Existence of closed geodesics on positively curved Finsler manifolds

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Abstract

For non-reversible Finsler metrics of positive flag curvature on spheres and projective spaces we present results about the number and the length of closed geodesics and about their stability properties.

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1 Introduction

For a Finsler metric F on a compact manifold we introduced in [Ra4] the concept of *reversibility* $\lambda := \max\{F(-X)\,;\, F(X) = 1\} \geq 1$. The reversibility attains its minimal value one if and only if the Finsler metric is *reversible,* i.e. $F(X) = F(-X)$ for all tangent vectors X. In particular Riemannian metrics are reversible. In this paper we investigate the consequences of the following length estimate for closed geodesics on a compact and positively curved Finsler manifold:

Theorem 1 [Ra4, Thm.1, Thm.4] *Let M be a compact and simply-connected manifold with a Finsler metric F with reversibility λ and flag curvature K satisfying $0 < K \leq 1$ resp. $\frac{\lambda^2}{(\lambda+1)^2} < K \leq 1$ if the dimension n is odd. Then the length of a closed geodesic is bounded from below by $\pi\frac{\lambda+1}{\lambda}$.*

It is a generalization of Klingenberg's injectivity radius estimate for compact Riemannian manifolds, cf. [Kl2, ch.2.6]. As announced in [Ra4, Rem.3] we apply Theorem 1 to obtain existence results for closed geodesics on positively curved manifolds carrying a non-reversible Finsler metric.
At first we consider the case of the 2-sphere. In the Riemannian case there are three geometrically distinct and simple closed geodesics on the 2-sphere with length in the interval $[2\pi, 2\pi/\sqrt{\delta}]$ if the Gaussian curvature K satisfies $1/4 < \delta \leq K \leq 1$. A closed curve is called *simple,* if it does not have self-intersections. This is a particular case

of an existence result for spheres in all dimensions, cf. [BTZ2, Thm.A].
The *Katok metrics* on the 2-sphere define a one-parameter family $F_\epsilon, \epsilon \in [0,1)$ of Finsler metrics of constant flag curvature 1 and reversibility $\lambda = (1+\epsilon)/(1-\epsilon)$, for $\epsilon = 0$ this is the standard Riemannian metric. For irrational ϵ there are exactly two geometrically distinct closed geodesics c_1, c_2 with lengths $L(c_1) = \pi(1+\lambda^{-1}); L(c_2) = \pi(\lambda+1)$, these geodesics only differ by orientation, cf. [Ra4, ch.5], [Zi, p.142].
Using the Morse inequalities, the topology of the space of unparametrized closed curves on S^2 and a detailed analysis of the sequence $\operatorname{ind}(c^m)$ of Morse indices of the m-fold covers c^m of a closed geodesic c we obtain the following existence result together with a length estimate:

Theorem 2 *Let F be a Finsler metric on the 2-sphere with reversibility λ and flag curvature K satisfying*

$$\left(\frac{\lambda}{\lambda+1}\right)^2 < \delta \leq K \leq 1$$

for some $\delta \in \mathbb{R}^+$. Then there are at least two *geometrically distinct closed geodesics c_1, c_2 whose lengths $L(c_1) \leq L(c_2)$ satisfy:*

$$L(c_1) \leq \frac{2\pi}{\sqrt{\delta}}\,;\, L(c_2) \leq \frac{\pi}{\sqrt{\delta}}\left(\frac{1}{\sqrt{\delta}\frac{\lambda+1}{\lambda}-1}+3\right).$$

In addition the shorter closed geodesic c_1 is simple.

Remark 1 (a) If we choose in particular $\delta := \left(\frac{2\lambda+1}{2\lambda+2}\right)^2 = \left(1-\frac{1}{2\lambda+2}\right)^2$ then $L(c_2) \leq 2\pi(\lambda+2)$

(b) If the metric has constant flag curvature, i.e. $\delta = 1$ then $L(c_2) \leq \pi(\lambda+3)$. In the above mentioned Katok examples $L(c_2) = \pi(\lambda+1)/\lambda\,;\, L(c_2) = \pi(\lambda+1)$. Equation 4 implies that for a bumpy Finsler metric of constant flag curvature 1 with only two geometrically distinct closed geodesics c_1, c_2 the following relation holds: $\frac{1}{L(c_1)} + \frac{1}{L(c_2)} = \frac{1}{\pi}$.

(c) The arguments in the Proof of Theorem 2 also show: If there is only one geometric closed geodesic on the 2-sphere, then its average index is at most 1. If the Finsler metric is *bumpy,* i.e. all closed geodesics are non-degenerate, then there are at least two geometrically distinct closed geodesics. This was shown by Ziller [Zi, p.149]. It follows from the results in [Ra1] that a bumpy Finsler metric on the 2-sphere with only finitely many geometrically distinct closed geodesics has at least two geometrically distinct *elliptic* closed geodesics, cf. [Ra1, Example 4.1]. Bangert and Long announced a proof that for *every* non-reversible Finsler metric on the 2-sphere there are two geometrically distinct closed geodesics, cf. [Lo1].

We also present applications of Theorem 1 in higher dimensions. We use an existence result for closed geodesics which the author derived in [Ra3] using the concept of the

Fadell-Rabinowitz index. We obtain a chain of subordinate cohomology classes in a quotient space of the space of closed curves. In the case of positive flag curvature we can estimate the number of geometrically distinct closed geodesics whose multiples are represented by the cohomology classes in this chain. As a general result we obtain Theorem 7 for metrics on compact and simply-connected manifolds of the rational homotopy type of a compact rank one symmetric space. Consequences for Finsler metrics on spheres are listed in the following

Theorem 3 *Let F be a Finsler metric on the n-sphere S^n with reversibility λ and flag curvature K satisfying $0 < \delta \le K \le 1$ for some $\delta \in \mathbb{R}^+$.*

(a) *The number of geometrically distinct closed geodesics with length $< 2n\pi$ is at least $n/2 - 1$, provided $\delta > \frac{\lambda^2}{(\lambda+1)^2}$.*

(b) *If $\lambda < \frac{n-1}{n-3}, n \ge 4$ and $\sqrt{\delta} > 2\frac{n-2}{n-1}\frac{\lambda}{1+\lambda}$ then there are at least $(n-2)$ geometrically distinct closed geodesics.*

(c) *If $n \ge 6$ is even and $\delta > 4/(n-2)^2$ then there are at least two geometrically distinct closed geodesics with length $\le n\,\pi$.*

There are Katok-metrics on the $2k$-sphere S^{2k} resp. the $(2k-1)$-sphere S^{2k-1} with $2k$ geometrically distinct closed geodesics (cf. [Zi, p.139]) and it is an open question whether there are always at least n closed geodesics on the n-sphere [Zi, p.156].

Finally we improve this result in Theorem 8 in the particular case of a *bumpy* metric, i.e. a metric all of whose closed geodesics are non-degenerate. For the m-dimensional complex projective space $\mathbb{C}P^m$ we obtain in Corollary 2 a lower bound for the number of geometrically distinct closed geodesics as well as for the number of non-hyperbolic closed geodesics provided there are only finitely many geometrically distinct ones.
One can study stability properties of a closed geodesic c with the help of the linearized Poincaré mapping P_c , which is a linear symplectic map of an $(2n-2)$-dimensional vector space. It can be defined using the Jacobi fields along this geodesic, cf. [BTZ1, ch.1], [Ra2]. In the most unstable case no (complex) eigenvalue of P_c lies on the unit circle, then the closed geodesic is called *hyperbolic.* For example all closed geodesics on a Finsler manifold with negative flag curvature are hyperbolic. We obtain a result similar to [BTZ1, Thm.B]:

Theorem 4 *Let F be a Finsler metric on a compact manifold with reversibility λ and flag curvature $0 < \delta \le K \le 1$. There exists a non-hyperbolic closed geodesic if the l–th homotopy group $\pi_l(M)$ is non-trivial for some $l \ge 2$ and $\sqrt{\delta} > \frac{l-1}{n-1}\frac{\lambda}{\lambda+1}$.*

In particular on the n-sphere S^n with a Finsler metric satisfying $\lambda^2/(\lambda+1)^2 < K \le 1$ there exists a non-hyperbolic closed geodesic. A more detailed analysis also produces existence results for closed geodesics of elliptic-parabolic type. Here a closed geodesic is called of *elliptic-parabolic type* if the linearized Poincaré map splits into two-dimensional rotations and a part whose eigenvalues are ± 1. Following the ideas of Thorbergsson [Th] and Ballmann, Thorbergsson and Ziller [BTZ1] we obtain as another consequence of the length estimate Theorem 1:

Theorem 5 *On a compact Finsler manifold M with reversibility λ and flag curvature $0 < \delta \le K \le 1$ there exists a closed geodesic of elliptic-parabolic type if one of the following conditions is satisfied:*

(a) $M = S^n$ *and* $\delta > \frac{9}{4}\frac{\lambda^2}{(\lambda+1)^2}$ *with* $\lambda < 2$.

(b) $M = \mathbb{R}P^n$ *and* $\delta > \frac{\lambda^2}{(\lambda+1)^2}$.

It is mentioned in [BTZ3, p.61] that most of the results presented in the Riemannian case generalize to Finsler metrics. For example it is stated that a Finsler metric with $9/16 < K \le 1$ carries a short closed geodesic of elliptic-parabolic type. But the arguments only work for *reversible* Finsler metrics respectively under the additional assumption that a shortest closed geodesic has length $\ge 2\pi$.
Another setting in which one can show the existence of a closed geodesic of elliptic-parabolic type is in the presence of an isometric S^1-action. For example the above mentioned Katok metrics F_ϵ on the 2-sphere carry an isometric S^1-action. For irrational parameter ϵ there are exactly two geometrically distinct closed geodesiscs which both are elliptic and invariant under the S^1-action. As an analogous result to [BTZ1, Theorem A(iii)] we obtain:

Theorem 6 *On a compact manifold with Finsler metric with an isometric S^1-action there exist at least two geometrically distinct closed geodesics. These closed geodesics are S^1-invariant and they are of elliptic-parabolic type.*

2 Critical Point Theory for Closed Geodesics

Here we list a couple of results of the critical point theory for closed geodesics, general references are the survey article [Ba] by Bangert, the book [Kl1] by Klingenberg and [Ra2].
If $c : S^1 := [0,1]/\{0,1\} \to M$ is a closed geodesic on the Finsler manifold (M,F) of length $L(c)$ then for every positive integer m the m-fold cover $c^m : S^1 \to M; c^m(t) = c(mt)$ is a closed geodesic, too. If $L(c) = \int_0^1 F(c'(t))\,dt$ denotes the *length,* then we have $L(c^m) = mL(c)$. We call a closed geodesic *prime* if it is not the cover c_0^m of another closed geodesic c_0 with $m > 1$. Closed geodesics are the critical points of the energy functional

$$E : \Lambda M \to \mathbb{R};\; E(c) = \frac{1}{2}\int_0^1 F^2\left(c'(t)\right) dt$$

on the *Hilbert manifold* ΛM of closed curves which is the set of all absolutely continuous closed curves with a square-integrable derivative.
The Morse index $\mathrm{ind}(c)$ of a closed geodesic is the index of the hessian $d^2E(c)$ of the energy functional. On the space ΛM there is a S^1-action given by changing the initial point. The energy functional is invariant under this group action. We call two closed geodesics c_1, c_2 of a non-reversible Finsler metric *geometrically equivalent,* if their traces $c_1(S^1) = c_2(S^1)$ and their orientation coincide. Otherwise we call them *geometrically distinct.* In contrast to the reversible case resp. the case of a Riemannian metric for a closed geodesic c the curve c^{-1} with $c^{-1}(t) = c(1-t)$ defined by reversing the orientation in general is not a geodesic. Hence a prime closed geodesic c produces infinitely many critical orbits $S^1.c^m; m \geq 1$ of the energy functional consisting of all geometrically equivalent closed geodesics. If a closed geodesic c is the m-fold cover $c = c_0^m$ of a prime closed geodesic c_0 then we call $m = \mathrm{mul}(c)$ the *multiplicity* of the closed geodesic c. Therefore a prime closed geodesic c produces a *tower* $S^1.c^m; m \geq 1$ of closed geodesics resp. critical orbits of the energy functional.
We can view the hessian of the energy functional also as a self-adjoint endomorphism. Then the index is the sum of the dimensions of negative eigenvalues and we call the *nullity* $\mathrm{null}(c)$ the dimension of the kernel $\ker d^2E(c)$ minus 1. Note that the dimension of the kernel is always at least 1 provided $L(c) > 0$ since there is a 1-dimensional group leaving the energy functional invariant. A closed geodesic c is called *non-degenerate* if $\mathrm{null}(c) = 0$. Geometrically the nullity is the dimension of periodic Jacobi fields along c which are orthogonal to the velocity field c'. Therefore $\mathrm{null}(c) \leq 2n-2$.
The sequence $\mathrm{ind}(c^m)$ grows almost linearly, we call the limit

$$\alpha_c := \lim_{m\to\infty} \frac{\mathrm{ind}(c^m)}{m}$$

introduced by Bott [Bo, Cor.1] the *average index* and $\overline{\alpha}_c = \alpha_c/L(c)$ the *mean average index.* We have the following estimate for the sequence $\mathrm{ind}(c^m)$:

$$|\mathrm{ind}(c^m) - m\alpha_c| \leq n-1\,, \tag{1}$$

cf. [Ra1, (1.4)]. By a Rauch comparison argument as in the Riemannian case one obtains

Lemma 1 ([Ra4, Lem.3]) *Let c be a closed geodesic on a Finsler manifold (M,F) of dimension n with positive flag curvature $K \geq \delta$ for some $\delta \in \mathbb{R}^+$.*

(a) The mean average index is bounded from below: $\overline{\alpha}_c \geq \sqrt{\delta}\,(n-1)/\pi$.

(b) If the length $L(c)$ satisfies $L(c) > k\pi/\sqrt{\delta}$ for some positive integer k then $\mathrm{ind}(c) \geq k(n-1)$.

Combining Lemma 1 with the length estimate Theorem 1 for a closed geodesic we obtain:

Lemma 2 *Let c be a closed geodesic on a compact and simply-connected Riemannian manifold of dimension n with a non-reversible Finsler metric with reversibility λ and flag curvature $0 < \delta \le K \le 1$ where $\delta > \frac{\lambda^2}{(\lambda+1)^2}$ if n is odd. Then*

$$\alpha_c \ge \sqrt{\delta}\,\frac{\lambda+1}{\lambda}\,(n-1)\,.$$

Proof. Since $L(c) \ge \pi\,(1+1/\lambda)$ by Theorem 1 the claim follows from Lemma 1. □

Now we come to the *Morse Inequalities* of the S^1-invariant functional $E : \Lambda M \to \mathbb{R}$. Let

$$\overline{b}_j := b_j\left(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbb{Q}\right)$$

where for $a \ge 0$ we denote $\Lambda^a M := \{\sigma \in \Lambda M \mid E(a) \le a\}$ the sublevel sets and b_j is the j-th Betti number. In particular $\Lambda^0 M$ is the set of point curves which can be identified with the manifold M. Since it is the fixed point set of the S^1-action one can also identify the quotient space $\Lambda^0 M/S^1$ with the manifold M. Given a closed geodesic c we use the following notation

$$\Lambda(c) := \{\sigma \in \Lambda M \mid E(\sigma) < E(c)\}\,.$$

Then we call

$$\overline{C}_*(c) = H_*\left(\left(\Lambda(c) \cup S^1.c\right)/S^1, \Lambda(c)/S^1; \mathbb{Q}\right)$$

the *S^1-critical group* of the closed geodesic c and let $\overline{b}_j(c) = \dim \overline{C}_j(c)$.
We collect the information about the S^1-critical groups in the following two lemmas:

Lemma 3 ([Ra1, Proof of Prop.2.2]) *Let c be a* non-degenerate *closed geodesic with $i = \mathrm{ind}(c), m = \mathrm{mul}(c)$. Then $c = c_1^m$ for a prime closed geodesic c_1 and*

$$\overline{b}_j(c) = \begin{cases} 1 \;;\; j = i; m \equiv 1 \pmod 2 \\ 1 \;;\; j = i; m \equiv 0 \pmod 2 \\ \qquad \text{and } \mathrm{ind}(c_1^2) \equiv \mathrm{ind}(c_1) \pmod 2 \\ 0 \;;\; \text{otherwise} \end{cases}$$

In the general case we obtain the following cases:

Lemma 4 [Ra2, Satz 6.13] *Let c be a closed geodesic with $i = \mathrm{ind}(c); l = \mathrm{null}(c)$. Then we have the following statements:*

(a) $\overline{b}_j(c) = 0$ *for* $j < i$ *or* $j > i + l$.

(b) $\overline{b}_i(c) + \overline{b}_{i+l} \le 1$ *and if* $\overline{b}_i(c) + \overline{b}_{i+l} = 1$ *then* $\overline{b}_j(c) = 0$ *for all* j *with* $i + 1 \le j \le i + l - 1$.

As a consequence of the formula for the sequence $\mathrm{ind}(c^m), m \ge 1$ given by Bott [Bo] we conclude:

Lemma 5 *Let c be a closed geodesic on a surface (i.e. $n = \dim M = 2$) with $\operatorname{ind}(c) = 1$ and average index $\alpha_c > 1$. Then for all $m \geq 1$ the indices $\operatorname{ind}(c^m)$ are odd.*

PROOF. Let P_c be the linearized Poincaré mapping, i.e. the linearization of the return map of the closed orbit of the geodesic flow corresponding to the closed geodesic. There is a function

$$I_c : S^1 := \{z \in \mathbb{C};\ |z| = 1\} \to \mathbb{Z}^{\geq 0}$$

with the following properties, cf. [Bo, Thm. A,C],[Lo, ch.9],[Ra2, ch.4]:

(a) $\operatorname{ind}(c^m) = \sum_{z^m=1} I_c(z)$

(b) Define $N_c : S^1 \to \mathbb{Z}^{\geq 0} : N_c(z) = \dim\ker(P_c - z\mathrm{Id})$. Then

$$\operatorname{null}(c^m) = \sum_{z^m=1} N_c(z)\,.$$

(c) The function I_c is constant in a neighborhood of points z with $N_c(z) = 0$. For the *splitting numbers*

$$S_c^{\pm}(z); = \lim_{\phi\to\pm 0} I_c(z\exp(i\phi)) - I_c(z)$$

of the function I_c the following estimate holds:

$$0 \leq S_c^{\pm}(z) \leq N_c(z)$$

(d) $I_c(z) = I_c(\bar{z}), N_c(z) = N_c(\bar{z})$

It follows that $\operatorname{ind}(c^m) \equiv \operatorname{ind}(c) \pmod 2$ for all odd m and $\operatorname{ind}(c^m) \equiv I_c(1) + I_c(-1) = \operatorname{ind}(c^2) \pmod 2$ for all even m. It was also shown by Bott that the splitting numbers only depend on the symplectic normal form of the linearized Poincaré map P_c, for a detailed discussion see [BTZ1, (2.13)], [Lo, ch.IV], [Ra2, ch.4].
Now we come to the case $n = 2$, then for the eigenvalues z of the linearized Poincaré map there are the following cases:

(a) $z \notin S^1$, i.e. z is a real number with $|z| \neq 1$. Then also z^{-1} is an eigenvalue, in this case the closed geodesic is called *hyperbolic* and $\operatorname{ind}(c^m) = m\operatorname{ind}(c)$. In particular the average index satisfies $\alpha_c = 1$ in contradiction to our assumption.

(b) If $z = 1$ then $S^+(1) = S^-(1)$ and $\alpha_c = I_c(-1) = \operatorname{ind}(c^2) - \operatorname{ind}(c) = \operatorname{ind}(c) + S^+(1) = 1 + S^+(1) > 1$ by assumption, hence we conclude $S^+(1) = 1$. It follows that $\operatorname{ind}(c^2) = 3$ and $\alpha_c = 2$.

(c) If $z = -1$ then $\alpha_c = \operatorname{ind}(c) = 1$ in contradiction to our assumption.

(d) If $z = \exp(\sqrt{-1}\pi\rho)$ with $\rho \in (0,1)$ then we conclude from [BTZ1, (2.13)] or [Ra2, Thm.4.3]: $S^+(z) + S^-(z) = 1$. Since $\alpha_c = \mathrm{ind}(c) + (S^+(z) - S^-(z))\rho > 1$ we conclude $S^+(z) = 1, S^-(z) = 0, \alpha_c \in (1,2)$ Hence in this case $\mathrm{ind}(c^2) = I_c(1) + I_c(-1) = 2\,\mathrm{ind}(c) + S^+(z) = 3$.

Therefore $\mathrm{ind}(c^2) = 3$ which implies that for all even m the indices $\mathrm{ind}(c^m)$ are odd, too. □

Now Lemma 4 and Lemma 5 imply the following

Corollary 1 *Let c be a prime closed geodesic on a surface (i.e. $n = 2$) with Finsler metric with index $\mathrm{ind}(c) = 1$ and average index $\alpha_c > 1$. Then for every $m \geq 1$:*

$$\sum_{j \equiv 1 \pmod 2} \overline{b}_j(c^m) \leq 1\,.$$

The Morse inequalities relate the critical groups as local information about the critical points with the global topological information given by the Betti numbers of the space on which the Morse function is defined.

Lemma 6 ([Ra1, 2.6]) *The rational Betti numbers $\beta_i := b_i\left(\Lambda S^2/S^1, \Lambda^0 S^2/S^1; \mathbb{Q}\right)$, of the pair of quotient spaces $\left(\Lambda S^2/S^1, \Lambda^0 S^2/S^1\right)$ are given by:*

$$\beta_i = \begin{cases} 2 & ; \quad i = 2m+1, m \geq 1 \\ 1 & ; \quad i = 1 \\ 0 & ; \quad i = 2m, m \geq 0 \end{cases}$$

3 Proof of Theorem 2

Let N be the odd integer satisfying

$$N - 2 \leq \frac{1}{\sqrt{\delta}\frac{\lambda+1}{\lambda} - 1} < N\,. \tag{2}$$

We assume that there is only one class of geometrically equivalent closed geodesics whose indices are bounded from above by N. Hence there is a prime closed geodesic c such that every closed geodesic $\tilde{c}$ with $\mathrm{ind}(\tilde{c}) \leq N$ is up to the choice of the initial point of the form c^m respectively $\tilde{c} \in S^1.c^m$ for some $m \geq 1$.

We define for all i with $0 \leq i \leq N$: $v_i := \sum_{m \geq 1} \overline{b}_i(c^m)$, then the Morse Inequalities for the S^1-invariant energy functional $E : \Lambda S^2 \to \mathbb{R}$ yield (cf. [Ra2, ch.6.1]):

$$v_i \geq \beta_i \tag{3}$$

for all i with $0 \leq i \leq N$. In particular we conclude from $\beta_1 = 1$ that for some $m \geq 1$: $\mathrm{ind}(c^m) \leq 1$. Since $L(c) \geq \pi\frac{\lambda+1}{\lambda}$ and $K \geq \delta > \left(\frac{\lambda}{\lambda+1}\right)^2$ we obtain from

Lemma 1 that $\mathrm{ind}(c^m) \geq 1$ for all $m \geq 1$. Hence we have finally shown: $\mathrm{ind}(c) = 1$. As an estimate for the average index we obtain from Lemma 1 : $\alpha_c \geq \sqrt{\delta}\,(\lambda+1)/\lambda > 1$. Inequality (1) and Inequality (2) imply:

$$\mathrm{ind}(c^N) \geq N\alpha_c - 1 \geq N\sqrt{\delta}\,\frac{\lambda+1}{\lambda} - 1 = N\left(\sqrt{\delta}\,\frac{\lambda+1}{\lambda} - 1\right) + N - 1 > N\,.$$

Therefore Corollary 1 implies:

$$\sum_{\substack{0\leq i\leq N \\ i\equiv 1 \pmod 2}} v_i = \sum_{\substack{0\leq i\leq N; m\geq 1 \\ i\equiv 1 \pmod 2}} \overline{b}_i(c^m) \leq \#\{m \mid \mathrm{ind}(c^m) \leq N\} < N\,.$$

This contradicts the Morse Inequalities (3) since by Lemma 6 for N odd:

$$\sum_{\substack{0\leq i\leq N \\ i\equiv 1 \pmod 2}} v_i \geq \sum_{\substack{0\leq i\leq N \\ i\equiv 1 \pmod 2}} \beta_i = N\,.$$

Hence there are at least two geometrically distinct closed geodesics c_1, c_2 with $\mathrm{ind}(c_1) = 1\,;\ \mathrm{ind}(c_2) \leq N$. We conclude from Lemma 1 and Inequality (2):

$$L(c_2) \leq \frac{\pi}{\sqrt{\delta}}(N+1) \leq \frac{\pi}{\sqrt{\delta}}\left(\frac{1}{\sqrt{\delta}\frac{\lambda+1}{\lambda} - 1} + 3\right)$$

Since $\mathrm{ind}(c_1) = 1$ Lemma 1(b) implies that $L(c_1) \leq 2\pi/\sqrt{\delta}$. For the given curvature bounds not only the length of a closed geodesic but also the length of a geodesic loop is bounded from above by $\pi(1+\lambda^{-1})$, cf. [Ra4, Thm. 1], hence c_1 is simple since $2\pi/\sqrt{\delta} < 2\pi(1+\lambda^{-1})$. □

4 Existence results in higher dimensions

We consider a compact and simply-connected manifold M whose rational cohomology algebra is generated by a single element $x \in H^d(M;\mathbb{Q})$ of degree d, with the relation $x^{m+1} = 0$. Hence the cohomology algebra $H^*(M;\mathbb{Q})$ is isomorphic to the truncated polynomial algebra $\cong T_{d,m+1}(x) = \mathbb{Q}[x]/(x^{m+1} = 0)$ and the dimension of M equals $d\,m$. The main examples are the compact rank one symmetric spaces, i.e. spheres S^d of dimension d (then $m = 1$), m-dimensional complex projective spaces $\mathbb{C}P^m$ with $d = 2$, m-dimensional quaternionic projective spaces $\mathbb{H}P^m$ with $d = 4$ and the Cayley plane $\mathbb{C}\mathrm{a}P^2$ with $d = 8, m = 2$. Then we obtain from [Ra3, Thm.5.11]:

Proposition 1 *Let M be a simply-connected and compact manifold whose rational cohomology algebra is generated by a single element of order d, i.e. $H^*(M;\mathbb{Q}) =$*

$T_{d,m+1}(x)$ endowed with a Finsler metric. Then there is a sequence $c_k; k \geq 1$ of prime closed geodesics and a sequence $m_k; k \geq 1$ of positive integers such that the sequence $S^1.c_k^{m_k}; k \geq 1$ is a sequence of S^1-orbits of closed geodesics which are pairwise distinct (although in general not geometrically distinct) and whose lengths and indices satisfy the following proerties for all $k \geq 1$: properties:

(a) $m_k L(c_k) = L\left(c_k^{m_k}\right) \leq L\left(c_{k+1}^{m_{k+1}}\right) = m_{k+1}L(c_{k+1})$

(b) $2k - (2m-1)d + 1 \leq \operatorname{ind}\left(c_k^{m_k}\right) \leq 2k + d - 1$

Theorem 7 *Let M be a simply-connected and compact manifold whose rational cohomology algebra is generated by a single element $x \in H^d(M;\mathbb{Q})$, i.e. $H^*(M;\mathbb{Q}) = T_{d,m+1}(x)$. We assume that the manifold M carries a Finsler metric with reversibility λ whose flag curvature K is positive and satisfies*

$$0 < \delta \leq K \leq 1\,,$$

where $\sqrt{\delta} > \lambda/(\lambda+1)$ if n is odd. Then the number of geometrically distinct closed geodesics of length $\leq L$ is bounded from below by

$$A(m,d,\delta,\lambda,L) := \frac{1}{2}\frac{\lambda+1}{\lambda}\sqrt{\delta}\left(md - 1 - \frac{\pi}{L}d\right)$$

Remark 2 (a) If we are not interested in the length of the closed geodesics we obtain as bound:

$$A(m,d,\delta,\lambda,\infty) = \lim_{L\to\infty} A(m,d,\delta,\lambda,L) = \frac{1}{2}(md-1)\sqrt{\delta}\,\frac{\lambda+1}{\lambda}$$

(b) The maximal value of the bound is attained if the flag curvature K is constant ($\delta = 1$) and the metric is reversible ($\lambda = 1$):

$$A(m,d,1,1,\infty) = md - 1 = n - 1$$

(c) Theorem 3 is a direct consequence for $m = 1, d = n$.

Proof of Theorem 7: We consider the sequence $\left(S^1.c_k^{m_k}\right)_{k\geq 1}$ of pairwise distinct critical orbits of closed geodesics satisfying the properties of Proposition 1. Hence $c_k, k \geq 1$ are prime closed geodesics and (m_k) is a sequence of positive integers. Fix a number $L > 0$ and let $a_L := \#\{k \geq 1 \,|\, L(c_k)m_k \leq L\}$. By the comparison result part (b) of Lemma 1:

$$a_L \geq b_L := \#\left\{k \geq 1 \;\middle|\; \operatorname{ind}\left(c_k^{m_k}\right) < \frac{L}{\pi}\sqrt{\delta}\,(n-1)\right\}.$$

We conclude from Proposition 1, (b) that for $2k < L\sqrt{\delta}\,(n-1)\pi^{-1} - (d-1)$: $\operatorname{ind}(c_k^{m_k}) < L\sqrt{\delta}\,(n-1)\pi^{-1}$, hence b_L is bounded from below by the integer part of

$L\sqrt{\delta}\,(n-1)\pi^{-1}-(d-1)$, hence $b_L \geq L\sqrt{\delta}\,(n-1)\pi^{-1}-d$. Since $L(c_k) \geq \pi(\lambda+1)/\lambda$ by Theorem 1 we obtain from Proposition 1, (a) that $m_k < L\pi^{-1}\lambda/(\lambda+1)$. Therefore the number of geometrically distinct closed geodesics in the set $c_1, c_2, \ldots, c_{a_L}$ is bounded from below by

$$\frac{a_L}{\frac{L}{\pi}\frac{\lambda}{\lambda+1}} \geq \frac{\frac{L}{\pi}\sqrt{\delta}\,(n-1)-d}{2\,\frac{L}{\pi}\frac{\lambda}{\lambda+1}} \geq \frac{1}{2}\frac{\lambda+1}{\lambda}\sqrt{\delta}\,\left(n-1-\frac{\pi}{L}\,d\right)$$

□

We call a Finsler metric *bumpy*, if all closed geodesics are non-degenerate. If the Finsler metric on a compact and simply-connected manifold is bumpy and has only finitely many geometric distinct prime closed geodesics $c_1, c_2, \ldots, c_r$ with average indices $\alpha_1, \alpha_2, \ldots, \alpha_r$ then the rational cohomology ring $H^*(M;\mathbb{Q})$ is generated by a single element x of degree d with the only relation $x^{m+1}=0$, i.e. $n = \dim M = md$. The invariants d, m determine the number

$$B(d,m) = \begin{cases} -\frac{m(m+1)d}{2d(m+1)-4} & ; d \text{ even} \\ \\ \frac{d+1}{2d-2} & ; d \text{ odd} \end{cases}.$$

for which the following formula is derived in [Ra1, Thm.3]:

$$B(d,m) = \sum_{i=1}^{r} \frac{\gamma_i}{\alpha_i} \tag{4}$$

Here $\gamma_i \in \{\pm 1/2, \pm 1\}$ is an invariant controling the parity of the sequence $\operatorname{ind}(c_i^m)$ and the orientability of the negative normal bundle c and c^2. Let $c_1, \ldots, c_s; s \leq r$ be the non-hyperbolic closed geodesics. Then for even d the following estimate holds with the same argument as in [Ra1, Thm.3.1(b)]:

$$\sum_{k=1}^{s} \left\{ |\gamma_k| \left(\frac{md-1}{\alpha_k} - 1 \right) + 2 \right\} \geq \frac{1}{4} m(m+1)d \tag{5}$$

As a consequence from this formula and Lemma 2 we obtain analogous to [Ra1, Cor.3.4]:

Theorem 8 *Let F be a Finsler metric on a compact and simply-connected manifold M with $H^*(M;\mathbb{Q}) = T_{d,m+1}(x)$ with reversibility λ and flag curvature K satisfying $0 < \delta \leq K \leq 1$, where $\sqrt{\delta} > \lambda/(\lambda+1)$ provided $m=1$ and d is odd.*

(a) If the metric is bumpy then there are at least $C(m,d,\delta,\lambda) := |B(d,m)|\sqrt{\delta}\frac{\lambda+1}{\lambda}(md-1)$ geometrically distinct closed geodesics.

(b) If the metric is bumpy and there exist only finitely many geometrically distinct closed geodesics then there are at least $m(m+1)d\left(\frac{4\lambda}{\sqrt{\delta}(\lambda+1)}+6\right)^{-1}$ non-hyperbolic closed geodesics.

For $d \in \{2,4,8\}$ and for a fixed value of the lower curvature bound δ the function $C(m,d,\delta,\lambda)$ grows quadratically in m. Let us consider Finsler metrics on the m-dimensional complex procjective space $\mathbb{C}P^m$. The flag curvature of the normalized Fubini-Study metric on $\mathbb{C}P^m$ satisfies $1/4 \le K \le 1$. For $\delta \to 1/4$ and $\lambda \to 1$ we obtain as maximal value $C(m,2,1/4,1) = (m+1)(m-1/2)$. It is very likely that this bound is not optimal since there are Finsler metrics of Katok type on $\mathbb{C}P^m$ with $m(m+1)$ geometrically distinct closed geodesics, cf. [Zi, p.139].
Another application is the following result analogous to [BTZ1, Cor.4]:

Corollary 2 *A bumpy Finsler metric on the m-dimensional complex projective space $\mathbb{C}P^m$ $(m \ge 7)$ with reversibility λ and flag curvature $0 < \delta \le K \le 1; \sqrt{\delta} = \frac{2}{m+1}\frac{\lambda}{1+\lambda}$ with only finitely many geometrically distinct closed geodesiscs carries at least $2m$ geometrically distinct closed geodesics. At least $(m-3)$ of these closed geodesics are non-hyperbolic.*

5 Stability properties of closed geodesics

A closed geodesic is called *hyperbolic* if all all eigenvalues of the linearized Poincaré map have modulus $\neq 1$. Then the sequence $\operatorname{ind}(c^m)$ is linear in m, i.e. $\operatorname{ind}(c^m) = m\operatorname{ind}(c)$. This was observed by Bott in [Bo] and shows immediately part (a) of the following

Lemma 7 *Let c be a closed geodesic of a Finsler manifold with average index α_c.*

(a) If $\operatorname{ind}(c) \neq \alpha_c$ then c is non-hyperbolic.

(b) If $\operatorname{ind}(c^2) - 2\operatorname{ind}(c) = n-1$ then c is of elliptic-parabolic type, the linearized Poincaré map splits into 2×2 blocks of the form

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and / or } \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \quad 0 \le \phi < \pi\,,$$

with respect to a symplectic basis $(X_1, Y_1, X_2, Y_2, \ldots, X_{n-1}, Y_{n-1})$ satisfying $\omega(X_i, Y_i) = \delta_{ij}; \omega(X_i, X_j) = \omega(Y_i, Y_j) = 0$ for the symplectic form ω.

Part (b) is shown in [BTZ1, Lemma 3.1], since $\operatorname{ind}(c^2) = I_c(1) + I_c(-1)$ and $\operatorname{ind}(c) = I_c(1)$, cf. the Proof of Lemma 5

Proof of Theorem 4: By considering the universal covering we can assume M to be simply-connected. Since $\pi_l(M) \neq 0$ one concludes that $\pi_{l-1}(\Lambda M) \neq 0$, hence there is a closed geodesic c with $\operatorname{ind}(c) \le l-1$. From Theorem 1 we conclude $L(c) \ge \pi(1+\lambda^{-1})$, hence Lemma 1 (b) implies that $\alpha_c > l-1$. Therefore c is non-hyperbolic by Lemma 7, part (a). □

Remark 3 On a compact and not-simply-connected Riemannian manifold of non-negative Ricci curvature there is a non-hyperbolic closed geodesic. The proof of this statement (cf. [BTZ1, Thm.B (a)]) carries over to the Finsler case without changes, here one does not need Klingenberg's injectivity radius estimate.

PROOF OF THEOREM 5:

(a) Since $b_{n-1}(\Lambda S^n, \Lambda^0 S^n) = 1$ there is a closed geodesic c with $\operatorname{ind}(c) \le n-1$. The length estimate Theorem 1 implies for the second cover $L(c^2) \ge 2\pi(1+\lambda^{-1}) > 3\pi/\sqrt{\delta}$. Hence by Lemma 1 we obtain $\operatorname{ind}(c^2) \ge 3(n-1)$ resp. $\operatorname{ind}(c^2) - 2\operatorname{ind}(c) \ge n-1$. We conclude from Lemma 7(b) that c is of elliptic-parabolic type.

(b) Let c be a shortest closed geodesic which is not homotopically trivial. Then c is a local minimum for the energy functional, hence $\operatorname{ind}(c) = 0$. Since c defines also a closed geodesic on the universal covering we conclude from Theorem 1 that $L(c^2) \ge \pi(1+\lambda^{-1}) > \pi/\sqrt{\delta}$. Hence by Lemma 1(b) we conclude $\operatorname{ind}(c^2) \ge (n-1)$ which shows that c is of elliptic-parabolic type by Lemma 7(b). $\square$

Now we come to a different setting in which one can show the existence of *two* geometrically distinct closed geodesics on a manifold with non-reversible Finsler metric.

An isometry $A: M \to M$ of finite order of a compact Finsler manifold (M, F) has *small displacement,* if for all points $p \in M$ the image point $A(p)$ does not lie in the cut locus. Let $\theta: M \times M \to \mathbb{R}$ be the distance function, i.e. $\theta(x,y)$ is the minimal length of a smooth curve $c: [0,1] \to M$ from $x = c(0)$ to $y = c(1)$. Note that this distance function in general is not symmetric since we consider non-reversible Finsler metrics. Then we define the function $f_A: M \to \mathbb{R}$ with $f_A(x) = \theta^2(x, Ax)$ which is smooth outside the fixed point set M_A of A. Then we obtain as in [BTZ1, p.239]: The point p is a critical point of f_A if and only if the unique minimal geodesic $\gamma: [0,1] \to M$ from $p = \gamma(0)$ to $A(p) = \gamma(1)$ is invariant under A, i.e. $A_*(\gamma'(0)) = \gamma'(1)$. It also follows that $\gamma(t)$ is a critical point of f_A for all $t \in [0,1]$. A critical point $p \notin M_A$ determines a closed geodesic, since A is of finite order. Analogous to [BTZ1, Thm.3.10] we obtain

Proposition 2 *Let A be an isometry of finite order and small displacement on a compact Finsler manifold. A local maximum of f_A determines a closed geodesic of elliptic-parabolic type if* $2\theta(p, A(p)) < \theta(p, \operatorname{Cut}(p))$.

PROOF OF THEOREM 6:

Let $\phi_t, t \in \mathbb{R}$ define the isometric S^1-action, i.e. $\phi_t : M \to M$ is a one-parameter group of isometries with ϕ_1 equals the identity. Then for a sufficiently large integer m the isometries $A = \phi_{1/m}$ and $A^{-1} = \phi_{-1/m}$ have small displacement and are of finite order. Then we conclude from Proposition 2 that there are two points $p_\pm$ such that $p_\pm$ is a maximum of f_A resp. $f_{A^{-1}}$. Then the closed geodesics $c_\pm$ with $c_\pm(0) = p_\pm$ and $c_\pm(1/m) = A^{\pm 1}(p_\pm)$ are of elliptic-parabolic type and satisfy $c(t) = \phi_t(p_\pm)$. If c_-, c_+ are geometrically equivalent, we can assume without loss of generality that $p = p_- = p_+$. Hence $c_\pm(t) = \phi_{\pm t}(p)$ which shows that $c_+'(0) = -c_-'(0)$. i.e. the closed geodesics $c_, c_+$ are geometrically distinct.

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