Łomnicki – Steinhaus – Kolmogorov : steps to a modern probability theory

Hans-Joachim Girlich (Leipzig)

Abstract

Ten years ago J. von Plato published a profound monograph on pathways to modern probability. We restrict our historical remarks to first but important steps to probability of three Eastern European mathematicians during the first quarter of the 20th century. We discuss early papers by A. Łomnicki, H.Steinhaus and A. Kolmogorov. Our aim is not only to stress new ideas and results, but also to uncover roots in common, motives, and connections amongst the authors. This special opportunity facilitates to recognise that the cradle of the famous Polish school of mathematics was standing not in Warsaw but in the former Austrian crownland Galicia.

1. Introduction

Hugo Steinhaus (1887-1972) published in the Polish monthly journal *Znak* in January of 1970 memories of his time as a student in Göttingen. He wrote especially about the period 1906/1907 (cf. [52] p.54/55) [in my translation , H.G.]:

> "In this time the papers by Georg Cantor, the brillant creator of set theory, were hardly known yet. As the young mathematician of Warsaw Wacław Sierpiński discovered: the position of a point in the plane may be described by only one number, he does not believe it. He wrote a letter to his friend Tadeusz Banachiewicz in Göttingen, This one went to the reading-room, looked out Cantor's corresponding paper and telegraphed to Sierpiński only: 'G. Cantor, Journal für Math, volume,pages…' That awoke our interest. Together with Antoni Łomnicki we went reading eagerly Borel's *Theorie des fonctions.* "

Steinhaus finished his study in Göttingen in May 1911 and obtained the PhD degree summa cum laude. His thesis *Neue Anwendungen des Dirichlet'schen Prinzips* written under David Hilbert's supervision connected and applied the two main streams in mathematical analysis created by his teacher and by H.Lebesgue in Paris (cf. [16], [47]). In the autumn of 1910 Nicolas Lusin (1883-1950) arrived Göttingen from Moscow. Edmund Landau recognised his talent for mathematical research but also his inhibition and he insist that Lusin wrote his first paper: *Über eine Potenzreihe* published in Palermo in 1911. It was the begin of two great developments. The first one leaded Lusin via Paris to his doctoral dissertation *The integral and trigonometric series,* to his professorship in 1917, and his "Lusitania " – the Moscow school of the theory of functions. The second one created a close co-operation between Lusin and Sierpiński which started with Steinhaus's reaction [48] to Lusin's paper edited and communicated by Sierpiński in Warsaw, strengthened and intensified during Sierpiński's internment in Moscow, 1914-1918, continued to the Congress of mathematicians from Slav countries in 1929. (cf. [31, [11], [43], [26]).

The paper is organised as follows.. We start in Section 2 with Łomnicki's background and environment in physics and logic with respect to probability. Section 3 is devoted to problems of mathematical analysis which were studied by Steinhaus, Kolmogorov and others. We consider a problem of number theory in Section 4 and its probabilistic treatment by Steinhaus in Section 5. From special random series in Section 6 we reach with Kolmogorov sums of independent random variables in Section 7. We conclude with a view of Steinhaus' work in Lwów at the beginning of the 1930s.

2. Sources of probability in Leopol

There is a town in Eastern Europe (49.50N,24.00E) which is known under the names Léopol, Lwów, Lemberg, Lvov, Lwiw. We are interested in the university which has changed the language of instruction from German to Polish in 1871. It was named Jan Kazimierz University in 1919 after the founder of a Jesuit academy in 1661 .

Marian von Smoluchowski (1872 – 1917) came to Leopol in 1899. Appointed professor in 1900, he held the chair of mathematical physics at the university from 1903 until 1913. His results include density variation, opalescence and Brownian motion.(cf. [46]). Following Boltzmann's probabilistic approach in physics (cf. [4]),he derived for the probability distribution of a Brownian particle under external forces the equation

$$W(x,x_0)_t\,dx = dx\int_{-\infty}^{\infty} W(\alpha,x_0)_\theta W(x,\alpha)_{t-\theta}\,d\alpha \qquad , \qquad (1)$$

the so-called *Smoluchowski's functional equation* (cf. [44], p.424, [15]). About his work and its scientific environment with respect to chance he wrote (cf. [45] , p. 253):

> "Trotz dieser enormen Ausdehnung des Anwendungsbereiches der Wahrscheinlichkeitsrechnung hat die exakte Analyse der ihr zugrunde liegenden Begriffe nur geringe Fortschritte gemacht;…Insbesondere ist auch eine allgemeine und mathematisch exakte Präzisierung der für die Anwendbarkeit dieser Rechnungsmethode charakteristischen Bedingungen noch immer ausständig, und man pflegt sich in dieser Hinsicht meist auf ein intuitives Wahrscheinlichkeitsgefühl zu verlassen. "

Probably, Smoluchowski thought on the multiplication law which he applied instinctively to obtain equation (1).

Anton Marjan Łomnicki (1881-1941) studied at the University of Leopol from 1899 until 1903 with J. Puzyna and M.v.Smoluchowski. He took an examination as a teacher of physics and mathematics, and worked at colleges in Leopol and Tarnowo for 16 years. His excellent textbooks in elementary geometry, trigonometry, and analytic geometry went through six editions. He habilitated in 1919 and became a professor for mathematics at the Polytechnicum of Lwów in 1920. His first assistant was S. Banach (cf. [35],[37]). In this year he finished his paper on the foundation of probability calculus published in the *Fundamenta Mathematicae* . His model based on the following idea (cf. [27], p.37):

> "Lorsqu'on parle de probabilité, on a toujours en vue deux ensembles m et M, dont le premier fait partie du second. … A chaque élément de M…on fait corréspondre un nombre positif fini,…appelons le : poids de l'élément. "

The textbooks by Markoff [32], Czuber [8], and Poincaré [36] were stimulating and provided that Łomnicki discussed systematically and in detail two cases. First, M is finite, and second, M is non countable. Conditions for the validity of the multiplication law in the finite case connected with a better understanding of dependence in probability are the most important result.

The logical structure of events in the second case (which Bohlmann in Rome in 1908 demanded) is not relevant to Łomnicki's analysis. He assumed only: m is measurable in the sense of Lebesgue or Carathéodory.

Jan Lukasiewicz (1878 –1956) gave lectures on probability theory at Leopol University in the winter semester of 1910/11 containing a logical foundation of probability in the finite case. It will be extended by Mazurkiewicz in 1932 (cf. [33]). But for an applied mathematician it seems to be not useful. On the other hand, Felix Hausdorff (1868-1942) presented in his lectures on probability theory at the University of Bonn in 1923 a surprising formalisation: An event m is subset of M and all events constitute a Borel field. The equivalence of an event field and a Borel field of sets is proved 13 years later by M. H. Stone (cf. [3], [29], [13], [10], [53]).

3. Special series of functions

The problem of representation of functions by series of simple function systems was studied in the theses of Riemann in 1854 and Cantor in 1870 as well as by Weierstrass for trigonometric functions, the latter as a special lacunary series in 1872. It was developed further in 1906 by Lebesgue's book *Lecons sur les séries trigonometriques* and Fatou's thesis *Séries trigonometriques et séries de Taylor* . As already mentioned in the Introduction Lusin considered in his first paper a power series. He constructed a special Taylor series which diverges in every boundary point of the disc of convergence, and an associated trigonometric series in the sense of Fatou which diverges almost everywhere. Soon afterwards, Steinhaus improved Lusin's result presenting a trigonometric series with coefficients tending to zero, divergent everywhere with no exception. He knew Lebesgue's book from Göttingen (cf. [18],[30],[48],[47]).

Stephan Banach (1892-1945) was Steinhaus' first doctoral student. In 1918 they constructed together a trigonometric series with coefficients tending to zero:

$$\sum_{n=2}^{\infty} \frac{\cos nx}{(\log n)^{1+\rho}} \quad , \quad \rho>0 \qquad (2)$$

which converges in the mean, that is in L^1 metric (cf. [16], [1]).

A few years later, Andrej Kolmogoroff (1903-1987) took up this result and studied the more general trigonometric series (cf. [22]).

$$\sum_{n=0}^{\infty} a_n \cos nx \quad . \tag{3}$$

He gave necessary and sufficient conditions for the convergence in the mean of series (3). Weierstrass' lacunary Fourier series

$$\sum a^k \cos b^k t \qquad a \in (0,1)\,,\, b \in \mathrm{N} \tag{4}$$

represent for *ab* large enough a continuous nowhere differentiable function (cf. [18]).
In 1922 Rademacher introduced the system of functions

$$r_n(t) = (-1)^{k+1} \quad \text{for} \quad \frac{k}{2^n} \le t < \frac{k+1}{2^n} \quad , k = 0,1,\ldots, 2^n\text{-}1 \ , \quad n \in \mathrm{N}. \tag{5}$$

We obtain $\quad r_n(t) = -sign \sin 2^n \pi t \qquad$ for $\quad t \neq \dfrac{k}{2^n}$.

He proved the *Theorem*.

$$\text{If} \quad \sum_{n=1}^{\infty} c_n^2 < \infty \quad , \text{ then} \quad \sum_{n=1}^{\infty} c_n r_n(t) \tag{6}$$

converges almost everywhere (cf. [39], [17]).

Under the same condition, in the same year, in the age of 19 Kolmogorov showed the convergence almost everywhere of the series

$$\sum_{k=1}^{\infty} A_{n_k}(t) \quad , \quad \text{where } n_k \in \mathrm{N} \qquad \text{with } n_{k+1} > q\cdot n_k \quad \text{and} \quad q>1, \tag{7}$$

$A_{n_k}(t) = a_k \cos n_k t + b_k \sin n_k t \quad , \ c_n^2 = a_n^2 + b_n^2 \quad$ (cf. [23],[55] V).

Now, we use the English spelling "Kolmogorov " but we prefer his own spelling "Kolmogoroff " because he has written his non-Russian papers and books in German, French, and Italian but not in English until 1941.

4. The asymptotic distribution of digits

Wacław Sierpiński (1882-1969) studied at the department of physics and mathematics of the (Russian) Imperial University of Warsaw from 1900 until 1904. He finished his studies with the degree of candidate of sciences and with a gold medal for his paper on number theory to a theme proposed by his teacher G.F. Woronoi (cf. [40],[43]).

In1917 Sierpiński published a simple proof of Borel's theorem that all real numbers are absolutely normal with exception of a set *S* of measure zero, that means the frequency of a single digit in the q-adic representation of almost every real number is asymptotically equal to $\frac{1}{q}$ for every integer q. For fixed q Borel's theorem on normal numbers has been proved for *q=2* by Hausdorff in 1914 also with estimation of sets of intervals covered *S* and for *q=10* by Rademacher with a new proof in 1918 (cf. [41], [12], [38]).

Let x=0. $\nu_1\nu_2\nu_3\ldots$be the binary representation of x$\in(0,1)$, $\nu_k \in \{0,1\}$. We describe results about the speed of the convergence of the digit frequency with the help of the expression

$$\mu_n(x) = \sum_{k=1}^{n} \nu_k - \frac{n}{2} \quad . \qquad (8)$$

Hausdorff obtained in [12] :

$$\mu_n(x) = o(n^\alpha) \qquad , \quad \alpha > \frac{1}{2} \quad , \qquad (9)$$

Aleksander Khintchine (1894-1959) improved (9) in 1922:

$$\limsup_{n\to\infty} \left| \frac{\mu(x)}{n \log\log n} \right| \le 1 \qquad (10)$$

for all *x* except a set of measure zero (cf. [19]).

In 1907-1908 Sierpiński started to work on the new field of set theory and stayed a few month with Banachiewicz and Steinhaus in Göttingen. He habilitated at the suggestion of J. Puzina in Leopol and become associated professor by emperor Franz Joseph in 1910. He lectured about set theory at the University of Leopol in 1909, and Hausdorff to the same topic already in 1901 at Leipzig University. But his book *Zarys teorii mnogości* was published in 1912, two years before Hausdorff's *Grundzüge der Mengenlehre* (cf. [14], [40], [12]).

5. Sequences of independent repeated trials

The binary representation of a real number may be interpreted as a sequence of repeated Bernoulli trials. James Bernoulli proved in *Ars Conjectandi* (published post mortem in 1713) his famous theorem about the relative frequency of an event ("success") which approximate the probability of this event ("probability of success") if we repeat a large number of trials under the same circumstances. But for the problem of the frequency of the digits in a dyadic system Laplace's calculus of probability with the weak law of large numbers is not sufficient. It was Emile Borel (1871-1956) who developed in 1909 a new approach connected with the term "denumerable probability" and the strong law of large numbers. Łomnicki seems to be mentioned shortcomings in Borel's paper [6] (cf. [2]). He cited [6] but in [27] he did not present this case standing between the finite case and geometric probability. Probably, he knew that his own paper [27] should be published in the same volume of *Fundamenta Mathematicae* as Steinhaus's paper *Les probabilités dénombrables et leur rapport à la théorie de la mesure* [49]. Here, Steinhaus gave a rigorous mathematical theory of sequences of independent repeated Bernoulli trials. He constructed a special probability space because he needed an axiomatic introduction of both "probability" and "event" . But for "measure" and "measurable set" Sierpiński has given such a fundament in [42] modifying Caratheodory's formal theory of measurability in [7] by elimination of the outer measure. Steinhaus worked with cylindrical sets as a generating system as Kolmogorov in the proof of his main theorem in [25] ten years later. As an application of his model Steinhaus proved Borel's strong law of large numbers. With Steinhaus' approach Khintchine [20] was able to transfer his inequality (10) to the law of iterative logarithm for a sequence of repeated Bernoulli trials.

6. Random series

Steinhaus introduced in [49] a special random series

$$\sum_{n=1}^{\infty} \pm c_n \quad , \tag{11}$$

Where { c_n } is a sequence of numbers and the sequence of signs will be generated by drawing with replacement a ball from a urn containing balls painted with sign + as much with sign –. It is evident that there is a one-to-one mapping between the sequences of signs and zero-one sequences and therefore with real numbers in *[0,1]* in dyadic representation. In this way, Steinhaus showed that the probability of convergence of the random series (11) is equal to the Lebesgue measure of all $x \in [0,1]$ with

$$\sum_{n=1}^{\infty} c_n r_n(x) \tag{12}$$

is convergent. He used Rademacher's Theorem (6) for a sufficient condition of convergence. But his zero-one law: the random series (11) converges with probability 0 or 1, is more important (cf.[9]).

Norbert Wiener (1894-1964) described in [54] p.570 in connection to [49] a random series

$$f(x) \approx a_0 + a_1\sqrt{2}\cos\pi x + \ldots + a_n\sqrt{2}\cos n\pi x + \ldots \tag{13}$$

as a model of Brownian motion. His Differential-space approach was not clear presented, It found interest at first after Kolmogorov's foundation of probability theory [25].

7. Sums of independent random variables

Kolmogorov's first paper on probability [21] was written together with Khintchine. The model for the study of the behaviour of a random series

$$\sum_{n=1}^{\infty} y_n \quad , \tag{14}$$

were y_n is a discrete random variable, was given by Khintchine in §1. He generalised Steinhaus' approach by transforming (14) into a series of functions

$$\sum_{n=1}^{\infty} \varphi_n(x) \quad . \tag{15}$$

The old conception of a discrete random variable (cf. [32])

$$y_n = y_n^{(i)} \qquad \text{with probability } p_n^{(i)} \quad , i \in \mathrm{N} \tag{16}$$

may be replaced by a function on $[0,1]$:

$$f_n(x) = y_n^{(i)} \quad \text{for } x \in I_n^{(i)} = (\sum_{k=1}^{i-1} p_n^{(k)}, \sum_{k=1}^{i} p_n^{(k)}) \,. \tag{17}$$

But Khintchine intended to preserve the properties which obtained Steinhaus with respect to (11) and (12). Therefore, he modelled a sequence of independent and centred random variables

$$z_n = y_n - a_n \quad , \qquad a_n = \sum_i y_n^{(i)} p_n^{(i)} \quad , \tag{18}$$

where the independence demanded to modify (17).

He took $\varphi_1(x) = z_1^{(i)} = y_1^{(i)} - a_1$ for $x \in I_1^{(i)}$ and divided every constant interval of φ_{n-1} in intervals with lengths proportional to $p_n^{(1)}$, $p_n^{(2)}, \ldots$ and set on the j-th partial interval: $\varphi_n(x) = z_n^{(j)} = y_n^{(j)} - a$.

Let be $$b_n = \sum_i (y_n^{(i)} - a_n)^2 p_n^{(i)} \quad , \tag{19}$$

Then Khintchine and Kolmogorov formulated and proved with different methods

Theorem. If the series $\sum_n a_n$ and $\sum_n b_n$ converge, then the series (15) converges almost everywhere.

In the case of bounded (discrete) random variables the convergence of the two series is also necessary for the convergence a.e. of (15). Three years later Kolmogorov generalised this Theorem to a Zero-one law for sums of independent random variables (cf. [24], [25]).

8. Conclusions

This note examined the beginning of modern probability theory on the basis of the papers written by four Eastern European mathematicians. A. Łomnicki [27] introduced the term "probability" as a set function, defined by a density and Lebesgue integral, on L-measurable subsets of a fixed part M of an Euclidean space. He considered separately a finite set M provided with weights for every point of M, and studied deviations from Laplace's calculus of probability.

H.Steinhaus [49] constructed a set-theoretical model for Bernoulli's sequence of independent repeated trials with equal chance of success or failure. He chose the set M of all zero-one sequences, and events as subsets of M which fulfil certain axioms. With a one-to-one mapping between M and the unit interval [0,1] he reached to the first probability space

$$\{[0,1],\mathrm{L},\lambda^{[1]}\}\quad , \tag{20}$$

where L denotes the set of all L-measurable subsets of [0,1] and $\lambda^{[1]}$ the Lebesgue measure in R^1. On this base he solved the problem of asymptotic distribution of digits and formulated a zero-one law of a special random series.

A. Khintchine and A. Kolmogorov [21] generalised Steinhaus' model to a sequence of independent discrete random variables. For sums of such variables they gave a criterion of convergence. This was the starting-point for a development of probability theory which found with *Grundbegriffe der Wahrscheinlichkeitsrechnung* acceptance and broadcasting in the general public of mathematicians (cf. [25], [11]).

Problems raised in [27] and [49] were studied and solved by Steinhaus and his doctoral students in Lwów in the 1930s. Z.Łomnicki and S.Ulam [28] introduced product measures in product spaces, M.Kac investigated stochastically independent functions, and further intensive research was published in the monograph *Theorie der Orthogonalreihen* by S.Kaczmarz and Steinhaus (cf. [38],[16],[17]).

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