Well-posedness and stability for a third-order Hele-Shaw problem with convected surface energy density

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Abstract

We investigate a moving boundary problem with a gradient flow structure which generalizes Hele-Shaw flow driven solely by surface tension to the case of nonconstant surface tension coefficient taken along with the liquid particles at boundary. The resulting evolution problem is first order in time, contains a third-order nonlinear pseudodifferential operator and is degenerate parabolic. Well-posedness of this problem in Sobolev scales is proved. The main tool is the construction of a variable symmetric bilinear form so that the third-order operator is semibounded with respect to it. Moreover, we show global existence and convergence to an equilibrium for solutions near trivial equilibria (balls with constant surface tension coefficient). Finally, numerical examples in 2D and 3D are given.

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1. Introduction

It is the aim of the present paper to consider the generalization of the well-investigated Hele-Shaw flow problem to the case of nonconstant surface tension coefficient (or surface energy density). While experiments on such situations have been reported in the literature (e.g. [10]), theoretical investigations of this seem to be lacking. A first step in this direction has been made in [8] where short-time solvability was proved for a Hele-Shaw problem with nonconstant surface tension coefficient and so-called kinetic undercooling. Here we discuss the problem without this regularization, using again the simple assumption that the surface energy density is convectively transported along the moving boundary.

This leads to the following moving boundary problem: For a given bounded domain $\Omega(0) \subset \mathbb{R}^m$ and a given non-negative function γ_0 defined on $\partial\Omega(0)$ one looks for a family of C^2-domains $\Omega(t) \subseteq \mathbb{R}^m$, $t > 0$ and functions $\varphi(\cdot,t) \in C^2\big(\overline{\Omega(t)}\big)$, $\psi(\cdot,t) \in C^2\big(\overline{\Omega(t)}\big)$, $\gamma_t \in C^2\big(\Gamma(t)\big)$ such that

$$\left.\begin{aligned} \Delta\varphi(\cdot,t) &= 0 && \text{in } \Omega(t),\\ \Delta\psi(\cdot,t) &= 0 && \text{in } \Omega(t),\\ \partial_n\psi(\cdot,t) &= \Delta_{\Gamma(t)}\gamma_t && \text{on } \Gamma(t),\\ \varphi(\cdot,t) &= \gamma_t\kappa(t) - \psi(\cdot,t) && \text{on } \Gamma(t),\\ V_n &= \partial_n\varphi(\cdot,t) && \text{on } \Gamma(t). \end{aligned}\right\} \tag{1.1}$$

Here $\kappa(t)$ is the $(m-1)$-fold mean curvature of $\Gamma(t)$, with the sign taken such that κ is negative for convex domains, ∂_n is the outer normal derivative and $V_n(t)$ is the (outer) normal velocity of $\Gamma(t)$, determining its time evolution.

This problem generalizes the well-known Hele-Shaw flow with surface tension regularization in the following way: Any solution represents a gradient flow with respect to the usual energy functional

$$E(\gamma,\Gamma) := \int_\Gamma \gamma\, d\Gamma,$$

where $\gamma > 0$ is now variable on Γ, and to the Riemannian metric g_Γ on the infinite-dimensional manifold $\mathscr{M}$ of surfaces Γ enclosing a fixed volume given by

$$g_\Gamma(v_1,v_2) := \int_\Omega \nabla\varphi_1\nabla\varphi_2\, dx \tag{1.2}$$

where the φ_i, $i = 1,2$ are (weak) solutions of the Neumann problems

$$\Delta\varphi_i = 0 \text{ in } \Omega, \quad \partial_n\varphi_i = v_i \text{ on } \Gamma.$$

The functions v_i can be identified with tangent vectors of $\mathscr{M}$; note that the conservation of volume implies $\int_\Gamma v_i = 0\, d\Gamma$. For more details and references see [1, 5, 8]. Kinetic undercooling regularization corresponds to adding in (1.2) a boundary integral term $\beta\int_\Gamma v_1v_2\, d\Gamma$ with $\beta > 0$, this case is discussed in [8].

As mentioned already, we assume that the values of the function Γ are transported with the liquid particles: Introducing Lagrangian coordinates $x = x(\xi,t)$, $\xi \in \Gamma(0)$ corresponding to the velocity field via

$$\partial_t x(\xi,t) = \nabla\varphi\big(x(\xi,t),t\big) \text{ for } t \geq 0, \quad x(\xi,0) = \xi, \tag{1.3}$$

we obtain that $x = x(\cdot,t)$ is a diffeomorphism from $\Gamma(0)$ onto $\Gamma(t)$, and the transport law for γ_t takes the form

$$\gamma_t\big(x(\xi,t)\big) = \gamma_0(\xi), \quad \xi \in \Gamma(0),\ t \geq 0. \tag{1.4}$$

This assumption is reasonable, for example, when γ depends on temperature and heat diffusion is negligible compared to convection. While it certainly oversimplifies the physical situation in the case when e.g. surfactants play a role, it seems that the mathematical character of the problem is essentially the same there as in our case. Note, however, that the situation here is qualitatively different from other models like anisotropic Hele-Shaw flow (cf. [4]) because in our case, the evolution is not determined solely by the shape of the evolving domain but there is a coupling with a transport problem in the moving boundary.

Our approach is based on reformulating (1.1) as a vector-valued evolution equation for a diffeomorphism mapping a fixed reference manifold to the moving boundary. In this way, the transport problem for γ is simply solved by prescribing a fixed smooth positive function on this reference manifold and pushing it forward to the moving boundary.

The paper is organized as follows:

After announcing our main results on short-time existence in Section 2, we start the proofs in Section 3 by investigating mapping properties of the occurring nonlocal operators in Sobolev scales. In particular, we derive flexible multilinear estimates for their Fréchet derivatives in low norms and extend them to higher norms by a generalized chain rule based on invariance properties. For related considerations concerning the analytic dependence of the Dirichlet-Neumann operator on the domain we refer to [3] and the references given there. Section 4 is devoted to the proof of the crucial estimate providing the semiboundedness of the evolution operator with respect to a specifically constructed variable inner product.

Technically, we use the natural decomposition of the right hand side into a second order operator mapping vectors to scalars and a first order operator mapping scalars to vectors. Furthermore, we use the fact that the right hand side is -in a sense to be made precise later - coercive with respect to the normal component. The semiboundedness enables us to invoke an abstract existence result based on Galerkin approximations and Rothe's method. This is done in Section 5. In this way, we prove our main result (Theorems 2.1 and 2.2) on short-time wellposedness of the moving boundary problem (1.1),(1.3),(1.4). We will omit certain details as they are parallel to the discussion in [8]. However, the right hand side of the evolution problem obtained there is of order two. As we are concerned here with an evolution equation whose right hand side is of order three, we have to refine the construction from [8] by including certain lower order terms. Differing from the situation there, here we have to demand strict positivity of γ because its inverse γ^{-1} enters one of these terms.

Finally, in Section 6 we investigate the evolution near the equilibrium solutions given by balls with constant γ. In this situation, Theorem 6.8 gives global existence in time and the evolving domain approaches a nontrivial equilibrium configuration depending on the given (nonconstant) γ and the initial domain. In contrast to the classical case of constant γ where the equilibria are given only by balls, any shape near a ball occurs as an equilibrium configuration for a certain function γ near the constant. Due to the degeneracy of our problem, the proof of long-time existence is more involved compared to the known proofs for the case of constant γ, cf. [6].

2. Statement of the local existence results

We list some notation. $C, C_1, \dots$ etc. denote generic constants; their dependences on other quantities is only indicated if not obvious from the context. Let $E \subseteq \mathbb{R}^m$, $m \geq 2$ be a bounded domain with smooth boundary $S := \partial E$ and ν the outer unit normal on S. For $M = S$ or $M = E$, we make constant use of the usual L^2-based Sobolev spaces $H^s(S)$, $H^s(S, \mathbb{R}^m)$ of order s with values in $\mathbb{R}$ and $\mathbb{R}^m$, respectively. The norms of these spaces will be denoted by $\|\cdot\|_s^M$; for $M = S$ the upper index M is dropped in most cases. When Fréchet derivatives of operator-valued mappings are considered, the additional arguments describing the variations are written in accolades ($\{\}$).

Now, as already mentioned in the introduction, we reformulate the moving boundary problem (1.1) - (1.4) by describing $\Gamma(t)$ as an embedding $u(\cdot,t) : S \to \mathbb{R}^m$ such that the curves $t \mapsto u(y,t)$ for fixed $y \in S$ are trajectories belonging to the velocity field and γ_t is constant along these curves. This approach enables us to consider γ_t as a known function during the evolution at the cost of describing the moving boundary by m functions. To do so, let

$$U := \big\{u : S \to \mathbb{R}^m \;\big|\; u = w|_S \text{ with } w \in \mathrm{Diff}(\bar{E}, \Omega_u \cup \Gamma_u)\big\} \tag{2.1}$$

where

$$\Omega_u = w(E) \quad \text{and} \quad \Gamma_u = \partial\Omega_u = u(S).$$

Throughout this paper, we use the abbreviation

$$U_s := U \cap H^s(S, \mathbb{R}^m).$$

Now, (1.1) - (1.4) is reduced to the following Cauchy problem, which will be investigated in the sequel: For given $u_0 \in U_s$, s sufficiently large, we look for $T > 0$ and a mapping $[0,T] \ni t \mapsto u(t) \in U_s$, such that

$$u'(t) = \mathscr{F}\big(u(t)\big), \quad t \in [0,T], \tag{2.2}$$

$$u(0) = u_0. \tag{2.3}$$

Thereby, for $u \in U$, we have set

$$\mathscr{F}(u) := F(u)\mathscr{G}(u) \quad \text{with} \quad \mathscr{G}(u) := H(u) + G(u), \tag{2.4}$$

where, for any given function f on S,

$$F(u)f := \nabla\varphi(u,f) \circ u \tag{2.5}$$

and $\varphi = \varphi(u,f)$ denotes the solution of the Dirichlet problem

$$\Delta\varphi = 0 \text{ in } \Omega_u, \quad \varphi = f \circ u^{-1} \text{ on } \Gamma_u. \tag{2.6}$$

Further, $H(u)$, $G(u)$ are given by

$$H(u) := \gamma(\kappa_{\Gamma_u} \circ u), \quad G(u) := -A(u)\Delta(u)\gamma. \tag{2.7}$$

Here $\gamma \in C^\infty(S)$ is a fixed and given positive function, κ_{Γ_u} denotes the mean curvature of Γ_u with sign and scaling conventions as above and

$$\Delta(u)w := \Delta_{\Gamma_u}(w \circ u^{-1}) \circ u \tag{2.8}$$

is the pullback to S of the Laplace-Beltrami operator Δ_{Γ_u} on Γ_u and

$$A(u)f := \varphi_N\big(u,f\big) \circ u \tag{2.9}$$

the Neumann-Dirichlet operator, i.e. $\varphi_N = \varphi_N(u,f)$ solves the Neumann problem

$$\Delta\varphi_N = 0 \text{ in } \Omega_u, \quad \partial_n\varphi_N = c + f \circ u^{-1} \text{ on } \Gamma_u, \quad \textstyle\int_{\Gamma_u} \varphi_N \, dx = 0. \tag{2.10}$$

The constant $c = c(u,f) \in \mathbb{R}$ in (2.10) is determined by the solvability condition

$$\textstyle\int_{\Gamma_u} (f \circ u + c)\, d\Gamma_u = 0; \tag{2.11}$$

clearly $c(u,f)=0$ for $f=\Delta(u)\gamma$. For fixed smooth γ on S, the mapping $u\mapsto H(u)$ constitutes a quasi-linear second order differential operator on S. Moreover, the solutions of the boundary value problems (2.6), (2.10) depend smoothly on the domain Ω_u, i.e. on $u\in H^s$, $s>(m+1)/2$ and $f\mapsto F(u)f$, $f\mapsto A(u)f$ represent pseudodifferential operators of order one and minus one, respectively. In particular, G is a pseudodifferential operator of lower order than H and may be considered as a correction term to ensure the gradient flow structure of the evolution problem. We will show later that

$$[u\mapsto \mathscr{F}(u)]\in C^\infty\big(U_s, H^{s-3}(S,\mathbb{R}^m)\big) \tag{2.12}$$

for $s>(m+3)/2$, $s\geq 3$. Now we are in position to formulate our main results.

Theorem 2.1. *(Short-time existence and uniqueness.)*
Fix an even integer $s_0>(m+7)/2$, $s_0\geq 6$ and assume $\gamma\in C^\infty(S)$ strictly positive on S. Let $s\geq s_0$ be an even integer and $u_0\in U_s$. Then there exist $T>0$ and an unique solution

$$u\in C\big([0,T],U_s\big)\cap C^1\big([0,T],H^{s-3}(S,\mathbb{R}^m)\big) \tag{2.13}$$

of the initial value problem (2.2), (2.3). Additionally, any given $\bar{u}_0\in U_{s_0}$ has a suitable H^{s_0}-neighborhood K, such that for initial values u_0 varying in $K\cap H^s$, there are $T>0$ and C independent of u_0 such that

$$\|u(t)\|_s\leq C(1+\|u(0)\|_s)\ \textit{ for all }\ t\in[0,T]. \tag{2.14}$$

Theorem 2.2. *(Regularity and continuous dependence on initial values.)*
Under the assumptions of Theorem 2.1 let u be a any solution to (2.2) in the class (2.13) with some $T>0$. Then there holds:

(i) $u(0)\in H^{s+1}(S,\mathbb{R}^m)$ implies $u(t)\in H^{s+1}(S,\mathbb{R}^m)$ for all $t\in[0,T]$.

(ii) Assume $u_0^n\to u_0$ in $H^s(S,\mathbb{R}^m)$ for $n\to\infty$. Then, for n sufficiently large, there exist solutions u_n of (2.2) in the class (2.13) with initial values $u_n(0)=u_0^n$ and there holds $u_n\to u$ in $C\big([0,T],H^s(S,\mathbb{R}^m)\big)$.

The proof of both theorems is given in Section 5.

Remarks: The restriction to even integers s is due purely to the construction of our bilinear form involving integer powers of a generalized Laplacian. This restriction can be lifted afterwards by using the nonlinear interpolation result given in [2], Proposition A.1 and Remark A.2. The dimension independent restriction $s_0\geq 6$ is needed as we use dual estimates for elliptic boundary value problems in norms with negative index.

3. Smooth domain dependence of the non-local operators

We start by gathering some properties of the nonlocal operators F, A, and $\mathscr{G}$ defined by (2.4)–(2.11). The multilinear estimates for the Fréchet derivatives can be seen as counterparts to the product estimate

$$\|u_1\dots u_k\|_t\leq C\|u_1\|_{s_1}\dots\|u_k\|_{s_k}$$

holding if $0\leq t\leq s_i\leq\sigma$, $\sigma>(m-1)/2$, $\sum_{i=1}^k s_i\geq t+(k-1)\sigma$. Here, however, we have to deal with nonlocal operators of various orders involving differentiations and the solution of elliptic BVP.

The statements and their proofs are essentially parallel to Corollary 4.4 and Lemma 4.5 in [8], therefore proofs will be omitted. Note, however, that $f \mapsto F(u)f$ is an operator of order one here as (2.6) is a Dirichlet problem. In fact, the normal component of F is given by the Dirichlet-Neumann operator while the tangential component is given by the tangential gradient of f.

Due to the variability in the choice of the s_i, the estimates will be flexible enough to control various lower order terms that will occur in the sequel.

Lemma 3.1. *(i) Let $s > (m+1)/2$ and $t \in [1, s]$ be given. Then*

$$F \in C^\infty\big(U_s, \mathscr{L}(H^t(S), H^{t-1}(S, \mathbb{R}^m))\big),$$
$$A \in C^\infty\big(U_s, \mathscr{L}(H^{t-1}(S), H^t(S, \mathbb{R}^m))\big)$$

and for any $u \in U_s$ and any choice of $s_1, \dots s_{k+1} \in [t, s]$ with $s_1 + \dots + s_{k+1} \geq t + ks$ there exists a constant $C > 0$ such that for all $f \in H^s(S)$, and all $u_1, \dots, u_k \in H^s(S, \mathbb{R}^m)$ there holds

$$\big\|F^{(k)}(u)\{u_1, \dots, u_k\}f\big\|_{t-1} \leq C\|u_1\|_{s_1} \cdots \|u_k\|_{s_k}\|f\|_{s_{k+1}} \tag{3.1}$$
$$\big\|A^{(k)}(u)\{u_1, \dots, u_k\}f\big\|_{t} \leq C\|u_1\|_{s_1} \cdots \|u_k\|_{s_k}\|f\|_{s_{k+1}-1} \tag{3.2}$$

(ii) Let $s > (m+3)/2$ and $t \in [2, s]$ be given. Then

$$\mathscr{G} \in C^\infty\big(U_s, H^{s-2}(S)\big)$$

and for any $u \in U_s$ and any choice of $s_1, \dots s_k \in [t, s]$ with $s_1 + \dots + s_{k+1} \geq t + ks$ there exists a constant $C > 0$ such that for all $u_1, \dots, u_k \in H^s(S, \mathbb{R}^m)$ there holds

$$\big\|\mathscr{G}^{(k)}(u)\{u_1, \dots, u_k\}\big\|_{t-1} \leq C\|u_1\|_{s_1} \cdots \|u_k\|_{s_k}. \tag{3.3}$$

The constants may be chosen independently of u as u varies in bounded and weakly closed subsets of U_s.

Remark 3.2. Note that a bounded subset of $H^s(S)$ is weakly closed if and only if it is closed in $H^t(S)$ for some $t < s$. Then it is compact in all $H^t(S)$ with $t < s$.

Remark 3.3. The estimate (3.3) is not optimal as we do not use the quasilinear character of $\mathscr{G}$. For our purposes, however, it will be sufficient.

Note that Lemma 3.1 implies the smoothness assertion (2.12).

Next, we prove some related estimates in norms with negative index. The use of such norms implies a loss of flexibility. Essentially, these estimates are parallel to product estimates of the type

$$\|u_1 \dots u_k\|_t \leq C\|u_1\|_s \dots \|u_{k-1}\|_s\|u_k\|_t,$$

$t \in [-s, s]$, $s > (m-1)/2$, which can be proved by duality arguments if $t < 0$.

Lemma 3.4. *Assume $s > (m+1)/2$, $s \geq 4$, $t \in [-3, s-1]$. Then*

$$F \in C^\infty\big(U_s, \mathscr{L}(H^{t+1}(S), H^t(S))\big)$$
$$A \in C^\infty\big(U_s, \mathscr{L}(H^{t-1}(S), H^t(S))\big)$$

and for $u \in U_s$

$$\|F'(u)u_1\|_t \le C\|u_1\|_s\|f\|_{t+1}, \tag{3.4}$$

$$\|A(u)f\|_t \le C\|f\|_{t-1}, \tag{3.5}$$

$$\|A^{(k)}(u)\{u_1,\dots,u_k\}f\|_t \le C\|u_1\|_s\dots\|u_k\|_s\|f\|_{t-1}, \tag{3.6}$$

$$\|A^{(k)}(u)\{u_1,\dots,u_k\}f\|_t \le C\|u_1\|_s\dots\|u_{k-1}\|_s\|u_k\|_t\|f\|_s, \tag{3.7}$$

$u_1,\dots,u_k \in H^s(S,\mathbb{R}^m)$, $k \in \mathbb{N}$. *The constants C can be chosen independently of u as u varies in bounded, weakly closed subsets of U_s.*

Proof. We will restrict ourselves to the assertions concerning A. Fix $s_0 \in ((m+1)/2, s)$ and an extension operator $\mathscr{E} \in \mathscr{L}(H^t(S), H^{t+1/2}(E))$, $t > 0$. Pick $v \in U_s$ and choose an H^{s_0}-neighborhood $V_{s_0} \subset U_s$ and $u_0 \in C^\infty(\overline{E}, \mathbb{R}^m)$ such that

$$\widetilde{u} := u_0 + \mathscr{E}(u-u_0) \in \operatorname{Diff}(\overline{E}, \overline{\Omega}_u).$$

This is possible by Lemma 4.1 in [8].

For $u \in V_{s_0}$, let the transformed operators $L(u)$ and $\mathscr{B}(u)$ be defined by

$$L(u)\psi := \partial_i(\sqrt{g}g^{ij}\partial_j\psi), \quad \mathscr{B}(u)\psi := \nu_i\sqrt{g}g^{ij}\partial_j\psi,$$

where $\sqrt{g}$, g^{ij} are the volume element and the (inverse) coefficients of the metric on E induced by $\widetilde{u}$, respectively, and ν is the outer unit normal on S. We consider the transformed boundary value problem

$$L(u)\psi = \Phi_1, \quad \mathscr{B}(u)\psi = \omega(u)(\Phi_2 + c), \quad \int_s \omega(u)(\Phi_2+c)\,dS = \int_E \sqrt{g}\Phi_1\,dx, \tag{3.8}$$

$c = c(u, \Phi_1, \Phi_2) \in \mathbb{R}$. Here $\omega(u) = d\Gamma_u/dS$ is the surface element belonging to the transformation induced by u which is given by a nonlinear first-order differential operator in $\widetilde{u}$.

For $\tau > 0$ and $v \in L^2(E)$ define

$$\|v\|_\tau := \sup_{z \in H^\tau(E), \|z\|_\tau = 1} \left| \int vz\,dx \right|.$$

(This differs from the usual norm in $H^{-\tau}(E) := (H_0^\tau(E))'$.) The BVP (3.8) is uniquely solvable and ψ satisfies an estimate

$$\|\psi\|_t + \|\psi\|^E_{t+1/2} \le C\big(\|\Phi_1\|^E_{t-3/2} + \|\Phi_2\|_{t-1}\big) \tag{3.9}$$

(cf. [7], Lemma 3.1).

As $A(u)f$ is the trace of the solution ψ of (3.8) with $\Phi_1 = 0$, $\Phi_2 = f$, we get (3.5) immediately from (3.9).

Note that $A'(u)\{u_1\}f$ is given as the solution ψ' of

$$L(u)\psi' = -L'(u)\{u_1\}\psi,$$
$$\mathscr{B}(u)\psi' = -\mathscr{B}'(u)\{u_1\}\psi + \omega'(u)\{u_1\}(f + c(u,0,f)) + \omega(u)\partial_u c(u,0,f)\{u_1\}.$$

As $f \mapsto c(u,0,f)$ and $v \mapsto \partial_u c(u,0,f)\{v\}$ are given by smoothing operators, to obtain (3.6) and (3.7) it is sufficient to use (3.9) and estimate either

$$\|L'(u)\{u_1\}\psi\|^\Omega_{t-3/2} \le C\big(\|\mathscr{E}u_1\|^\Omega_{t+1/2} + \|u_1\|_t\big)\|\psi\|_{s+1} \le C\|u_1\|_t\|f\|_s$$

or

$$\|L'(u)\{u_1\}\psi\|^{\Omega}_{t-3/2} \le C\big(\|\psi\|^{\Omega}_{t+\frac{1}{2}} + \|\psi\|_t\big)\|u_1\|_s \le C\|u_1\|_s\|f\|_{t-1},$$

together with analogous estimates for $\|\mathscr{B}'(u)\psi\|_{t-1}$ and $\|\omega'(u)\{u_1\}f\|_{t-1}$. The general case follows now by induction over k, cf. [8], Lemma 4.5.

The estimate (3.4) can be obtained in a similar fashion, discussing a Dirichlet problem instead of (3.8).

Finally, the uniformity of the estimates follows from the fact that bounded, weakly closed subsets of U_s are compact in $H^{s_0}(S)$. □

We choose m smooth vector fields $D_1, \dots, D_m$ on S such that

$$\operatorname{span}\{D_1, \dots, D_m\} = T_x \ \text{ for all } \ x \in S$$

and use the multi-index notation $D^\alpha = D_1^{\alpha_1} \dots D_m^{\alpha_m}$, $\alpha = (\alpha_1, \dots, \alpha_m)$ for higher order derivatives; for simplicity we assume that $(D_1, \dots, D_m)$ coincides with the tangential gradient on S. Note that, for $s \ge 0$ integer, we can use

$$(u, v)_s = \sum_{|\alpha| \le s} (D^\alpha u, D^\alpha v)_{L^2(S)}$$

as scalar product generating the norm in $H^s(S)$. Moreover, as an immediate consequence of the invariance properties

$$\big(F(u)f\big) \circ \tau = F(u \circ \tau)(f \circ \tau)$$

for any diffeomorphism τ on S, we have a differentiation rule which resembles Leibniz' rule at an abstract level, cf. [8]: For any multi-index α and $u \in U_s$, $f \in H^s(S)$, $s > |\alpha| + (m+1)/2$ there holds

$$D^\alpha F(u)f = \sum c_{\beta_1, \dots, \beta_{k+1}} F^{(k)}(u)\{D^{\beta_1}u, \dots, D^{\beta_k}u\} D^{\beta_{k+1}} f \tag{3.10}$$

where the sum has to be extended over all integers k and systems of non-negative multi-indices $\beta_1, \dots, \beta_{k+1}$ with

$$0 \le k \le |\alpha|, \quad 1 \le |\beta_1|, \dots, |\beta_k|, \quad \beta_1 + \dots + \beta_{k+1} = \alpha. \tag{3.11}$$

The coefficients are non-negative integers, in particular, $c_\alpha = c_{\alpha,0} = 1$.

Combining the differentiation rule for F with the estimate of the derivatives in lower norms we obtain

Proposition 3.5. *(i) Let $s \ge s_0 > (m+1)/2$, s integer, $u \in U_s$. Then*

$$\|F(u)f\|_{s-1} \le C\big(\|u\|_s\|f\|_{s_0} + \|f\|_s\big) \tag{3.12}$$

with an uniform constant as long as u varies in H^{s_0}- bounded and weakly H^{s_0}- closed subsets of U_s.
(ii) Assume additionally $s \ge s_0 + 2$ and let α be any multi-index with $|\alpha| = s$. Writing $D^\alpha = D^{\alpha_1} \dots D^{\alpha_s}$ with $|\alpha_1| = \dots = |\alpha_s| = 1$, we have

$$D^\alpha F(u)f = F(u)D^\alpha f + F'(u)\{D^\alpha u\}f + \sum_{i=1}^{s} F'(u)\{D^{\alpha_i}u\}D^{\beta_i}f + R_\alpha(u)f \tag{3.13}$$

where $\alpha = \alpha_i + \beta_i$ and the remainder term allows the estimate

$$\|R_\alpha(u)f\|_0 \le C\big(\|u\|_s\|f\|_{s_0+1} + \|f\|_{s-1}\big).$$

The constant can be chosen uniformly as u varies in H^{s_0+2}- bounded, weakly H^{s_0+2}-closed subsets of U_s.

Proof. We consider the more complicated situation (ii) only. According to (3.10), the remainder term has a representation as a sum of terms

$$I_\beta = F^{(k)}(u)\{D^{\beta_1}u, \dots, D^{\beta_k}u\}D^{\beta_{k+1}}f,$$

where the multi-indices satisfy (3.11) and additionally

$$|\beta_1|, \dots, |\beta_k| \le s-1, \quad |\beta_{k+1}| \le s-2.$$

Hence $k \ge 1$. For each of the terms I_β, we will choose numbers $\theta_1, \dots, \theta_{k+1} \in [0,1]$ such that $\theta_1 + \ldots + \theta_{k+1} = 1$ and set

$$s_i := (1-\theta_i)s_0 + \theta_i.$$

If $k = 1$ we choose θ_1, θ_2 such that $\theta_1 + \theta_2 = 1$ and $|\beta_2| = \theta_1 + \theta_2(s-2)$. If $k = 2$ and $|\beta_3| = 0$ we choose $\theta_i := (|\beta_i| - 1)/(s-2)$ for $i = 1, 2$ and $\theta_3 := 0$. If $k = 2$ and $|\beta_3| \ge 1$ or $k \ge 3$ we choose

$$\theta_i := (|\beta_i| - 1)/(s-3) \text{ for } i = 1,2,3, \quad \theta_i := |\beta_i|/(s-3) \text{ for } i \ge 4.$$

In all cases, we have

$$\left.\begin{aligned} |\beta_i| + s_i &\le (1-\theta_i)(s_0+2) + \theta_i s, \qquad i = 1, \dots, k, \\ |\beta_{k+1}| + s_{k+1} &\le (1-\theta_{k+1})(s_0+1) + \theta_{k+1}(s-1). \end{aligned}\right\} \tag{3.14}$$

Set $\lambda := \theta_1 + \ldots + \theta_k$. Using (3.1) with $t = 1$, $s = s_0$, (3.14), norm convexity, and Young's inequality we get

$$\begin{aligned} \|I_\beta\|_0 &\le C\|u\|_{|\beta_1|+s_1} \cdots \|u\|_{|\beta_1|+s_k}\|f\|_{|\beta_{k+1}|+s_{k+1}} \\ &\le C\|u\|_{s_0+2}^{k-1}\big(\|u\|_{s_0+2}\|f\|_{s-1}\big)^{1-\lambda}\big(\|u\|_s\|f\|_{s_0+1}\big)^\lambda \\ &\le C\|u\|_{s_0+2}^{k-1}\big(\|u\|_{s_0+2}\|f\|_{s-1} + \|u\|_s\|f\|_{s_0+1}\big), \end{aligned}$$

and the result follows. □

The following lemma provides an explicit characterization for the linearization of F, namely, up to terms of order zero in v,

$$F'(u)f \approx -F(u)(v \cdot F(u)f).$$

This structure will be important later. It can be verified in an informal way by performing the variation on Ω_u itself instead of transforming the problem to the reference domain.

Lemma 3.6. *Let $s > (m+3)/2$. Then for $u \in U_s$, $v \in H^s(S, \mathbb{R}^m)$ and $f \in H^s(S)$ there holds*

$$\|F'(u)\{v\}f + F(u)(v \cdot F(u)f)\|_0 \le C\|f\|_s\|v\|_0.$$

Proof. From (2.5) we get

$$F_i'(u)\{v\}f = \partial_i\phi' \circ u + v_j \partial_i\partial_j\phi \circ u$$

with $\phi = \phi(u,f)$ from (2.6) and $\phi' = \phi'(u,f)\{v\}$ given by

$$\phi'(u,f)\{v\}(x) = \partial_\varepsilon(\phi(u+\varepsilon v, f)(x))|_{\varepsilon=0}, \quad x \in \Omega_u.$$

The function ϕ' satisfies

$$\Delta\phi' = 0 \text{ in } \Omega_u, \quad \phi' = -\nabla\phi \cdot v \ \text{ on } \ \Gamma_u, \tag{3.15}$$

therefore $\partial_i\phi' \circ u = -F_i(u)(v \cdot F(u)f)$.

Parallel to the proof of Lemma 5.1 in [8] one obtains

$$\|v_j\partial_i\partial_j\phi \circ u\|_0 \le C\|v\|_0\|\phi(u,f)\|_{C^2(\overline{\Omega}_u)} \le C\|f\|_s\|v\|_0.$$

This proves the assertion. □

4. The main estimate

In this section we prove H^s- a priori estimates for the non-linear operator $\mathscr{F}$ w.r. to variable bilinear forms which we define in the sequel. As already mentioned in the introduction, these estimates are the main ingredient in the existence proof.

To begin with, for $u \in U_s$, $s > (m+1)/2$ we define

$$P(u)v := v \cdot \big(n(u) \circ u\big), \quad N(u)w := w\,\big(n(u) \circ u\big), \tag{4.1}$$

$$\Lambda(u)w := \nabla_{\Gamma_u}(w \circ u^{-1}) \circ u \tag{4.2}$$

as the euclidean scalar product and multiplication with outer normal $n(u)$ of Γ_u and pullback of tangential gradient ∇_{Γ_u} along Γ_u, respectively. Considered as operators in v and w, the coefficients of $P(u)$, $N(u)$ and $\Lambda(u)$ are smooth functions of u and its first derivatives. Thus,

$$P(u) \in \mathscr{L}\big(H^t(S,\mathbb{R}^m), H^t(S)\big), \quad N(u) \in \mathscr{L}\big(H^t(S), H^t(S,\mathbb{R}^m)\big), \tag{4.3}$$

$$\Lambda(u) \in \mathscr{L}\big(H^t(S), H^{t-1}(S,\mathbb{R}^m)\big) \tag{4.4}$$

depend smoothly on $u \in U_s$ for $|t| \le s-1$ and $|t-1| \le s-1$, respectively. Clearly, the operators P, N, Λ satisfy invariance properties as stated for F in [8]. As a consequence, the differentiation rule (3.10) is also true for P, N, Λ; we make use of that without explicit mention. Further recall that the pullback $\Delta(u)w$ of the Laplace Beltrami operator Δ_{Γ_u} on Γ_u according to (2.8) and the operator $H(u)$ according to (2.7) may be expressed as

$$\Delta(u)w = \Lambda_i(u)\big(\Lambda_i(u)w\big), \quad H(u) = -\gamma\Lambda_i(u)\big(n_i(u) \circ u\big) \tag{4.5}$$

respectively.

In the further considerations of this section we fix s_0 to be the smallest integer such that $s_0 \ge 6$ and $s_0 > (m+7)/2$ and set

$$\widetilde{U}_s := U_s \cap K \ \text{ for all } \ s \ge s_0$$

with a fixed H^{s_0}- bounded and weakly H^{s_0}-closed subset $K \subseteq U_{s_0}$. Note that

$$1 \le C\|u\|_{s_0} \le C'\|u\|_s, \quad \|u\|_{C^3(S)} \le C$$

for all $u \in \widetilde{U}_s$, $s \geq s_0$.

Furthermore, we have the estimates

$$\|\mathscr{G}(u)\|_{s-2}, \|\mathscr{F}(u)\|_{s-3} \leq C\|u\|_s \text{ for all } u \in \widetilde{U}_s, s \geq s_0,$$

and the operators defined in (4.3), (4.4) are bounded uniformly with respect to $u \in \tilde{U}_s$.

Due to our choice of the differential operators D_i the Laplace-Beltrami operator on the compact reference manifold S is given by $\Delta_0 := D_i D_i$. It has an approximate inverse, i.e. there is an operator $\Delta_0^+ \in \mathscr{L}\big(H^\tau(S), H^{\tau+2}(S)\big)$, $\tau \in \mathbb{R}$, such that

$$\Delta_0 \Delta_0^+ = \Delta_0^+ \Delta_0 = \mathrm{id} + Q_0,$$

with a smoothing operator Q_0 simply given by orthogonal projection in $L^2(S)$ onto the subspace of functions which are constant on each connectivity component of S; in particular, $Q_0 \in \mathscr{L}(H^\tau(S), H^\sigma(S))$ for any $\sigma, \tau \in \mathbb{R}$. In the same manner, we define the approximate inverse $\Delta^+(u)$ for $\Delta(u)$. In this case we have

$$[u \mapsto \Delta^+(u)] \in C^\infty\big(U_s, \mathscr{L}\big(H^t(S), H^{t+2}(S)\big)\big), \quad t \in [0, s-2]$$

and

$$\Delta(u)\Delta(u)^+ = \Delta(u)^+\Delta(u) = \mathrm{id} + Q(u), \tag{4.6}$$

where $Q(u) \in \mathscr{L}\big(H^\tau(S), H^\sigma(S)\big)$ for any $\sigma \in \mathbb{R}$, $\tau \geq 1-s$, and the corresponding norms are bounded independently of $u \in \tilde{U}_s$.

Lemma 4.1. *Let $s \geq s_0$ with $s = 2k$, $k \in \mathbb{N}$ and $u \in U_s$. Then we have*

$$\Delta_0^k \mathscr{F}(u) = \tilde{F}(u)\big(\tilde{G}(u)(\Delta_0^k u)\big) + F(u)\big(R_a(u)\big) + R_b(u) \tag{4.7}$$

where the abbreviations

$$\tilde{F}(u)f := F(u)f + F_0(u)f, \quad \tilde{G}(u)v := \gamma\Delta(u)(P(u)v) + G_1(u)v$$

have been used. Here $f \mapsto F_0(u)f$ and $v \mapsto G_1(u)v$ are operators of order zero and one, respectively,

$$F_0(u) \in \mathscr{L}\big(H^t(S), H^t(S, \mathbb{R}^m)\big), \quad G_1(u) \in \mathscr{L}\big(H^t(S, \mathbb{R}^m), H^{t-1}(S)\big), \tag{4.8}$$

$t \in [-1, s-1]$ and $t \in [-2, s-3]$, respectively, and the remainder terms R_a, R_b satisfy

$$\|R_a(u)\|_0 \leq C\|u\|_{s-1}, \tag{4.9}$$

$$\|R_b(u)\|_0 \leq C\big(\|\tilde{G}(u)(\Delta_0^k u)\|_{-1} + \|\mathscr{F}(u)\|_{s_0-3}\|u\|_s + \|u\|_{s-1}\big). \tag{4.10}$$

The constants are independent of u and and the operator norms of $F_0(u)$ and $G_1(u)$ are bounded independently of u as long as u varies in a set $\tilde{U}_s$.

Proof. The operator $F(u)$ vanishes on constants and, by elliptic regularity,

$$\|f\|_{s_0-2} \leq C\|F(u)f\|_{s_0-3}, \qquad u \in \tilde{U}_s \tag{4.11}$$

if f has zero mean value over S. We define

$$\tilde{\mathscr{G}}(u) := \mathscr{G}(u) - \frac{1}{|S|}\int_S \mathscr{G}(u)\, dS. \tag{4.12}$$

Using Proposition 3.5, (ii), we write $\Delta_0^k \mathscr{F}(u) = \Delta_0^k F(u)\tilde{\mathscr{G}}(u)$ in the form

$$F(u)\big(\Delta_0^k \tilde{\mathscr{G}}(u)\big) + F'(u)\big\{\Delta_0^k u\big\}\tilde{\mathscr{G}}(u) + 2\sum_{j=0}^{k-1} F'(u)\big\{D_i u\big\}\Delta_0^j D_i \Delta_0^{k-1-j}\tilde{\mathscr{G}}(u) + R_1(u)$$

$$= F(u)\big(\Delta_0^k \mathscr{G}(u)\big) + F'(u)\big\{\Delta_0^k u\big\}\tilde{\mathscr{G}}(u) + 2kF'(u)\big\{D_i u\big\}D_i\Delta_0^{k-1}\mathscr{G}(u) + R_1(u) + R_2(u).$$

According to this proposition and (4.11) $R_1(u)$ allows the estimate

$$\|R_1(u)\|_0 \le C\big(\|u\|_s\|\tilde{\mathscr{G}}(u)\|_{s_0-2} + \|\tilde{\mathscr{G}}(u)\|_{s-1}\big) \le C\big(\|u\|_s\|\mathscr{F}(u)\|_{s_0-3} + \|\mathscr{G}(u)\|_{s-1}\big)$$

For $R_2(u)$ we find from Lemma 3.1 (with $t=1$)

$$\begin{aligned}\|R_2(u)\|_0 &\le 2\sum_{j=0}^{k-1}\big\|F'(u)\big\{D_i u, [\Delta_0^j, D_i]\Delta_0^{k-1-j}\mathscr{G}(u)\big\}\big\|_0\\ &\le C\sum_{i,j}\big\|[\Delta_0^j, D_i]\Delta_0^{k-1-j}\mathscr{G}(u)\big\|_1 \le C\|\mathscr{G}(u)\|_{s-1}.\end{aligned}$$

Further, by Lemma 3.6 we have

$$F'(u)\big\{\Delta_0^k u\big\}\tilde{\mathscr{G}}(u) = -F(u)\big(\Delta_0^k u\cdot\mathscr{F}(u)\big) + R_3(u)$$

with

$$\|R_3(u)\|_0 \le C\|\Delta_0^k u\|_0\|\tilde{\mathscr{G}}(u)\|_{s_0-2} \le C\|u\|_s\|\mathscr{F}(u)\|_{s_0-3}.$$

Defining $F_0(u)$ by

$$F_0(u)v := 2kF'(u)\big\{D_i u\big\}D_i\Delta_0^+ v,$$

we get

$$\Delta_0^k\mathscr{F}(u) = \big(F(u)+F_0(u)\big)\big(\Delta_0^k\mathscr{G}(u) - \Delta_0^k u\cdot\mathscr{F}(u)\big) + R_4(u)$$

with a remainder term

$$R_4(u) = F_0(u)\big(\Delta_0^k u\cdot\mathscr{F}(u)\big) + R_1(u) + R_2(u) + R_3(u) - 2kF'(u)\{D_i u\}D_i Q_0\Delta_0^{k-1}\widetilde{\mathscr{G}}(u).$$

Hence, using

$$\big\|F_0(u)\big(\Delta_0^k u\cdot\mathscr{F}(u)\big)\big\|_0 \le C\|u\|_s\|\mathscr{F}(u)\|_{s_0-3}$$

and the above estimates for R_1, R_2, R_3 we obtain

$$\|R_4(u)\|_0 \le C\big(\|\mathscr{G}(u)\|_{s-1} + \|u\|_s\|\mathscr{F}(u)\|_{s_0-3}\big). \tag{4.13}$$

Recall that $\mathscr{G}(u)$ depends linearly on γ. Slightly abusing notation, we write $\mathscr{G}(u)\gamma$ etc. in the remaining part of this proof (see (6.6) below). Note that in analogy to (3.10), we get (for sufficiently smooth u)

$$D^\alpha\mathscr{G}(u)\gamma = \sum c_{\beta_1,\dots,\beta_{k+1}}\mathscr{G}^{(k)}(u)\{D^{\beta_1}u,\dots,D^{\beta_k}u\}D^{\beta_{k+1}}\gamma$$

with (3.11) holding for k and the multiindices $\beta_1,\dots,\beta_{k+1}$. Applying this differentiation rule, we get

$$\Delta_0^k\mathscr{G}(u)\gamma = \mathscr{G}'(u)\big\{\Delta_0^k u\big\}\gamma + \mathscr{G}_1(u)\gamma + R_5(u),$$

where $\mathscr{G}_1(u)$ contains all terms where derivatives of u of order $s-1$ and $s-2$ occur.

Consequently, it is a sum of terms of the forms

$$\mathscr{G}'(u)\big\{\Delta_0^j D_i \Delta_0^{k-1-j} u\big\} D_i\gamma, \quad \mathscr{G}''(u)\big\{D_i u, \Delta_0^j D_i \Delta_0^{k-1-j} u\big\}\gamma,$$

with $j \in \{0, \dots, k-1\}$, and

$$\mathscr{G}'(u)\{z\} D_i D_l \gamma, \quad \mathscr{G}''(u)\{D_i u, z\} D_l \gamma, \quad \mathscr{G}''(u)\{D_i D_l u, z\}\gamma, \quad \mathscr{G}'''(u)\{D_i u, D_l u, z\}\gamma,$$

with $z := \Delta_0^j D_i \Delta_0^\mu D_l \Delta_0^{k-2-j-\mu} u$ and $j, \mu \in \{0, \dots, k-2\}$, $j + \mu \le k-2$. Using the estimates (3.3) and the assumption $s \ge s_0 > (m+7)/2$, one obtains from analogous arguments as in the proof of Proposition 3.5, (ii)

$$\|R_5(u)\|_0 \le C\|u\|_{s-1}. \tag{4.14}$$

Writing in the above terms

$$\Delta_0^{k-1-j} u \approx (\Delta_0^+)^{j+1} \Delta_0^k u, \quad \Delta_0^{k-2-j-\mu} u \approx (\Delta_0^+)^{j+\mu+2} \Delta_0^k u \tag{4.15}$$

up to smoothing remainder terms, we get

$$\Delta_0^k \mathscr{G}(u)\gamma = \mathscr{G}'(u)\big\{\Delta_0^k u\big\}\gamma + \mathscr{G}_2(u)\{\Delta_0^k u\}\gamma + R_6(u) \tag{4.16}$$

with a first-order operator $v \mapsto \mathscr{G}_2(u)\{v\}\gamma$ and a remainder term $R_6(u)$ satisfying the estimate (4.14) again. Hence, using that the linearization of the mean curvature $H(u)$ has $\Delta(u)(P(u)v)$ as main part, i.e.

$$\mathscr{G}'(u)\{v\}\gamma = \gamma\Delta(u)P(u)v + \mathscr{G}_3(u)\{v\}\gamma \tag{4.17}$$

with a first-order operator $v \mapsto \mathscr{G}_3(u)\{v\}\gamma$, we get the representation (4.7) with

$$G_1(u)v := \mathscr{G}_2(u)\{v\}\gamma + \mathscr{G}_3(u)\{v\}\gamma - v \cdot \mathscr{F}(u)$$

and with the remainder terms

$$R_a(u) := R_6(u), \quad R_b(u) := R_4(u) + F_0(u)\big(R_6(u)\big).$$

Now the estimate (4.9) of R_a coincides with (4.14), whereas the estimate (4.10) of R_b follows from

$$\begin{aligned} \|R_b(u)\|_0 &\le C\big(\|R_4(u)\|_0 + \|R_6(u)\|_0\big) \\ &\le C\big(\|\mathscr{G}(u)\|_{s-1} + \|u\|_s \|\mathscr{F}(u)\|_{s_0-3} + \|u\|_{s-1}\big) \end{aligned}$$

by (4.13), (4.14) and

$$\begin{aligned} \|\mathscr{G}(u)\|_{s-1} &\le C\big(\|\Delta_0^k \mathscr{G}(u)\|_{-1} + \|\mathscr{G}(u)\|_0\big) \\ &= C\big(\|\tilde{G}(u)(\Delta_0^k u) + R_6(u)\|_{-1} + \|\mathscr{G}(u)\|_0\big) \\ &\le C\big(\|\tilde{G}(u)(\Delta_0^k u)\|_{-1} + \|u\|_{s-1}\big). \end{aligned}$$

The statements (4.8) are consequences of Lemma 3.4 and of (3.4). This becomes clear if G_1 is written out explicitly in terms of differential operators and Fréchet derivatives of A. □

Now fix $s \ge s_0$ with $s = 2k$, $k \in \mathbb{N}$. Letting F_0 and G_1 as in Lemma 4.1 we set for $u \in U_s$

$$\tilde{\mathscr{F}}(u)v := \big(F(u) + F_0(u)\big)\tilde{G}(u)v.$$

Further, for $u \in U_s$ let $M(u)$ be the operator defined by

$$M(u)v := M_0(u)v + \tilde{M}_0(u)v, \quad \tilde{M}_0(u)v = M_1(u)P(u)v + N(u)M_2(u)v \tag{4.18}$$

Here, the main part M_0 of M is given by

$$M_0(u)v := v - \Lambda(u)A(u)P(u)v \tag{4.19}$$

with A from (2.9) (cf. [8]), whereas the lower order terms are given by

$$M_1(u)w := -M_0(u)F_0(u)A(u)w, \tag{4.20}$$

$$M_2(u)v := \Delta(u)^+\big(\gamma^{-1}G_1(u)v\big) \tag{4.21}$$

From (3.5) and (4.8) we get

$$\begin{aligned}
&M_0(u) \in \mathscr{L}\big(H^t(S,\mathbb{R}^m), H^t(S,\mathbb{R}^m)\big), \quad -4 \le t \le s-2,\\
&M_1(u) \in \mathscr{L}\big(H^t(S), H^{t+1}(S,\mathbb{R}^m)\big), \quad -2 \le t \le s-3\\
&M_2(u) \in \mathscr{L}\big(H^t(S,\mathbb{R}^m), H^{t+1}(S)\big) \quad -2 \le t \le s-3.
\end{aligned}$$

The operators depend smoothly on $u \in U_s$ and have uniformly bounded norms as u varies in $\widetilde{U}_s$.

Because of $P(u)\Lambda(u) = 0$ the operator $M_0(u)$ constitutes an isomorphism in $L^2(S,\mathbb{R}^m)$ with inverse

$$M_0(u)^{-1}v = v + \Lambda(u)A(u)P(u)v. \tag{4.22}$$

In particular, we have

$$c\|v\|_0 \le \|M_0(u)v\|_0 \le C\|v\|_0, \tag{4.23}$$

$$c\|v\|_0 - C\|v\|_{-1} \le \|M(u)v\|_0 \le C\|v\|_0 \tag{4.24}$$

with suitable positive constants c, C independent of $u \in \tilde{U}_s$ and $v \in L^2$. Moreover, after a simple calculation we obtain

$$\big(M_0(u)F(u)f, M_0(u)v\big)_0 = \big(B(u)f, P(u)v\big)_0, \tag{4.25}$$

where $f \mapsto B(u)f := P(u)(F(u)f)$ is the Dirichlet-Neumann operator.

For the sake of completeness, we gather some properties of B which we will need in the sequel. We will use the commutator notation $[Q_1, Q_2] := Q_1Q_2 - Q_2Q_1$ for operators, in particular, if f is a function we will write $[f, Q]w := fQw - Q(fw)$. Note that property b) is in fact the L^2-symmetry of $B(u)$ with respect to the measure induced from Γ_u.

Lemma 4.2. *Assume $u \in \tilde{U}_s$, $f \in C^1(S)$, $w \in H^2(S)$, $v \in H^1(S)$. Then:*

a) *If $f \ge \alpha > 0$ then*

$$\int_S fwB(u)w\,dS \ge c\|w\|_{1/2}^2 - C\|w\|_0^2$$

with $c = c(\alpha) > 0$, $C = C(\|f\|_{C^1})$. Moreover,

b)

$$\int_S w\,B(u)v\,dS = \int_S \omega(u)v\,B(u)(\omega(u)^{-1}w)\,dS,$$

c)

$$\|B(u)w\|_{-2} \leq C\|w\|_{-1},$$

d)

$$\|[f, B(u)]w\|_0 \leq C\|w\|_0$$

with $C = C(\|f\|_{C^1})$,

e)

$$\|[\Lambda_i(u), B(u)]w\|_0 \leq C\|w\|_1.$$

All constants are independent of $u \in \tilde{U}_s$.

Proof. a) As in the proof of Lemma 3.4 we extend u to a diffeomorphism from E to Ω_u and denote the coefficients of the corresponding induced metric by g^{ij} and the corresponding volume element by $\sqrt{g}$. Let ν denote the outer unit normal on S and let $\mathscr{E}$ denote the harmonic extension from S into E. Let ϕ be the solution of the Dirichlet problem

$$L(u)\phi := \partial_i(\sqrt{g}g^{ij}\partial_j\phi) = 0 \text{ in } E, \quad \phi|_S = w.$$

Then

$$B(u)w = \omega(u)^{-1}\nu_i\sqrt{g}g^{ij}\partial_j\phi,$$

and by integration by parts

$$\begin{aligned}
&\int_S fw\, B(u)w\, dS \\
&= \int_S f\phi\omega(u)^{-1}\nu_i\sqrt{g}g^{ij}\partial_j\phi\, dS = \int_E \partial_i(\mathscr{E}(f\omega(u)^{-1})\phi\sqrt{g}g^{ij}\partial_j\phi)\, dx \\
&= \int_E \mathscr{E}(f\omega(u)^{-1})\sqrt{g}g^{ij}\partial_i\phi\partial_j\phi\, dx + \int_E \partial_i(\mathscr{E}(f\omega(u)^{-1})\phi\sqrt{g}g^{ij}\partial_j\phi\, dx \\
&\geq c\|\phi\|_1^{E^2} - C\|\phi\|_1^E\|\phi\|_0^E \geq c\|\phi\|_1^{E^2} - C\|\phi\|_0^{E^2} \geq c\|w\|_{1/2}^2 - C\|w\|_0^2.
\end{aligned}$$

The uniformity of these estimates with respect to $u \in \tilde{U}_s$ follows by a compactness argument as in [8].

b) The assertion follows from transforming the integral to Γ_u, applying Green's formula and transforming back.

c) Using b), the assertion follows from a standard duality argument and the fact that $B(u) \in \mathscr{L}(H^2(S), H^1(S))$.

d) Maintaining the notation from the proof of a), we have

$$[f, B(u)]w = \omega(u)^{-1}\nu_i\sqrt{g}g^{ij}(f\partial_j\phi - \partial_j\psi)$$

where ψ satisfies

$$L(u)\psi = 0 \text{ in } E, \quad \psi|_S = fw = f\phi|_S.$$

Therefore, by estimates parallel to Lemma 4.3 in [8],

$$\begin{aligned}
\|[f, B(u)]w\|_{1/2} &\leq \|\omega(u)^{-1}\nu_i\sqrt{g}g^{ij}\partial_j(\phi\mathscr{E}f - \psi)\|_{1/2} + C\|\phi\|_{1/2} \\
&\leq C\left(\|L(u)(\phi\mathscr{E}f)\|_0^E + \|\phi\|_{1/2}\right) \leq C\left(\|\phi\|_1^E + \|\phi\|_{1/2}\right) \leq C\|w\|_{1/2}.
\end{aligned}$$

As both multiplication by f and $B(u)$ are symmetric with respect to the L^2-inner product induced from Γ_u, we get by duality

$$\|[f, B(u)]w\|_{-1/2} \le C\|w\|_{-1/2},$$

and the result follows by interpolation.

e) We have, by the chain rule for the operators D_k,

$$\begin{aligned}[\Lambda_i(u), B(u)] &= [\alpha_k^i(u)D_k, B(u)] = [\alpha_k^i(u), B(u)]D_k + \alpha_k^i(u)[D_k, B(u)] \\ &= [\alpha_k^i(u), B(u)]D_k + \alpha_k^i(u)B'(u)\{D_k u\}.\end{aligned}$$

The result follows now from b) and the estimate

$$\|B'(u)\{D_k u\}w\|_0 \le C\|w\|_1,$$

which is a simple consequence of (3.1). □

The next lemma will be crucial in the proof of the main estimate as it will provide coercivity for the normal component.

Lemma 4.3. *There are positive constants c, C such that*

$$(\Delta(u)^+(\gamma^{-1}w), B(u)w)_0 \le -c\|w\|^2_{-1/2} + C\|w\|^2_{-2}$$

for all $u \in \tilde{U}_s$, $w \in H^1(S)$.

Proof. Set $z := \Delta(u)^+(\gamma^{-1}w)$. Then $w = \gamma\Delta(u)z + \gamma Q(u)(\gamma^{-1}z)$, see (4.6). By Lemma 4.2, b),

$$\begin{aligned}I &:= \big(\Delta(u)^+(\gamma^{-1}w), B(u)w\big)_0 \le \big(z, B(u)\gamma\Delta(u)z\big)_0 + \|z\|_1\|B(u)(\gamma Q(u)(\gamma^{-1}z)\|_{-1} \\ &\le \big(\omega(u)\gamma\Delta(u)z, B(u)(\omega(u)^{-1}z)\big)_0 + C\|w\|^2_{-1}.\end{aligned}$$

Setting $\tilde{z} := \omega(u)^{-1}z$, $\tilde{\gamma} := \omega(u)^2\gamma$ and using (4.5) we get

$$\begin{aligned}I &\le \big(\tilde{\gamma}\Delta(u)\tilde{z}, B(u)\tilde{z}\big)_0 + \big(\omega(u)\gamma[\omega(u), \Delta(u)]\tilde{z}, B(u)\tilde{z}\big)_0 + C\|w\|^2_{-1} \\ &\le \big(\tilde{\gamma}\Lambda_i(u)\Lambda_i(u)\tilde{z}, B(u)\tilde{z}\big)_0 + C\|z\|_1^2 + C\|w\|^2_{-1}.\end{aligned}$$

By integration by parts, one obtains an estimate

$$\left|\int_S \Lambda_i(u)f\,dS\right| \le C\int_S |f|\,dS,$$

cf. [8], Eq. (5.5). This yields

$$\begin{aligned}I &\le -(\Lambda_i(u)\tilde{z}, \Lambda_i(u)\tilde{\gamma}B(u)\tilde{z})_0 + C\|z\|_1^2 + C\|w\|^2_{-1} \\ &\le -(\Lambda_i(u)\tilde{z}, \tilde{\gamma}\Lambda_i(u)B(u)\tilde{z})_0 + C\|z\|_1\sum_i \|[\Lambda_i(u), \gamma]B(u)\tilde{z}\|_0 + C\|w\|^2_{-1} \\ &\le -(\Lambda_i(u)\tilde{z}, \tilde{\gamma}B(u)\Lambda_i(u)\tilde{z})_0 + C\|z\|_1\sum_i \|[B(u), \Lambda_i(u)]\tilde{z}\|_0 + C\|w\|^2_{-1} \\ &\le -c\|\Lambda_i(u)\tilde{z}\|^2_{1/2} + C\|z\|^2_1 + C\|w\|^2_{-1} \\ &\le -c\|\tilde{z}\|^2_{3/2} + C\|w\|^2_{-1} \le -c\|w\|^2_{-1/2} + C\|w\|^2_{-2},\end{aligned}$$

where parts a) and e) of Lemma 4.2 have been used together with interpolation in the scale $H^t(S)$. □

In view of (4.23), (4.24) for every fixed $u \in U_s$, $s \geq s_0$, $s = 2k$, $k \in \mathbb{N}$ and λ sufficiently large (independent of $u \in \widetilde{U}_s$)

$$(v, w)_{s,u} := \lambda\big(M_0(u)v, M_0(u)w\big)_0 + \big(M(u)\Delta_0^k v, M(u)\Delta_0^k w\big)_0 \tag{4.26}$$

defines a scalar product on $H^s(S, \mathbb{R}^m)$ which is equivalent to the usual one.

The next two lemmas provide properties of the inner product $(\cdot,\cdot)_{s,u}$ which will be used when we apply the abstract existence result of Theorem 5.2 to our situation. They are parallel to Lemmas 5.3 and 5.4 in [8], therefore the proofs are omitted here. Note the uniformity of all estimates with respect to $u \in \tilde{U}_s$.

Lemma 4.4. *Assume $s \geq s_0$.*

(i) There exists a $C > 0$ such that for all $v \in H^{s+3}(S, \mathbb{R}^m)$, $w \in H^s(S, \mathbb{R})$, $u \in \tilde{U}_s$

$$(v, w)_{s,u} \leq C\|v\|_{s+3}\|w\|_{s-3}.$$

(ii) There exist λ_0, $c_0 > 0$ such that for all $v \in H^{s+6}(S, \mathbb{R}^m)$, $\lambda \geq \lambda_0$

$$\big(v, (-\Delta_0^3 + \lambda)v\big)_{s,u} \geq c_0\|v\|_{s+3}^2.$$

As an immediate consequence of Lemma 4.4 (i) we get the existence of a continuous bilinear form $\langle\cdot,\cdot\rangle_{s,u}$ on $H^{s+3}(S, \mathbb{R}^m) \times H^{s-3}(S, \mathbb{R}^m)$ compatible with $(\cdot,\cdot)_{s,u}$, i.e. there holds $\langle v, w\rangle_{s,u} = (v, w)_{s,u}$ for all $v, w \in H^{s+3}(S, \mathbb{R}^m)$. Further, we put for $\varepsilon \in (0, 1]$

$$\langle v, w\rangle_{s,u}^{\varepsilon} := \langle v, w\rangle_{s_0,u} + \varepsilon^2\langle v, w\rangle_{s,u}. \tag{4.27}$$

Lemma 4.5. *We assume as above $s \geq s_0$, $\varepsilon \in (0, 1]$.*

(i) For fixed $u \in U_s$, the mapping $\langle\cdot,\cdot\rangle_{s,u}^{\varepsilon} : H^{s+3}(S, \mathbb{R}^m) \times H^{s-3}(S, \mathbb{R}^m) \to \mathbb{R}$ constitutes a continuous, nondegenerate bilinear form whose restriction to $H^{s+3}(S, \mathbb{R}^m) \times H^{s+3}(S, \mathbb{R}^m)$ is symmetric.

(ii) With constants $C > 0$ independent of ε, u, v, w, one has for $u, w \in \tilde{U}_s$ and $v \in H^{s+3}(S, \mathbb{R}^m)$:

$$C^{-1}\big(\|v\|_{s_0}^2 + \varepsilon^2\|v\|_s^2\big) \leq \langle v, v\rangle_{s,u}^{\varepsilon} \leq C\big(\|v\|_{s_0}^2 + \varepsilon^2\|v\|_s^2\big), \tag{4.28}$$

$$\langle v, v\rangle_{s,u}^{\varepsilon} \leq \langle v, v\rangle_{s,w}^{\varepsilon}\big(1 + C\|u - w\|_{s_0-3}\big). \tag{4.29}$$

(iii) Weak convergences $u_n \rightharpoonup u$ in H^s, $w_n \rightharpoonup w$ in H^{s-3} imply

$$\langle v, w_n\rangle_{s,u_n}^{\varepsilon} \to \langle v, w\rangle_{s,u}^{\varepsilon}$$

for all $v \in H^{s+3}$.

Now we are prepared to formulate and prove the following a-priori estimates for $\mathscr{F}$ w.r. to the bilinear forms $(\cdot,\cdot)_{s,u}$.

Proposition 4.6. *Let $s \geq s_0$ be an even integer. Then*

$$\big(u, \mathscr{F}(u)\big)_{s,u} \leq C\|u\|_s^2 \tag{4.30}$$

for all $u \in \widetilde{U}_s \cap C^\infty(S, \mathbb{R}^m)$ with a constant C independent of u.

Proof. For later use, we prove the estimate in the following stronger form: For every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\big(u, \mathscr{F}(u)\big)_{s,u} \le C\|u\|_s\big((\varepsilon + \|\mathscr{F}(u)\|_{s_0-3})\|u\|_s + C(\varepsilon)\|u\|_{s-1}\big). \tag{4.31}$$

Setting $v := \Delta_0^k u$ and using the notations of Lemma 4.1 we have

$$\big(M(u)\Delta_0^k u, M(u)\Delta_0^k \mathscr{F}(u)\big)_0 = I(u)v^2 + J(u) + \big(M(u)v, M(u)R_b(u)\big)_0$$

with

$$I(u)v^2 := \big(M(u)v, M(u)\tilde{\mathscr{F}}(u)v\big)_0 \quad J(u) := \big(M(u)v, M(u)F(u)R_a(u)\big)_0$$

¿From (4.10) we obtain

$$\big(M(u)v, M(u)R_b(u)\big)_0 \le C\|u\|_s\big(\|\tilde{G}(u)v\|_{-1} + \|\mathscr{F}(u)\|_{s_0-3}\|u\|_s + \|u\|_{s-1}\big).$$

To estimate $J(u)$, we write this term as $J_1(u) + \ldots + J_4(u)$ with

$$\begin{aligned} J_1(u) &= \big(M_0(u)v, M_0(u)F(u)R_a(u)\big)_0, \quad J_2(u) = \big(\tilde{M}_0(u)v, M_0(u)F(u)R_a(u)\big)_0, \\ J_3(u) &= \big(M_0(u)v, \tilde{M}_0(u)F(u)R_a(u)\big)_0, \quad J_4(u) = \big(\tilde{M}_0(u)v, \tilde{M}_0(u)F(u)R_a(u)\big)_0. \end{aligned}$$

Using (4.25) and (4.9) we obtain for J_1

$$J_1(u) = \big((B(u)R_a(u), P(u)v\big)_0 \le C\|R_a(u)\|_0\|P(u)v\|_1 \le C\|u\|_{s-1}\|P(u)v\|_1$$

As

$$P(u)v = \Delta(u)^+\big(\gamma^{-1}(\tilde{G}(u)v - G_1(u)v)\big) - Q(u)(P(u)v) \tag{4.32}$$

we see from (4.8)

$$\|P(u)v\|_1 \le C\big(\|\tilde{G}(u)v\|_{-1} + \|v\|_0\big), \tag{4.33}$$

and consequently

$$J_1(u) \le C\|u\|_{s-1}\big(\|\tilde{G}(u)v\|_{-1} + \|u\|_s\big).$$

For J_2 we have

$$J_2(u) \le C\|\tilde{M}_0(u)v\|_1\|M_0(u)F(u)R_a(u)\|_{-1} \le C\|v\|_0\|R_a(u)\|_0 \le C\|u\|_s\|u\|_{s-1},$$

and the same estimates are valid for J_3, J_4, thus

$$J(u) \le C\|u\|_{s-1}\big(\|\tilde{G}(u)v\|_{-1} + \|u\|_s\big).$$

Further, we decompose I according to $I(u)v^2 = I_1(u)v^2 + \ldots + I_4(u)v^2$ in the same manner as J, i.e.

$$\begin{aligned} I_1(u)v^2 &= \big(M_0(u)v, M_0(u)\tilde{\mathscr{F}}(u)v\big)_0, \quad I_2(u)v^2 = \big(\tilde{M}_0(u)v, M_0(u)\tilde{\mathscr{F}}(u)v\big)_0, \\ I_3(u)v^2 &= \big(M_0(u)v, \tilde{M}_0(u)\tilde{\mathscr{F}}(u)v\big)_0, \quad I_4(u)v^2 = \big(\tilde{M}_0(u)v, \tilde{M}_0(u)\tilde{\mathscr{F}}(u)v\big)_0 \end{aligned}$$

and estimate each term separately. Using (4.25) again, the term I_1 may be written as

$$\begin{aligned} I_1(u)v^2 &= \big(M_0(u)v, M_0(u)F(u)\tilde{G}(u)v\big)_0 + \big(M_0(u)v, M_0(u)F_0(u)\tilde{G}(u)v\big)_0 \\ &= \big(P(u)v, B(u)\tilde{G}(u)v\big)_0 + \big(M_0(u)v, M_0(u)F_0(u)\tilde{G}(u)v\big)_0. \end{aligned}$$

In the first summand we insert (4.32) and use

$$\big|\big(Q(u)P(u)v, B(u)\tilde{G}(u)v\big)_0\big| \le C\|v\|_0\|\tilde{G}(u)v\|_{-1},$$

and by Lemma 4.3

$$\big(\Delta(u)^+(\gamma^{-1}\tilde G(u)v), B(u)\tilde G(u)v\big)_0 \le -c_0\|\tilde G(u)v\|^2_{-1/2} + C\|\tilde G(u)v\|^2_{-1}.$$

Remembering the definitions (4.20), (4.21) of M_1 and M_2, we have

$$\begin{aligned}\big(M_0(u)v, M_0(u)F_0(u)\tilde G(u)v\big)_0 &= -\big(M_0(u)v, M_1(u)B(u)\tilde G(u)v\big)_0,\\ \big(\Delta(u)^+(\gamma^{-1}G_1(u)v), B(u)\tilde G(u)v\big)_0 &= \big(M_2(u)v, B(u)\tilde G(u)v\big)_0,\end{aligned}$$

and consequently we arrive at

$$\begin{aligned} I_1(u)^2v \le &-\big(M_2(u)v, B(u)\tilde G(u)v\big)_0 - \big(M_0(u)v, M_1(u)B(u)\tilde G(u)v\big)_0\\ &- c_0\|\tilde G(u)v\|^2_{-1/2} + C\big(\|v\|_0 + \|\tilde G(u)v\|_{-1}\big)\|\tilde G(u)v\|_{-1}\end{aligned}\tag{4.34}$$

From Lemma 4.2, c) we get $F \in \mathscr{L}(H^{-1}(S), H^{-2}(S,\mathbb{R}^m))$ and therefore for I_4 we have the estimate

$$\begin{aligned} |I_4(u)v^2| &\le C\|\tilde M_0(u)v\|_1\|\tilde M_0(u)\tilde{\mathscr{F}}(u)v\|_{-1}\\ &\le C\|v\|_0\|(F(u)+F_0(u))\tilde G(u)v\|_{-2} \le C\|v\|_0\|\tilde G(u)v\|_{-1},\end{aligned}\tag{4.35}$$

where (4.8) has been applied again. Further, concerning I_3 we have

$$I_3(u)v^2 = \big(M_0(u)v, \tilde M_0 F(u)\tilde G(u)v\big)_0 + \big(M_0(u)v, \tilde M_0 F_0(u)\tilde G(u)v\big)_0$$

where the last summand allows the estimate

$$\big|\big(M_0(u)v, \tilde M_0(u)F_0(u)\tilde G(u)v\big)_0\big| \le C\|v\|_0\|\|\tilde G(u)v\|_{-1}.$$

Remembering $\tilde M_0(u) = M_1(u)P(u) + N(u)M_2(u)$, the first summand is written

$$\begin{aligned}&\big(M_0(u)v, \tilde M_0(u)F(u)\tilde G(u)v\big)_0\\ &\qquad = \big(M_0(u)v, M_1(u)B(u)\tilde G(u)v\big)_0 + \big(P(u)v, M_2(u)F(u)\tilde G(u)v\big)_0.\end{aligned}$$

Using (4.33) we get

$$\begin{aligned}\big|\big(P(u)v, M_2(u)F(u)\tilde G(u)v\big)_0\big| &\le C\|P(u)v\|_1\|\tilde G(u)v\|_{-1}\\ &\le C\big(\|v\|_0 + \|\tilde G(u)v\|_{-1}\big)\|\tilde G(u)v\|_{-1},\end{aligned}$$

and consequently

$$I_3(u)v^2 \le \big(M_0(u)v, M_1(u)B(u)\tilde G(u)v\big)_0 + C\big(\|v\|_0 + \|\tilde G(u)v\|_{-1}\big)\|\tilde G(u)v\|_{-1}.\tag{4.36}$$

Arguing along the same lines for I_2 we obtain

$$I_2(u)v^2 = \big(\tilde M_0(u)v, M_0(u)F(u)\tilde G(u)v\big)_0 + \big(\tilde M_0(u)v, M_0(u)F_0(u)\tilde G(u)v\big)_0,$$

where again

$$\big|\big(\tilde M_0(u)v, M_0F_0(u)\tilde G(u)v\big)_0\big| \le C\|v\|_0\|\tilde G(u)v\|_{-1}$$

and

$$\begin{aligned}&\big(\tilde M_0(u)v, M_0(u)F(u)\tilde G(u)v\big)_0\\ &\qquad = \big(M_2(u)v, B(u)\tilde G(u)v\big)_0 + \big(M_1(u)P(u)v, M_0(u)F(u)\tilde G(u)v\big)_0\end{aligned}$$

with

$$\begin{aligned}&\big|\big(M_1(u)P(u)v, M_0(u)F(u)\tilde{G}(u)v\big)_0\big|\\ &\qquad\le \|M_1(u)P(u)v\|_2\|M_0(u)F(u)\tilde{G}(u)v\|_{-2} \le C\|P(u)v\|_1\|\tilde{G}(u)v\|_{-1}\\ &\qquad\le C\big(\|v\|_0 + \|\tilde{G}(u)v\|_{-1}\big)\|\tilde{G}(u)v\|_{-1}.\end{aligned}$$

Thus we have

$$I_2(u)v^2 \le \big(M_2(u)v, B(u)\tilde{G}(u)v\big)_0 + C\big(\|v\|_0 + \|\tilde{G}(u)v\|_{-1}\big)\|\tilde{G}(u)v\|_{-1}. \tag{4.37}$$

Summarizing, we get

$$\begin{aligned}(u, \mathscr{F}(u))_{s,u} \le\; &-c_1\|\tilde{G}(u)v\|_{-1/2}^2 + C_1\big(\|\tilde{G}(u)v\|_{-1}^2 + \|u\|_s(\|\tilde{G}(u)v\|_{-1}\\ &+\|\mathscr{F}(u)\|_{s_0-3}\|u\|_s + \|u\|_{s-1})\big)\end{aligned}$$

and, estimating further

$$\begin{aligned}\|u\|_s\|\tilde{G}(u)v\|_{-1} &\le \frac{\varepsilon}{C_1}\|u\|_s^2 + C_2(\varepsilon)\|\tilde{G}(u)v\|_{-1}^2,\\ \|\tilde{G}(u)v\|_{-1}^2 &\le \frac{c_1}{C_1(1+C_2(\varepsilon))}\|\tilde{G}(u)v\|_{-1/2}^2 + C(\varepsilon)\|\tilde{G}(u)v\|_{-3}^2,\\ \|\tilde{G}(u)v\|_{-3}^2 &\le C\|u\|_{s-1}^2,\end{aligned}$$

we obtain (4.31). □

Remark 4.7. Reinspecting the estimates in the previous proofs it is straightforward to check that for fixed $s \ge s_0$ the occurring constants, in particular in (4.30), (4.31), are independent of γ as long as γ varies in some fixed set

$$\big\{\gamma \in C^\infty(S) \mid \gamma \ge \gamma^* > 0, \|\gamma\|_{s_1} \le M\big\} \tag{4.38}$$

with some sufficiently large $s_1 = s_1(s)$.

Using Lemma 3.6, we write for $u \in \tilde{U}_s$, $v \in H^s$, $s \ge s_0$

$$\mathscr{F}'(u)v = F(u)\big(\gamma\Delta(u)(P(u)v) + G_2(u)v\big) + R(u)v \tag{4.39}$$

where $v \mapsto G_2(u)v := \mathscr{G}_3(u)v - \mathscr{F}(u)\cdot v$ and $v \mapsto R(u)v$ are operators of order one and zero, respectively. Note that $\mathscr{F}'(u)$ coincides with $\tilde{\mathscr{F}}(u)$ if G_1 is replaced by G_2 and F_0 is replaced by 0. Hence, defining (cf. (4.18)-(4.21))

$$M_3(u) := \Delta(u)^+(\gamma^{-1}G_2(u)v), \quad \tilde{M}(u) := M_0(u) + M_3(u),$$

we find that $\tilde{M}$ has the same properties as M above, and we obtain, parallel to (4.31), the following estimate which will be used in the uniqueness proof:

Lemma 4.8. *Let $s \ge s_0$. Then there exists a constant C such that for all $u \in \tilde{U}_s$, $v \in H^s$ we have*

$$\big(\tilde{M}(u)\mathscr{F}'(u)v, \tilde{M}(u)v\big)_0 \le C\|v\|_0^2.$$

5. Proof of short time existence and uniqueness

We are ready now to prove our main results as announced in Theorems 2.1 and 2.2. As the existence proof is in some respects analogous to the corresponding considerations in [8], we restrict ourselves to an outline and refer to that paper for the details.

Fix an even integer $s_0 > (m+7)/2$, $s_0 \geq 6$ and let $s \geq s_0$ be an even integer as well. Let $\tilde{U}_s$ be defined as above. The notations $C_w([0,T],X)$ and $C^1_w([0,T],X)$ will denote the spaces of weakly continuous and weakly continuously differentiable functions, respectively, with values in some subset X of a normed space.

At first an estimate which provides uniqueness and Lipschitz continuous dependence on the initial value in the L^2-norm is given:

Proposition 5.1. *Let $u, v \in C_w([0,T], \tilde{U}_{s_0}) \cap C^1_w([0,T], H^{s_0-3}(S,\mathbb{R}^m))$ be two solutions of (2.2). Then*

$$\|v(t) - u(t)\|_0 \leq C\|v(0) - u(0)\|_0 \tag{5.1}$$

with C depending only on $\tilde{U}_{s_0}$ and on T.

Proof. Set $w(t) := v(t) - u(t)$ and note that

$$w(t) \in C([0,T], H^\sigma(S,\mathbb{R}^m)) \cap C^1([0,T], H^{\sigma-3}(S,\mathbb{R}^m))$$

for $\sigma < s_0$. In particular, the map

$$t \mapsto g(t) := \|\tilde{M}(u(t))w(t)\|_0^2$$

is differentiable and has the derivative

$$\begin{aligned} g'(t) \;=\; & 2\big(\tilde{M}'(u(t))\{u'(t)\}w(t), \tilde{M}(u(t))w(t)\big)_0 \\ & +2\big(\tilde{M}(u(t))(\mathscr{F}(v(t)) - \mathscr{F}(u(t)), \tilde{M}(u(t))w(t)\big)_0. \end{aligned}$$

To estimate the first term we note that, parallel to the estimates in Lemma 3.1,

$$\begin{aligned} \|\tilde{M}'(u(t))\{u'(t)\}w(t)\|_0 \;\leq\; & C\|u'(t)\|_{s_0-3}\|w(t)\|_0 \\ \leq\; & C\|\mathscr{F}(u(t))\|_{s_0-3}\|w(t)\|_0 \leq C\|w(t)\|_0. \end{aligned}$$

The second term can be estimated by using

$$\mathscr{F}(v(t)) - \mathscr{F}(u(t)) = \mathscr{F}'(u(t))w(t) + R,$$

where

$$R := \int_0^1 \int_0^\tau \mathscr{F}''(\theta v(t) + (1-\theta)u(t))\{w(t), w(t)\}\, d\theta d\tau$$

allows an estimate

$$\|R\|_0 \leq C\|w(t)\|_3\|w(t)\|_{s_0-3} \leq C\|w(t)\|_{s_0}\|w(t)\|_0 \leq C\|w(t)\|_0,$$

where estimates on $\mathscr{F}''$ parallel to Lemma 3.1 and norm convexity have been used. Thus, by Lemma 4.8,

$$\begin{aligned} g'(t) \;\leq\; & 2\big(\tilde{M}(u(t))\mathscr{F}'(u(t))w(t), \tilde{M}(u(t))w(t)\big)_0 + C\|w(t)\|_0^2 \\ \leq\; & \|w(t)\|_0^2 \leq Cg(t). \end{aligned}$$

Therefore, by Gronwall's inequality,

$$\|v(t) - u(t)\|_0^2 \leq Cg(t) \leq Cg(0) \leq C\|v(0) - u(0)\|_0^2.$$

□

To prove existence, we will rely on an abstract existence theorem whose proof has been given in [8]. It generalizes an existence theorem concerning evolution equations with semibounded operators by Kato and Lai [9] to the case of variable bilinear forms. The setting is the following:

$$\left.\begin{array}{l}
\text{Let } X \subseteq Y \subseteq Z \text{ be real, separable Banach spaces with dense and continuous embeddings and } \mathscr{U} \subseteq Y \text{ open. For every } u \in \mathscr{U} \text{ let } \langle\cdot,\cdot\rangle_u : X \times Z \to \mathbb{R} \text{ be a continuous and nondegenerate bilinear form, such that with fixed constants } C \geq 1,\ M \geq 0\text{:}\\
\text{(H1)}\quad \langle v, w\rangle_u = \langle w, v\rangle_u \text{ for all } v, w \in X;\\
\text{(H2)}\quad C^{-1}\|v\|_Y^2 \leq \langle v, v\rangle_u \leq C\|v\|_Y^2 \text{ for all } v \in X,\ u \in \mathscr{U};\\
\text{(H3)}\quad \langle v, v\rangle_u \leq \langle v, v\rangle_w \big(1 + M\|u - w\|_Z\big) \text{ for all } v \in X,\ u, w \in \mathscr{U};\\
\text{(H4)}\quad \text{weak convergences } u_n \rightharpoonup u \text{ in } Y,\ u_n, u \in \mathscr{U}, \text{ and } w_n \rightharpoonup w \text{ in } Z \text{ imply } \langle v, w_n\rangle_{u_n} \to \langle v, w\rangle_u \text{ for all } v \in X.
\end{array}\right\} \quad \text{(H)}$$

Assuming (H) to hold, by the dense embedding $X \subseteq Y$ and

$$\big|\langle v, w\rangle_u\big|^2 \leq \langle v, v\rangle_u \langle w, w\rangle_u \leq C^2\|v\|_Y^2\|w\|_Y^2 \ \text{ for } \ v, w \in X$$

there exists to each $u \in \mathscr{U}$ a scalar product $(\cdot,\cdot)_u$ on Y, which is compatible with $\langle\cdot,\cdot\rangle_u$, i.e. we have

$$(v, w)_u = \langle v, w\rangle_u \ \text{ for } \ v \in X, w \in Y.$$

Moreover, for $u_n, u \in \mathscr{U}$, $u_n \rightharpoonup u$, $w_n \rightharpoonup w$ in Y implies

$$(v, w_n)_{u_n} \to (v, w)_u \ \text{ for all } \ v \in X.$$

For the sake of brevity we put

$$\|v\|_u = (v, v)_u^{1/2}, \quad |||u||| = (u, u)_u^{1/2}\ .$$

Theorem 5.2. *Assume* (H) *is satisfied with some ball*

$$\mathscr{U} = B := \big\{u \in Y \,\big|\, \|u\|_Y < R\big\}, \quad R > 0,$$

and $\mathscr{F} : B \to Z$ *is a weakly sequentially continuous mapping such that*

$$2\langle u, \mathscr{F}(u)\rangle_u + M\,\|\mathscr{F}(u)\,\|_Z|||u||| \leq \beta\big(|||u|||^2\big) \ \textit{for all} \ u \in X \cap B \tag{5.2}$$

with a C^1*-function* $\beta : \mathbb{R}_+ \to \mathbb{R}_+ = [0, \infty)$*. Let* $u_0 \in B$*,*

$$|||u_0||| < r := R/(2C^3)^{1/2},$$

and $T > 0$ *such that the solution* ρ *of the scalar Cauchy problem*

$$d\rho/dt = \beta\big(\rho(t)\big), \quad \rho(0) = |||u_0|||^2 \tag{5.3}$$

exists on $[0,T]$ *and satisfies* $\rho(t) < r^2$ *there. Then the Cauchy problem*

$$u'(t) = \mathscr{F}(u(t)), \qquad u(0) = u_0, \tag{5.4}$$

possesses a solution $u \in C_w([0,T], \mathscr{U}) \cap C_w^1([0,T], Z)$ *for which additionally*

$$\begin{aligned} &|||u(t)|||^2 \le \rho(t) \ \text{ for all } \ t \in [0,T], \\ &u(t) \to u_0 \ \text{ in } \ Y \ \text{ for } \ t \to +0. \end{aligned}$$

Proof of Theorems 2.1, 2.2 (outline): Instead of (2.2), (2.3) we consider the Cauchy problem

$$\left.\begin{aligned} v'(t) &= \hat{\mathscr{F}}(v) := \mathscr{F}(v + w_0), \\ v(0) &= u_0 - w_0, \end{aligned}\right\} \tag{5.5}$$

where w_0 is smooth and near u_0.

To apply Theorem 5.2, we set for $\varepsilon \in (0,1]$

$$\begin{aligned} X &:= H^{s+3}(S,\mathbb{R}^m),\ \|\cdot\|_X := \|\cdot\|_{s_0+3} + \varepsilon\|\cdot\|_{s+3}, \\ Y &:= H^{s}(S,\mathbb{R}^m),\quad\ \ \|\cdot\|_X := \|\cdot\|_{s_0} + \varepsilon\|\cdot\|_{s}, \\ Z &:= H^{s-3}(S,\mathbb{R}^m),\ \|\cdot\|_X := \|\cdot\|_{s_0-3} + \varepsilon\|\cdot\|_{s-3}. \end{aligned}$$

For $u \in \tilde{U}_{s_0}$, let $\langle\cdot,\cdot\rangle_u^\varepsilon$ be the bilinear form compatible to the inner product on Y given by

$$(v,w)_u^\varepsilon := (v,w)_{s_0,u} + \varepsilon^2 (v,w)_{s,u}$$

with $(v,w)_{s_0,u}$, $(v,w)_{s,u}$ given by (4.26). Lemma 4.5 ensures that this bilinear form satisfies the assumptions (H), with constants independent of ε. Thus Theorem 5.2 yields existence of a solution

$$u \in C_w([0,T], \tilde{U}_s) \cap C_w^1([0,T], H^{s-3}(S,\mathbb{R}^m))$$

and an estimate

$$\|u(t)\|_s \le C(1 + \|u_0\|_s) \tag{5.6}$$

with C independent of u_0 and t.

The uniqueness result from Proposition 5.1 enables us to define an evolution operator T_t by setting $T_t u_0 := u(t)$. By a nonlinear interpolation result given in [2], Proposition A.1 and Remark A.2, the estimates (5.1) and (5.6) imply H^τ-continuity of T_t for $\tau \in [0,s)$, uniformly in $t \in [0,T]$. Approximation of the initial value u_0 by $u_0^n \in H^{s+1}$ and of the solution u by the corresponding solutions $u_0^n \in C([0,T], \tilde{U}_s) \cap C^1([0,T], H^{s-3}(S,\mathbb{R}^m))$ yields then

$$u \in C([0,T], \tilde{U}_s) \cap C^1([0,T], H^{s-3}(S,\mathbb{R}^m))$$

by uniform convergence. Finally, the existence time T can be shown to be independent of s by standard continuation arguments. For further details we refer to [8]. □

6. Nontrivial equilibria and long-time existence

In this section we will investigate the existence of equilibrium points and the long-time dynamic of the evolution problem (2.2). Our considerations are restricted to situations near trivial equilibria; i.e. we will assume that the domain is near a ball and γ is near a constant. Therefore in the following we specialize the reference domain to

$$E := \{x \in \mathbb{R}^m \mid |x| < 1\}, \quad S := \partial E = \{x \in \mathbb{R}^m \mid |x| = 1\}.$$

First we show that for any given domain Ω_u near a ball there exists a corresponding γ such that (u, γ) yields an equilibrium point for (2.2). Of course, the opposite question is more interesting for our evolution: Given a surface energy density, find a corresponding class of equilibrium shapes and for given initial shape, show global existence of the solution in time and convergence to some member of this class. Our proof of this is organized as follows. Using the refined semiboundedness estimate of Proposition 4.6 we obtain weak exponential growth of a solution in higher Sobolev norms provided the solution remains near the trivial equilibrium with respect to some lower norms. This enables us to show that the scalar function $f(t)$, which is defined by (6.11) below, controls the evolution. Then a simple dicussion of the spectral properties of the evolution equation for f yields global existence.

We start by stating some simple integral identities needed later on; in particular, assertion (ii) of following lemma together with volume conservation implies that the center of gravity remains fixed during evolutions under consideration.

Lemma 6.1. *(i) We have*

$$\int_S \omega(u) N_i(u) u_j \, dS = \delta_{ij} |\Omega_u|, \tag{6.1}$$

where δ_{ij} denotes the Kronecker symbol, and

$$\int_S \omega(u) N(u) \big(\mathscr{G}(u)\gamma\big) \, dS = 0. \tag{6.2}$$

(ii) For any solution $u = u(t)$ of (2.2) the vector of first moments

$$M(t) := \int_{\Omega_{u(t)}} x \, dx \tag{6.3}$$

is independent of t.

Proof. (i) After retransformation onto Γ_u with outer normal n, the equation (6.1) follows from

$$\int_{\Gamma_u} n_i x_j \, d\Gamma_u = \int_{\Omega_u} \partial_i x_j \, dx = \delta_{ij} |\Omega_u|,$$

whereas (6.2) reads

$$\int_{\Gamma_u} n(\gamma\kappa - \psi) \, d\Gamma_u = 0,$$

where ψ is harmonic in Ω_u with Neumann boundary condition $\partial_n \psi = \Delta_{\Gamma_u} \gamma$ on Γ_u. By Green's formula we get

$$\int_{\Gamma_u} n\psi \, d\Gamma_u = \int_{\Gamma_u} x \partial_n \psi \, d\Gamma_u,$$

hence writing $n\kappa = \Delta_{\Gamma_u} x$ on Γ_u we obtain

$$\int_{\Gamma_u} n(\gamma\kappa - \psi) \, d\Gamma_u = \int_{\Gamma_u} \big(\gamma \Delta_{\Gamma_u} x - x \Delta_{\Gamma_u} \gamma\big) \, d\Gamma_u = 0 \tag{6.4}$$

by an integration by parts.
(ii) Consider the solution to (1.1) corresponding to u. We have, using Green's formula and (6.4),

$$\begin{aligned}\dot{M}(t) &= \int_{\Gamma(t)} x V_n \, d\Gamma(t) = \int_{\Gamma(t)} x \, \partial_n \phi \, d\Gamma(t)\\ &= \int_{\Gamma(t)} \partial_n x \, \phi \, d\Gamma(t) = \int_{\Gamma(t)} n(\gamma\kappa - \psi) \, d\Gamma(t) = 0,\end{aligned}$$

which is the assertion. □

Remark 6.2. Note that the presence of the correction term G is crucial not only for the generalized gradient flow property but also for the validity of above Lemma.

In further considerations we assume $s \geq s_0$ with $s_0 \in \mathbb{N}$ fixed as in Section 4, set

$$\tilde{U}_s := \big\{ u \in H^s(S, \mathbb{R}^m) \bigm| \|u - w_0\|_{s_0} \leq \delta_0 \big\}, \quad w_0(x) := x \;\text{ for }\; x \in S \tag{6.5}$$

and assume $\delta_0 > 0$ sufficiently small, whenever necessary. Moreover, to stress the dependency on γ, we consider now $\mathscr{F}(u)$ and $\mathscr{G}(u)$ as linear operators defined by

$$\mathscr{F}(u)v := F(u)\big(\mathscr{G}(u)v\big), \quad \mathscr{G}(u)v := -v\Lambda_i(u)\big(n_i(u) \circ u\big) - A(u)\Delta(u)v; \tag{6.6}$$

for $s \geq s_0$ und $2 \leq t \leq s$ the operators

$$\mathscr{G}(u) \in \mathscr{L}\big(H^t(S), H^{t-1}(S)\big), \quad \mathscr{F}(u) \in \mathscr{L}\big(H^t(S), H^{t-2}(S)\big) \tag{6.7}$$

depend smoothly on $u \in \tilde{U}_s$. For a given surface energy density $\gamma \in C^\infty(S)$ and $u_0 \in \tilde{U}_s$ the Cauchy problem (2.2), (2.3) reappears as

$$\dot{u} = \mathscr{F}(u)\gamma, \quad u(0) = u_0. \tag{6.8}$$

We call a function γ on S an *equilibrium surface energy density* for a given $u \in \tilde{U}_s$ iff

$$\mathscr{F}(u)\gamma = 0 \;\text{ on }\; S \quad \text{or equivalently} \quad \mathscr{G}(u)\gamma = const. \;\text{ on }\; S. \tag{6.9}$$

The latter condition yields a nonlocal first-order elliptic equation for the determination of an equilibrium surface energy density γ. By straightforward perturbation arguments and expansion into spherical harmonics in case of $u = w_0$, the next lemma ensures the existence of a solution of this equation, uniquely determined up to a scaling factor and a linear combination of m functions.

Lemma 6.3. *Assume $\delta_0 > 0$ sufficiently small. Then for any given $u \in \tilde{U}_s$, $s \geq s_0$ there exists a uniquely determined positive function $\gamma(u) \in H^{s-1}(S)$ such that*

$$\mathscr{G}(u)\gamma(u) = -1 \;\text{ on }\; S, \qquad \int_S N(u)\gamma(u) \, dS = 0.$$

Proof. By elliptic regularity it suffices to consider the case $s = s_0$. If $a \in \mathbb{R}^m$, $u \in \tilde{U}_s$ and $\gamma \in H^{s-1}(S)$ such that

$$\mathscr{G}(u)\gamma = -1 + a \cdot u \;\text{ on }\; S,$$

then (6.2) implies

$$\int_S \omega(u) N(u)(a \cdot u) \, dS = 0,$$

and further $a_1 = \ldots = a_m = 0$ by (6.1). Hence it suffices to show the invertibility of the operator

$$[(\gamma, a) \mapsto L(u)(\gamma, a)] \in \mathscr{L}\big(H^{s-1}(S) \times \mathbb{R}^m, H^{s-2}(S) \times \mathbb{R}^m\big)$$

given by

$$L(u)(\gamma, a) := \big(\mathscr{G}(u)\gamma - a \cdot u, c\big), \quad c := \int_S N(u)\gamma \, dS.$$

As $L(u)$ depends smoothly on $u \in \tilde{U}_s$ it remains to show the existence of

$$L(w_0)^{-1} \in \mathscr{L}\big(H^{s-2}(S) \times \mathbb{R}^m, H^{s-1}(S) \times \mathbb{R}^m\big). \tag{6.10}$$

In this case we have

$$\mathscr{G}(w_0)\gamma = -(m-1)\gamma - A_S \Delta_S \gamma,$$

where A_S and Δ_S denote the Neumann-Dirichlet operator and the Laplace-Beltrami operator on the unit sphere S, respectively. Hence, if we expand

$$\gamma = \sum_{l=0}^{\infty} \gamma_l, \quad \gamma_1 = b \cdot x,$$

where γ_l is a spherical harmonic of degree l and $b \in \mathbb{R}^m$, it follows from

$$\Delta_S \gamma_l = -l(l+m-2)\gamma_l, \qquad A_S \gamma_l = l^{-1}\gamma_l \quad (l > 0)$$

that

$$L(w_0)(\gamma, a) = \Big(-(m-1)\gamma_0 - a \cdot x + \sum_{l=2}^{\infty} (l-1)\gamma_l, \frac{|S|}{m} b\Big).$$

This gives immediately (6.10). Clearly, the equilibrium surface energy density belonging to w_0 is the constant function

$$\overline{\gamma} := \gamma(w_0) = L(w_0)^{-1}(-1, 0) = -1/\kappa_0 \text{ on } S$$

with the curvature $\kappa_0 = -(m-1)$ on S. The proof is complete. $\square$

In the following considerations, to derive a-priori estimates independent of the existence interval $[0, T]$, let

$$u \in C^0\big([0, T], \tilde{U}_{s_0+4}\big) \cap C^1\big([0, T], H^{s_0+1}(S, \mathbb{R}^m)\big)$$

be any solution of (6.8). Thereby, without explicit mentioning, we always assume that

$$\|u_0\|_{s_0+4} = \|u(0)\|_{s_0+4} \le M$$

and γ is taken from some set of the form (4.38) with fixed positive constants γ^*, M and with a sufficiently large $s_1 \in \mathbb{N}$, such that, in view of Remark 4.7, the constants in the estimates of Section 4 are independent of γ for $s \le s_0 + 4$. Further we define (cf. (4.12))

$$f(t) := \tilde{\mathscr{G}}\big(u(t)\big)\gamma = \mathscr{G}\big(u(t)\big)\gamma - \frac{1}{|S|} \int_S \mathscr{G}\big(u(t)\big)\gamma \, dS. \tag{6.11}$$

Note that $F\big(u(t)\big)v \equiv 0$ for $v \equiv const$ implies

$$\dot{u}(t) = \mathscr{F}\big(u(t)\big)\gamma = F\big(u(t)\big)f(t). \tag{6.12}$$

The following Lemmas 6.4, 6.5 show in which sense the evolution of u can be controlled by $f(t)$, $t \in [0,T]$ with constants independent of the existence time T. As main step, we find from the improved semi-boundedness estimate (4.31) that $\|u(t)\|$ has only slow exponential growth, more precisely we have

Lemma 6.4. *Let $\varepsilon > 0$, $a > 0$, $c > 0$ be given. There are constants $C = C(a,c,\varepsilon)$, $\delta = \delta(\varepsilon) > 0$ such that each of the assumptions*

(i) $\|f(t)\|_{s_0-2} \le \delta$, $t \in [0,T]$, *or*

(ii) $\|f(t)\|_{s_0-2} \le ce^{-at}$, $t \in [0,T]$,

implies

$$\|u(t)\|_{s_0+4} \le Ce^{\varepsilon t} \text{ for all } t \in [0,T]. \tag{6.13}$$

The constants C and δ may be chosen independently of u and T.

Proof. In view of Theorem 2.2 it is sufficient to prove (6.13) for any sufficiently regular solution $u = u(t)$ of (6.8). In particular, the mapping $t \mapsto g(t) := \big(u(t), u(t)\big)_{s,u(t)}$ with $s := s_0 + 4$ may be assumed to be differentiable. From Proposition 4.6, estimate (4.31) we get for any given $\varepsilon > 0$

$$\big(u(t), \mathscr{F}(u(t))\gamma\big)_{s,u(t)} \le \Big(C\|\mathscr{F}(u(t))\gamma\|_{s_0-3} + \frac{\varepsilon}{2}\Big)\, g(t) + C(\varepsilon)$$

and, using $D(u)\{w\}v^2$ as abbreviation for the derivative of the mapping $u \mapsto (v,v)_{s,u}$,

$$\big|D(u(t))\{\mathscr{F}(u(t))\gamma\}u(t)^2\big| \le C\|\mathscr{F}(u(t))\gamma\|_{s_0-3}\|u(t)\|_s^2.$$

Consequently, by differentiating g, we have

$$\begin{aligned} g'(t) &= 2\big(u(t), \dot u(t)\big)_{s,u(t)} + D\big(u(t)\big)\big\{\dot u(t)\big\}u(t)^2 \\ &\le \Big(C\|\mathscr{F}(u(t))\gamma\|_{s_0-3} + \frac{\varepsilon}{2}\Big)\, g(t) + C(\varepsilon) \le \Big(C_1\|f(t)\|_{s_0-2} + \frac{\varepsilon}{2}\Big)\, g(t) + C(\varepsilon). \end{aligned}$$

Defining

$$\alpha(t) := \frac{\varepsilon}{2}t + C_1\int_0^t \|f(s)\|_{s_0-2}\, ds$$

and noting that under the assumptions (i) or (ii) we have

$$0 \le \alpha(t) \le \varepsilon t + C(a,c)$$

we get from Gronwall's inequality

$$\begin{aligned} \|u(t)\|_s^2 \le Cg(t) &\le Ce^{\alpha t}\left(g(0) + C(\varepsilon)\int_0^t e^{-\alpha(s)}\, ds\right) \\ &\le C(\varepsilon)e^{\varepsilon t}(g(0)+t) \le C(\varepsilon)e^{2\varepsilon t}(g(0)+1) \le C(\varepsilon)(M^2+1)e^{2\varepsilon t}. \end{aligned}$$

This implies the assertion. ∎

Lemma 6.5. *Let $\varepsilon > 0$ and $a > 0$ be given. Then there exists $\delta > 0$ such that*

$$\|u(0) - w_0\|_{s_0} \le \delta, \quad \|f(t)\|_{s_0+1} \le \delta e^{-at} \text{ for all } t \in [0,T] \tag{6.14}$$

imply

$$\|u(t) - w_0\|_{s_0} \le \varepsilon \text{ for all } t \in [0,T].$$

The constant δ may be chosen independently of T and u.

Proof. We assume according to Lemma 6.4

$$\|u(t)\|_{s_0+1} \le Ce^{at/2} \quad \text{for} \quad t \in [0,T].$$

Define $g(t) := \|u(t) - w_0\|_{s_0}^2$. Then

$$g'(t) = 2\big(\mathscr{F}(u(t))\gamma, u(t) - w_0\big)_{s_0} \le C\|\mathscr{F}(u(t))\gamma\|_{s_0} = C\|F(u(t))f(t)\|_{s_0},$$

and consequently,

$$g'(t) \le C\|u(t)\|_{s_0+1}\|f(t)\|_{s_0+1} \le C'\delta e^{at/2}e^{-at}$$

Hence, for δ sufficiently small,

$$g'(t) \le \tfrac{1}{4}a\varepsilon^2 e^{-at/2} \quad \text{and} \quad g(0) \le \tfrac{1}{2}\varepsilon^2.$$

This implies

$$g(t) \le \tfrac{1}{2}\varepsilon^2(2 - e^{-at/2}) \le \varepsilon^2,$$

which is the assertion. □

Now, to obtain estimates of $f(t)$ we derive the evolution equation satisfied by f. Differentiation of (6.11) with respect to t gives

$$\dot f(t) = \mathscr{G}\big(u(t)\big)\big\{\dot u(t)\big\}\gamma - \frac{1}{|S|}\int_S \mathscr{G}\big(u(t)\big)\big\{\dot u(t)\big\}\gamma\, dS,$$

hence inserting (6.12) we obtain

$$\dot f(t) = \mathscr{H}\big(u(t),\gamma\big)f(t) \tag{6.15}$$

where the operator $\mathscr{H}$ is given by

$$\mathscr{H}(u,\gamma)w := \mathscr{G}'(u)\{F(u)w\}\gamma - \frac{1}{|S|}\int_S \mathscr{G}'(u)\{F(u)w\}\gamma\, dS. \tag{6.16}$$

The operator $\mathscr{H}$ is negative semi-bounded in L^2 in the following sense:

Lemma 6.6. *For $\|u - w_0\|_{s_0}$ and $\|\gamma - \overline{\gamma}\|_{s_0}$ sufficiently small we have with a positive constant c independent of u and γ:*

$$\big(\mathscr{H}(u,\gamma)w, w\big)_0 \le -c\|w\|_{3/2}^2 \tag{6.17}$$

for all $w \in C^\infty(S)$ with

$$\int_S w\, dS = 0, \quad \int_S \omega(u)N_i(u)w\, dS = 0.$$

Proof. Instead of (6.17) we prove the estimate in the form

$$\big(\mathscr{H}(u,\gamma)w, w\big)_0 \le -c_1\|w\|_{3/2}^2 + c_2 R(u,w) \tag{6.18}$$

for *all* $w \in C^\infty(S)$ with some constants $c_1, c_2 > 0$, where

$$R(u,w) := \left(\int_S w\, dS\right)^2 + \sum_{i=1}^m \left(\int_S \omega(u)N_i(u)w\, dS\right)^2.$$

Further, by perturbation arguments using

$$\|\mathscr{H}(u,\gamma)w - \mathscr{H}(w_0,\bar{\gamma})w\|_{-3/2} \le C\|w\|_{3/2}\big(\|u-w_0\|_{s_0} + \|\gamma-\overline{\gamma}\|_{s_0}\big),$$

it suffices to show (6.18) for $u = w_0$, $\gamma = \bar{\gamma}$. For $\gamma = \bar{\gamma}$ we have

$$\mathscr{G}(u)\overline{\gamma} = \overline{\gamma}H(u),$$

therefore the linearization of the mean curvature at a sphere yields

$$\mathscr{G}'(w_0)\{v\}\overline{\gamma} = \overline{\gamma}\big((m-1)(x\cdot v) + \Delta_S(x\cdot v)\big),$$

As $[w \mapsto x\cdot F(w_0)w] = B_S$ is the Dirichlet-Neumann operator on the unit sphere S, this implies (6.18). $\square$

Lemma 6.7. *There exists $a > 0$ with the property that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|u(0) - w_0\|_{s_0} \le \delta$ and $\|\gamma - \bar{\gamma}\|_{s_0} \le \delta$ imply*

$$\|f(t)\|_{s_0+1} \le \varepsilon e^{-at} \ \textit{ for all } \ t \in [0,T].$$

δ may be chosen independently of u and $T > 0$.

Proof. First note, that by definition of $f(t)$ and Lemma 6.1, (i) we have

$$\int_S f(t)\,dS = 0, \quad \int_S \omega\big(u(t)\big)N_i\big(u(t)\big)f(t)\,dS = 0.$$

Consequently (6.15) and Lemma 6.6 imply

$$\frac{d}{dt}\big(\|f(t)\|_0^2\big) = 2\big(\mathscr{H}(u(t),\gamma)f(t),f(t)\big)_0 \le -c\|f(t)\|_0^2,$$

with some $c > 0$, hence

$$\|f(t)\|_0 \le e^{-ct}\|f(0)\|_0.$$

Further, as

$$\|f(t)\|_{s_0-2} \le C\big(\|u(t)-w_0\|_{s_0} + \|\gamma-\bar{\gamma}\|_{s_0}\big) \le C(\delta_0+\delta),$$

we get from Lemma 6.4 (i) by assuming δ and the constant δ_0 in the definition (6.5) of $\tilde{U}_s$ sufficiently small,

$$\|u(t)\|_{s_0+4} \le Ce^{\mu t}, \quad \mu := c/(2(s_0+1))$$

and moreover, using the estimate (2.14),

$$\|f(t)\|_{s_0+2} = \|\mathscr{G}(u(t))\gamma\|_{s_0+2} \le C\|u(t)\|_{s_0+4} \le C'e^{\mu t}.$$

Now we have by interpolation

$$\|f(t)\|_{s_0+1} \le C\big(e^{\mu t}\big)^{\frac{s_0+1}{s_0+2}}\big(e^{-ct}\|f(0)\|_0\big)^{\frac{1}{s_0+2}} = C\|f(0)\|_0^{\frac{1}{s_0+2}}e^{-at}$$

with $a = c/(2(s_0+2))$. This implies the assertion. $\square$

Now we are in position to formulate our main result about the long-time existence and convergence to an equilibrium configuration for $t \to \infty$.

Theorem 6.8. *Let $M > 0$ be given. Then there exists an $\varepsilon > 0$ such that for $\gamma \in C^\infty(S)$ with $\|\gamma\|_{s_1} \le M$, $\|\gamma - \bar\gamma\|_{s_0} \le \varepsilon$ and for any initial value $u_0 \in H^{s_0+4}$ with $\|u_0\|_{s_0+4} \le M$, $\|u_0 - w_0\|_{s_0} \le \varepsilon$ the solution of the Cauchy problem* (6.8) *exists for all $t > 0$. Moreover, $u(t)$ converges exponentially to some $u^\star = u^\star(u_0, \gamma)$ in H^s, $s < s_0 + 4$ for $t \to \infty$, i.e.*

$$\|u(t) - u^\star\|_s \le Ce^{-at} \tag{6.19}$$

with suitable $C, a > 0$ (depending on $s < s_0 + 4$). Finally we have $\mathscr{G}(u^\star)\gamma = const.$ on S, i.e. γ is an equilibrium surface energy density for $u^\star$.

Proof. First, choose $\delta \in (0, \delta_0)$ and $T > 0$, such that, by our local existence theorems, initial values $u_0 \in H^{s_0+4}$ with $\|u_0 - w_0\|_{s_0} \le \delta$ guarantees the (unique) solvability of (6.8) on the time interval $[0, T]$ with $u(t) \in \tilde U_{s_0+4}$, $t \in [0, T]$. Then, for $\varepsilon > 0$ sufficiently small, Lemmas 6.5, 6.7 ensure $\|u(T) - w_0\|_{s_0} \le \delta$, hence the solution can be continued to the interval $[T, 2T]$ with $u(t) \in \tilde U_{s_0+4}$, $t \in [0, 2T]$. Applying now Lemma 6.5 and 6.7 to the time interval $[0, 2T]$ (note the independence of the constants in these lemmas of the time interval length) we obtain $\|u(2T) - w_0\|_{s_0} \le \delta_0$ again and the solution can be continued to the interval $[2T, 3T]$. Repeating these arguments yields global existence of the solution. Moreover, Lemma 6.7 implies

$$\|f(t)\|_{s_0+1} \le Ce^{-at} \ \text{ for all } \ t \ge 0$$

with $a > 0$. Consequently for any $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\|u(t)\|_{s_0+4} \le C(\varepsilon)e^{\varepsilon t} \ \text{ for all } \ t \ge 0 \tag{6.20}$$

by Lemma 6.4 (ii). Further, for $0 \le t \le \theta < \infty$,

$$\|u(t) - u(\theta)\|_0 \le \int_t^\theta \|F(u(\tau))f(\tau)\|_0 \, d\tau \le C \int_t^\theta \|f(\tau)\|_{s_0+1} \, d\tau \le Ce^{-at},$$

and by interpolation using (6.20) with ε sufficiently small,

$$\|u(t) - u(\theta)\|_s \le C_s e^{-a_s t} \ \text{ for } \ s < s_0 + 4$$

with suitable constants $C_s, a_s > 0$. This implies convergence of $u(t)$ to some $u^\star$ in $H^s(S)$, $s < s_0 + 4$ as $t \to \infty$ and the estimate (6.19). The final statement follows from letting $t \to \infty$ in (6.11). □

Remark 6.9. It is not hard to see the following regularity property of $u^\star$: if the initial value u_0 belongs additionally to H^s with some $s > s_0 + 4$ then $u^\star(u_0, \gamma) \in H^{s'}$ for $s' < s$ (recall that we have always assumed $\gamma \in C^\infty$). The question whether or not $u^\star$ belongs to H^s remains open and requires more sophisticated estimates.

To illustrate possible equilibrium shapes $\Gamma_{u^\star}$ according to Theorem 6.8 we have performed several numerical test calculations for $m = 2, 3$. In a 2D situation, starting from a circle $S = \Gamma_{w_0}$ and a surface energy density of form

$$\gamma(x) = 1 + 0.8\cos(6\varphi), \quad x = (\cos\varphi, \sin\varphi) \in S, \ 0 \le \varphi < 2\pi,$$

we obtain an equilibrium shape $\Gamma_{u^\star}$ as pictured by the solid line in Figure 1. In contrast, if the correction term $G(u)$, which ensures the gradient flow structure of the evolution problem, is dropped in the definition (2.4), then the resulting shape $\Gamma_{u^\star}$ is given by the dotted line in Figure 1. Clearly, in the latter situation the center of gravity remains fixed due to the symmetries of the chosen initial values and every equilibrium configuration (u, γ) is characterized by $\gamma\kappa = const.$ on Γ_u, hence Γ_u must be convex. (This is similar to a Hele-Shaw evolution where the values of γ are transported only in normal direction to the moving boundary, as this also leads to a dropping of the term $G(u)$.) As Figure 1 shows, this convexity is not true for the full problem. The second example concerns an axisymmetric situation in 3D. Here the evolution starts from the unit sphere $S = \Gamma_{w_0}$ with the surface energy density

$$\gamma(x) = 1.0 + 0.8x_1(4.0x_1^2 - 3.0), \quad x = (x_1, x_2, x_3) \in S$$

and results in a equilibrium shape as shown in Figure 2. To indicate the length scale we have added grid lines with distance 0.25 in each direction.

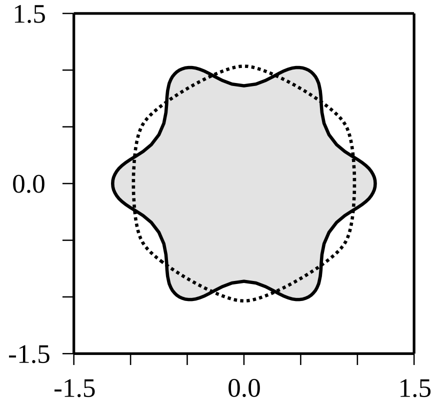

Figure 1: 2D examples

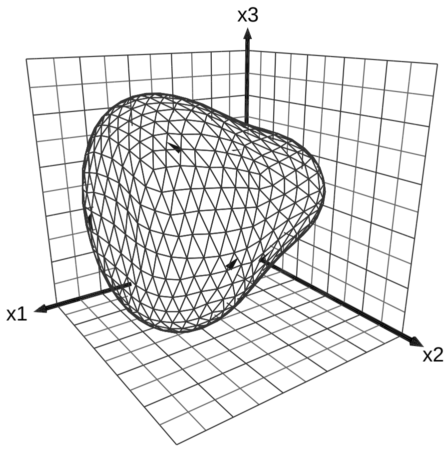

Figure 2: 3D example

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