmatrix} \beta & 1 & 0 \\ \lambda & 0 & \delta \\ \delta & 0 & 0 \end{pmatrix} \quad \text{with } 0 < \delta_0 \leq \delta \leq 1, -2 \leq \beta \leq 2, |\lambda| \leq 1, \Re\lambda \geq 0. \tag{2.9}$$

Straightforward inspection shows that any such matrix has one simple eigenvalue of strictly positive real part and two simple, or one double, eigenvalue(s) of strictly negative real part. Assume now there were a heteroclinic unstable-to-stable connection (p,q,w); necessarily, it behaves exponentially at both infinities. Considering

$$\tilde{p} := \int_{-\infty} p, \quad \tilde{w} := \int_{-\infty} w,$$

we find

$$\begin{aligned}(\tilde{p}' + 2\alpha\tau\tilde{p})' &= \lambda\tilde{p} + \tilde{w} \\ \tilde{w}' &= \delta\tilde{p}.\end{aligned} \tag{2.10}$$

Multiplying with $\bar{\tilde{p}}$ and integrating by parts, we obtain

$$-\int \tilde{p}'\bar{\tilde{p}}' - \alpha\int \tau'\tilde{p}\bar{\tilde{p}} = \lambda\int \tilde{p}\bar{\tilde{p}} + \delta\int \tilde{w}\bar{\tilde{w}}'$$

which implies

$$\Re\lambda \int |\tilde{p}|^2 < 0, \tag{2.11}$$

a contradiction. The ω-limit of the orbit whose α-limit is the unstable bundle at -1 is the unstable bundle at $+1$. As the latter is an attractor, this unstable-to-unstable bundle connection is robust.

Subregime $0 \le \delta \le \delta_0$, $|\lambda| \ge \gamma_0$ *with sufficiently small* $\delta_0 > 0$ *and some* $\gamma_0 > 0$. Assume first that $\delta = 0$. In that subcase, any heteroclinic unstable-to-stable connection (p,q,w) would have $w = 0$ and satisfy

$$(\tilde{p}' + 2\alpha\tau\tilde{p})' = \lambda\tilde{p}. \tag{2.12}$$

We readily reach the same conclusions as in the previous case. Due to its robustness, the unstable-to-unstable heteroclinic connection persists for small $\delta > 0$. I. e., again no USBC is possible in the whole subregime, if $\delta_0 > 0$ is chosen sufficiently small.

Subregime $0 \le \delta \le \delta_0$, $0 \le |\lambda| \le \gamma_0$ *with sufficiently small* $\delta_0, \gamma_0 > 0$. For $(\alpha,\delta,\lambda) = (\alpha_*,\delta_*,\lambda_*) = (1,0,0)$, there is the unique USBC given by

$$(p_*,q_*,w_*) = (\tau',0,0).$$

We parametrize points in $\mathcal{S}$ near $(\alpha_*,\delta_*,\lambda_*)$ as

$$(\alpha,\delta,\lambda) = (\sqrt{1-\mu^2}, \mu\hat{\delta}, \mu\hat{\lambda}) \ \text{ with } \hat{\delta}^2 + |\hat{\lambda}|^2 = 1$$

and investigate what happens to (p_*,q_*,w_*) upon perturbing μ in

$$\begin{aligned}\dot{p} &= -2\alpha\tau p + q \\ \dot{q} &= \mu\hat{\lambda}p + \mu\hat{\delta}w \\ \dot{w} &= \mu\hat{\delta}p.\end{aligned} \tag{2.13}$$

away from 0. We apply Lemma 1. In its terminology,

$$E_0^- = (0,0)^\top, \quad D_\mu[q]|_{\mu=0} = 2(\hat{\lambda},\hat{\delta})^\top \notin E_0^+,$$

the latter as the right-end slow-flow ($p = 2q$) coefficient matrix,

$$\begin{pmatrix} \hat{\lambda}/2 & \hat{\delta} \\ \hat{\delta}/2 & 0 \end{pmatrix}, \tag{2.14}$$

satisfies

$$\begin{pmatrix} \hat{\lambda}/2 & \hat{\delta} \\ \hat{\delta}/2 & 0 \end{pmatrix} \begin{pmatrix} \hat{\lambda} \\ \hat{\delta} \end{pmatrix} = (\hat{\lambda}/2) \begin{pmatrix} \hat{\lambda} \\ \hat{\delta} \end{pmatrix} + \begin{pmatrix} \hat{\delta}^2 \\ 0 \end{pmatrix}. \tag{2.15}$$

The unstable and stable manifolds thus break away from each other transversely upon increasing μ away from 0. Lemma 7 is proved. □

Lemma 2 is an immediate consequence of

Lemma 8. *If $\alpha_0, \delta_0 > 0$ are sufficiently small, then the only parameter values $((\lambda, \omega), \alpha, \delta) \in S \times [0, \alpha_0] \times [0, \delta_0]$ for which (2.4) has an unstable-to-stable bundle connection are given by*

$$\alpha = 1, \delta = 0, |\omega| = 1, \lambda = 0.$$

Upon variation of $((\lambda, \omega), \alpha, \delta)$ away from these critical values, the stable and unstable manifolds whose intersections these connections are move away from each other transversely.

Proof. We first note that for any $c > 0$ there is a $\tilde{c} > 0$ such that for all parameter values satisfying $|\lambda| \geq c > 0$ and $0 \leq \alpha, \rho < \tilde{c}$, there can be no USBCs. To see this, we consider (2.5) for $(\alpha, \delta) = (0, 0)$,

$$\begin{aligned} \dot{p}_1 &= q_1 \\ \dot{q}_1 &= \lambda p_1 \\ \dot{\tilde{q}}_j &= 0; \end{aligned} \tag{2.16}$$

for this constant-coefficients system the one-dimensional left-end unstable bundle connects to the right-end stable bundle — this connection, and thus the non-existence of an USBC, are robust.

We can hence restrict attention to those values $((\lambda, \omega), \alpha, \delta) \in S \times [0, \infty) \times [0, \infty)$, for which λ, α, δ are near 0 (and correspondingly $|\omega|$ is almost equal to 1). Now, writing such values as

$$(\alpha, \delta, \lambda) = (r^2 \bar{\alpha}, r^3 \bar{\delta}, r^4 \bar{\lambda})$$

with

$$\bar{\alpha}^2 + \bar{\delta}^2 + |\bar{\lambda}|^2 = 1 \quad \text{and} \quad r \geq 0 \quad \text{sufficiently small} \tag{2.17}$$

and using scaling property (2.8), we write (2.6) as

$$\begin{aligned} \dot{p} &= -2\bar{\alpha}\tau p + q + O(r\bar{\delta}) \\ \dot{q} &= \bar{\lambda} p + \bar{\delta} w + O(r\bar{\delta}) \\ \dot{w} &= \bar{\delta} p + O(r\bar{\delta}) \\ \dot{z} &= O(r\bar{\delta}) \end{aligned} \tag{2.18}$$

and augment this system by adding the trivial equations

$$\begin{aligned}\dot{\bar{\alpha}} &= 0,\\ \dot{\bar{\delta}} &= 0,\\ \dot{\bar{\lambda}} &= 0,\\ \dot{r} &= 0.\end{aligned} \tag{2.19}$$

The augmented system (2.18),(2.19) has the bundle connections corresponding to

$$(p, q, w, z, \bar{\alpha}, \bar{\delta}, \bar{\lambda}, r) \equiv (\tau', 0, 0, 0, 1, 0, 0, r)$$

and we will be done once we know that this intersection of smooth invariant manifolds is transverse for sufficiently small r. Now, Lemma 7 means exactly this transversality. Lemma 8 and thus Proposition 2 are proved. □

3 Evans function: Outmost regime

In this section we study (0.9) in the case (0.14). The coefficient matrix of the (p, q)-part is

$$\begin{pmatrix} A(u_\epsilon) & I \\ \rho(\lambda I + iB^\omega(u_\epsilon) + \rho|\omega|^2 I) & 0 \end{pmatrix}. \tag{3.1}$$

We consider several subregimes.

Inner part of outmost regime: $r_1 \le \rho \le r_2$ *with* r_1, r_2 *arbitrary such that* $0 < r_1 < r_2$. In this regime (3.1) is uniformly close, for sufficiently small $\epsilon > 0$, to the constant-coefficients hyperbolic family

$$\begin{pmatrix} A^0 & I \\ \rho(\lambda I + iB^{\omega,0} + \rho|\omega|^2 I) & 0 \end{pmatrix}, \quad ((\lambda, \omega), \rho) \in S_+ \times [r_1, r_2]. \tag{3.2}$$

Outer part of outmost regime: $\rho \ge r_2$ *with* r_2 *sufficiently large.*
We consider two subcases.

Subcase $|\omega| > \delta_0$ *with arbitrary* δ_0. By rescaling $q = \rho|\omega|\tilde{q}$ and dividing the resulting vectorfield by $\rho|\omega|$, i.e. rescaling the independent variable, (3.1) can be written as

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + O(1/\rho), \tag{3.3}$$

uniformly. Matrices (3.3) are uniformly hyperbolic, if r_2 is large enough.

Subcase $|\omega| \le \delta_0$ *with sufficiently small* $\delta_0 > 0$. Scaling $q = (\rho|\lambda + \rho|\omega|^2|)^{1/2}\tilde{q}$ and dividing the resulting vectorfield by $(\rho|\lambda + \rho|\omega|^2|)^{1/2}$ turns (3.1) into

$$\begin{pmatrix} 0 & I \\ \zeta I & 0 \end{pmatrix} + O(|\omega| + \frac{1}{\sqrt{\rho}}), \tag{3.4}$$

with $\zeta = (\lambda + \rho|\omega|^2)|\lambda + \rho|\omega|^2|^{-1}$, again uniformly. For $(\lambda, \omega) \in S$, matrices (3.4) are uniformly hyperbolic, if δ_0 and r_2 are sufficiently small respectively large.

Proposition 3 is proved.

4 Lopatinski condition and transversal breaking

We first recapitulate a theorem by Zumbrun and Serre by formulating its proof via a slow-fast-dynamics argument.

Theorem 2. *[ZS] Consider the profile u of any Lax shock wave. Its Evans function $\mathcal{E}(\rho\lambda, \rho\omega)$ and Lopatinski-Kreiss-Majda function $\Delta(\lambda, \rho)$ are related to each other by the identity*

$$(\frac{\partial}{\partial\rho}\mathcal{E}(\rho\lambda, \rho\omega))|_{\rho=0} = \Gamma\Delta(\lambda, \omega), \quad (\lambda, \omega) \in S \tag{4.1}$$

with $\Gamma \neq 0$ if and only if the intersection of the unstable and stable manifolds of the profile equation along the profile is transverse. There are $2n$ $\mathbb{C}^{2n}$-valued continuous functions

$$v_1^-, \dots, v_l^-, \quad w_1^-, \dots, w_{n-l}^- \quad \text{and} \quad v_l^+, \dots, v_n^+, \quad w_1^+, \dots, w_{l-1}^+$$

of $(\rho, (\lambda, \omega)) \in [0, \infty) \times S$ such that

$$\mathcal{E} = \det(v_1^-, \dots, v_l^-, w_1^-, \dots, w_{n-l}^-, v_l^+, \dots, v_n^+, w_1^+, \dots, w_{l-1}^+)$$

and

$$\frac{\partial}{\partial\rho}\det(v_1^-, \dots, v_l^-, w_1^-, \dots, w_{n-l}^-, v_l^+, \dots, v_n^+, w_1^+, \dots, w_{l-1}^+)|_{\rho=0} \tag{4.2}$$

exists and is equal to the product of the two $n \times n$ determinants

$$\Gamma = \pm\det(\pi_p(v_1^-), \dots, \pi_p(v_l^-), \pi_p(v_{l+1}^+), \dots, \pi_p(v_n^+))|_{\rho=0} \tag{4.3}$$

and

$$\det(\lambda[u] + i[f^\omega(u)], \pi_q(w_1^-), \dots, \pi_q(w_{n-l}^-), \pi_q(w_1^+), \dots, \pi_q(w_{l-1}^+))|_{\rho=0},$$

while for $\rho = 0$, $\pi_q(w_1^-), \dots, \pi_q(w_{n-l}^-)$ and $\pi_q(w_1^+), \dots, \pi_q(w_{l-1}^+)$ span the unstable resp. stable space of, in the notation of Sec. 0,

$$-(\lambda I + iB^\omega(u^\pm))(A(u^\pm) - sI)^{-1},$$

respectively.

Proof. The equations for the shock profile and the eigenvalue problem are

$$\begin{aligned} u' &= f(u) - su - c \\ p' &= (A(u) - sI)p + q \\ q' &= \rho(\lambda I + iB^\omega(u) + \rho|\omega|^2 I)p. \end{aligned} \tag{4.4}$$

Consider the unstable and stable solution spaces via the values $U(0), S(0) \in \mathcal{G}_n^{2n}$ of their (p, q)-component at 0 (as a value of the independent variable. By the general theory [AGaJ, Z, ZS, GMWZ1] and the continuous dependence of the frozen-end invariant spaces with respect to the parameters, $U(0)$ and $S(0)$ can be represented by bases

$$v_1^-, \dots, v_l^-, \quad w_1^-, \dots, w_{n-l}^- \quad \text{and} \quad v_l^+, \dots, v_n^+, \quad w_1^+, \dots, w_{l-1}^+$$

which are continuous functions of $(\rho, (\lambda, \omega))$. We use a particular decomposition into "fast" vectors $v_i^\pm$ and "slow" vectors $w_j^\pm$ which is adapted to the analysis of the limit $\rho \to 0$. The starting point of this decomposition is the observation that system (4.4) is singularly perturbed for ρ small. Standard geometric singular perturbation theory [F] implies the existence of two n-dimensional normally hyperbolic slow manifolds

$$u = u^\pm, \quad p = -(A^\pm - sI)^{-1} q + O(\rho), \quad q \in \mathbb{C}^n \ .$$

The corresponding slow flows on $\mathbb{C}^n$ are described by

$$\dot{q} = -(\lambda I + iB^\omega(u^\pm))(A(u^\pm) - sI)^{-1} q + O(\rho) \ . \tag{4.5}$$

The fast flow away from the slow manifolds is a smooth $O(\rho)$ perturbation of the layer problem

$$\begin{aligned} u' &= f(u) - su - c \\ p' &= (A(u) - sI)p + q \\ q' &= 0. \end{aligned} \tag{4.6}$$

In particular, the q-components of the unstable resp. stable solutions at 0 differ from their asymptotic values at u^- resp. u^+ by $O(\rho)$.
In the terminology of geometric singular perturbation theory [F, Sz], the unstable / stable solution spaces correspond to fast unstable / stable fibres, of the slow unstable space $E^{-,u}_{slow}$ at u^- and the slow stable space $E^{+,s}_{slow}$ at u^+, respectively; the corresponding sections at 0 are

$$U(0) = (\mathcal{F}^u(E_u^-))|_0, \quad S(0) = (\mathcal{F}^s(E_s^+))|_0.$$

We choose $v_1^-, \dots, v_l^-$, and $v_l^+, \dots, v_n^+$ as bases for the corresponding subfibres

$$U_{fast}(0) = \mathcal{F}^u(\{0\})|_0 \ , \quad S_{fast}(0) = \mathcal{F}^s(\{0\})|_0 \ ;$$

as the fast flow depends smoothly on the parameters, these vector fields can be assumed to be smooth functions of $(\rho, (\lambda, \omega))$. Complementing this, the above considerations imply that $w_1^-, \dots, w_{n-l}^-$ and $w_1^+, \dots, w_{l-1}^+$ can be chosen such that

$$\pi_q(w_i^\pm) = r_i^\pm + O(\rho),$$

where

$$r_1^-, \dots r_{n-l}^- \quad \text{and} \quad r_1^+, \dots r_{l-1}^+$$

are bases of the unstable / stable spaces of the matrices

$$-(\lambda I + iB^\omega(u^\pm))(A(u^\pm) - sI)^{-1},$$

respectively. Recall that the vectors $r_i^\pm$ are precisely the vectors used in the definiton of the Lopatinski-Kreiss-Majda function.
For the computation of (4.2), recall now that — the profile u lying in the intersection of the unstable manifold of u^- and the stable manifold of u^+ — for $\rho = 0$, the vector $u'(0)$ belongs to both $U_{fast}(0)$ and $S_{fast}(0)$; we express this as

$$v_l^-|_{\rho=0} = v_l^+|_{\rho=0} = u'(0). \tag{4.7}$$

Up to a sign, $\mathcal{E}$ is equal to

$$\det(v_1^-, \ldots, v_l^-, v_{l+1}^+, \ldots, v_n^+, v_l^- - v_l^+, w_1^-, \ldots, w_{n-l}^-, w_1^+, \ldots, w_{l-1}^+)$$

By virtue of (4.7), the derivative $\frac{\partial}{\partial\rho}\mathcal{E}$ at $\rho = 0$ exists even though the vectors $w_i^\pm$ are only continuous in ρ, and is equal to the product of (4.3) and

$$\det(\pi_q(\frac{\partial}{\partial\rho}(v_l^- - v_l^+))|_{\rho=0}, r_1^-, \ldots, r_{n-l}^-, r_1^+, \ldots, r_{l-1}^+)$$

The exponential decay of u, $v_l^\pm$ at $\pm\infty$, the smoothness of $v_l^\pm$ with respect to ρ, and the form of the equations (4.4) imply

$$\pi_q(\frac{\partial}{\partial\rho}v_l^-)|_{\rho=0} = \int_{-\infty}^0 (\lambda I + iB^\omega(u))u' \quad \text{and} \quad \pi_q(\frac{\partial}{\partial\rho}v_l^+)|_{\rho=0} = \int_{\infty}^0 (\lambda I + iB^\omega(u))u'$$

and thus

$$\pi_q(\frac{\partial}{\partial\rho}(v_l^- - v_l^+))|_{\rho=0} = \lambda[u] + i[f^\omega] \ .$$

Finally, as for $q = 0$ the second equation in (4.6) is the variational equation associated with the first one, Γ does indeed play the claimed rôle as a transversality coefficient. The Theorem is proved. $\square$

By a close analogy, we can now give a quick

Proof of Lemma 1. As in the specific situation of Theorem 2, we now find $k+m$ $\mathbb{C}^{k+m}$-valued continuous functions

$$v_1^-, \ldots, v_l^-, \quad w_1^-, \ldots, w_r^- \quad \text{and} \quad v_l^+, \ldots, v_k^+, \quad w_1^+, \ldots, w_{m-r-1}^+$$

of $\mu \in [0, \mu_0]$, with analogous meanings — notably the $w_j^\pm$ now spanning the $E_\mu^\pm$ —, such that

$$\mathcal{D} = \frac{\partial}{\partial\mu}\det(v_1^-, \ldots, v_l^-, w_1^-, \ldots, w_r^-, v_l^+, \ldots, v_k^+, w_1^+, \ldots, w_{m-r-1}^+)|_{\mu=0}$$

exists, $\mathcal{D}$ equals the product of the $k \times k$ determinant

$$\mathcal{D}_1 = \pm\det(\pi_p(v_1^-), \ldots, \pi_p(v_l^-), \pi_q(v_{l+1}^+), \ldots, \pi_p(v_k^+))|_{\rho=0}$$

times the $m \times m$ determinant

$$\mathcal{D}_2 = \det(D_\mu[q]|_{\mu=0}, \pi_q(w_1^-), \ldots, \pi_q(w_r^-), \pi_q(w_1^+), \ldots, \pi_q(w_{m-r-1}^+))|_{\rho=0},$$

and

the desired transversality holds if $\mathcal{D} \neq 0$.

Now, $\mathcal{D}_1$ does not vanish because the orbit as such was assumed to be transverse. The desired transversality thus holds if

$$\mathcal{D}_2 \neq 0.$$

However, this inequality just rephrases the condition

$$E_0^- \oplus \mathbb{C}D_\mu[q]|_{\mu=0} \oplus E_0^+ = \mathbb{C}^m$$

mentioned at the end of the statement of Lemma 1. Part (i) of Theorem 1 is proved.

In view of Theorem 2 and the transversality of profiles for small Lax shocks, computations we have carried out in various applications of Lemma 1 — notably those below (1.9),(1.14) — amount to part (ii) of Theorem 1.

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