

CONFORMAL HOLONOMY OF BI-INVARIANT METRICS

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ABSTRACT. We discuss in this paper the conformal geometry of bi-invariant metrics on compact semisimple Lie groups. For this purpose we develop a conformal Cartan calculus adapted to this problem. In particular, we derive an explicit formula for the holonomy algebra of the normal conformal Cartan connection of a bi-invariant metric. As an example, we apply this calculus to the group $SO(4)$. Its conformal holonomy group is calculated to be $SO(7)$.

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1. INTRODUCTION

In Riemannian geometry the concept of holonomy is a well studied problem. The holonomy groups arise from affine connections on Riemannian manifolds. Most famous is the holonomy theory of the Levi-Civita connection, which is the canonical connection to a Riemannian metric. A list of possible holonomy groups in this case was first established in [Ber55].

In conformal geometry, there exists no canonical affine connection. Any choice of a torsion-free affine connection preserving the structure group (Weyl connection) is an additional datum to the conformal structure on a manifold. However, taking into consideration that conformal geometry should be understood as a 'second order' structure, it turns out and is a well-known fact that there exists a canonical Cartan connection on the prolonged principal fibre bundle of second order, which has a parabolic of the Möbius group as structure group. This canonical connection solves Cartan's equivalence problem for conformal geometry, which says that two conformal structures are equivalent if and only if the canonical connections coincide. The canonical connection on a conformal space takes in general values in the Lie algebra $\mathfrak{so}(1, n + 1)$ of the Möbius group $SO(1, n + 1)$. As in the case of Riemannian geometry, the knowledge of the holonomy algebra and group defined to this canonical conformal Cartan connection is a significant invariant of a space with conformal structure. In particular, it describes invariant structures and solutions of invariant differential operators on a conformal space.

The aim of this paper is to develop and apply a conformal Cartan calculus to a very simple situation, namely the conformal geometry of bi-invariant metrics

on compact semisimple Lie groups, in order to calculate the conformal holonomy explicitly. We proceed as follows. In the next two paragraphs, we recall briefly the basic concept of conformal Cartan geometry, in general, and the well-known notion of bi-invariant metrics. In paragraph 4, we develop the conformal Cartan calculus on bi-invariant metrics. In particular, we will describe the canonical connection and its curvature on a semisimple group N by certain maps γ_{nor} and κ , which live on the flat model $(\mathfrak{so}(1, n+1), \mathfrak{p})$. Then we discuss properties of these maps and derive an explicit formula for the holonomy algebra (paragraph 5). Finally, we make explicit calculations for the bi-invariant metric on $SO(4)$, in particular, we derive the holonomy group.

It is the project of a forthcoming paper (cf. [Lei04b]) to investigate the conformal holonomy in more general situations e.g. like invariant metrics on (reductive) homogenous Riemannian spaces. Even more generally, it is the idea to find explicit formulas and calculations for the holonomy of homogenous parabolic geometries admitting a canonical Cartan connection (remember that conformal geometry is an example for a $|1|$ -graded parabolic geometry). With homogenous space we do not mean here a space with flat canonical connection, but a space with a parabolic geometry whose automorphism group acts transitively on the base space. Beside conformal geometry, a further classical example for this approach would be CR-geometry, which is a $|2|$ -graded parabolic geometry.

2. CONFORMAL CARTAN GEOMETRY

We describe here briefly the conformal structure of a smooth manifold uniquely determined by a normal Cartan connection on a second order principal fibre bundle with parabolic structure group (cf. [Kob72], [CSS97]). We start with recalling the flat homogenous model of conformal geometry.

Let $G = SO(1, n+1)$ be the Lorentzian group, which acts on the $(n+2)$ -dimensional Minkowski space $\mathbb{R}^{1, n+1}$ equipped with the scalar product

$$\langle x, x \rangle_{1, n+1} = 2x_0x_{n+1} + \sum_{i=1}^n x_i^2 .$$

Its Lie algebra $\mathfrak{g} = \mathfrak{so}(1, n+1)$ is $|1|$ -graded by

$$\mathfrak{g} = \mathfrak{m}_{-1} \oplus \mathfrak{co}(n) \oplus \mathfrak{m}_1 ,$$

where $\mathfrak{m}_{-1} = \mathbb{R}^n$ and $\mathfrak{m}_1 = \mathbb{R}^{n*}$ are dual vector spaces via the Killing form on \mathfrak{g} . The 0-part $\mathfrak{p}_0 := \mathfrak{co}(n)$ of this grading decomposes to the semisimple part $\mathfrak{so}(n)$ and the center \mathbb{R} , which is responsible for the conformal weight of a representation of \mathfrak{g} . In matrix form, we have

$$\begin{pmatrix} 0 & 0 & 0 \\ m & 0 & 0 \\ 0 & -{}^t m & 0 \end{pmatrix} \in \mathfrak{m}_{-1} , \quad \begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \in \mathfrak{p}_0 , \quad \begin{pmatrix} 0 & l & 0 \\ 0 & 0 & -{}^t l \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{m}_1 .$$

The commutators are then given by

$$\begin{aligned} [,] : \mathfrak{p}_0 \times \mathfrak{p}_0 &\rightarrow \mathfrak{p}_0 , & [(A, a), (A', a')] &= (AA' - A'A, 0) \\ [,] : \mathfrak{p}_0 \times \mathfrak{m}_{-1} &\rightarrow \mathfrak{m}_{-1} , & [(A, a), m] &= Am + am \\ [,] : \mathfrak{m}_1 \times \mathfrak{p}_0 &\rightarrow \mathfrak{m}_1 , & [l, (A, a)] &= lA + al \\ [,] : \mathfrak{m}_{-1} \times \mathfrak{m}_1 &\rightarrow \mathfrak{p}_0 , & [m, l] &= (ml - {}^t(ml), lm) , \end{aligned}$$

where $(A, a), (A', a') \in \mathfrak{so} \oplus \mathbb{R}$, $m \in \mathbb{R}^n$, $l \in \mathbb{R}^{n*}$. The subalgebra

$$\mathfrak{p} := \mathfrak{co}(n) \oplus \mathfrak{m}_1$$

is parabolic, i.e., it contains a maximal solvable subalgebra of $\mathfrak{so}(1, n+1)$.

Let P be the closed subgroup of G consisting of the matrices

$$\left\{ \left(\begin{array}{ccc} a^{-1} & v & b \\ 0 & A & r \\ 0 & 0 & a \end{array} \right) \middle| \begin{array}{l} A \in SO(n), a \in \mathbb{R} \setminus 0, v \in \mathbb{R}^{n*}, \\ r = -aA^t v, b = -\frac{a}{2} \langle {}^t v, {}^t v \rangle_n \end{array} \right\}.$$

The group P is the stabilizer of the point $o = [1 : 0 : \cdots : 0]$ in the projectivation $P_{n+1}(\mathbb{R})$ of the Minkowski space $\mathbb{R}^{1,n+1}$ and the Lie algebra of P is the parabolic \mathfrak{p} . The homogenous space G/P is the n -dimensional Möbius sphere S^n with standard conformally flat structure, which is induced here from the scalar product on $\mathbb{R}^{1,n+1}$, and G acts by conformal automorphisms on S^n . Furthermore, let P_1 be the closed subgroup of P given by the matrices

$$\left\{ \left(\begin{array}{ccc} 1 & v & b \\ 0 & I_n & -{}^t v \\ 0 & 0 & 1 \end{array} \right) \middle| v \in \mathbb{R}^{n*}, b = -\frac{1}{2} \langle {}^t v, {}^t v \rangle \right\}.$$

This is the vector group $\exp \mathfrak{m}_1$ with Lie algebra \mathfrak{m}_1 and it is the kernel of the linear isotropy representation of P acting on $T_o S^n$ (the point o can be thought of as the point at infinity when conformally compactifying the Euclidean space \mathbb{R}^n). Since P_1 is normal, the quotient P/P_1 is a group itself, which is isomorphic to $CO(n)$. Any element $p \in P$ can be represented in a unique way by a product $p_0 \cdot \exp l$, where $p_0 \in CO(n)$ and $l \in \mathfrak{m}_1$.

The Maurer-Cartan form

$$\omega_G : TG \rightarrow \mathfrak{g}$$

is a Cartan connection on S^n with parabolic structure group P . The Maurer-Cartan equation

$$d\omega_G = -\frac{1}{2}[\omega_G, \omega_G]$$

shows that the curvature of the Cartan connection ω_G vanishes, i.e., ω_G is a flat connection.

We apply now the flat model (G, P) to (conformally) curved spaces. Let (M^n, c) be a smooth manifold of dimension $n \geq 3$ with conformal structure c . The conformal structure c is usually given as an equivalence class of metrics $[g]$, which differ from each other only by multiplication with a positive smooth function on M (scaling function). The conformal structure c defines a first order reduction $CO(M)$ of the general linear frame bundle $GL(M)$ to the structure group $CO(n)$ in the usual manner. On the other side, the conformal structure also induces a reduction $P(M)$ of the general linear frame bundle $GL^2(M)$ of second order to the structure group P . It is

$$P(M)/P_1 \cong CO(M).$$

Then the $CO(n)$ -invariant lifts from $CO(M)$ to $P(M)$ are in 1-to-1 correspondence with Weyl connections on (M, c) , i.e., affine connections with structure group $CO(n)$ and without torsion.

The principal P -fibre bundle $P(M)$ admits a canonical Cartan connection ω_{nor} , which, in particular, describes the embedding of $P(M)$ into $GL^2(M)$, and therefore determines the conformal structure c on M . The canonical connection is made unique by imposing a normalisation condition. To say it in detail, the Cartan connection

$$\omega_{nor} : TP(M) \rightarrow \mathfrak{g}$$

has the properties:

- (1) $\omega_{nor}(p) : T_p P(M) \rightarrow \mathfrak{g}$ is an isomorphism for all $p \in P(M)$, which is along the fibre the tautological map, and
- (2) $R_p^* \omega_{nor} = \text{Ad}(p^{-1}) \omega_{nor}$, where R_p denotes right translation by p on $P(M)$.

The curvature of ω_{nor} is defined by

$$K := d\omega_{nor} + \frac{1}{2}[\omega_{nor}, \omega_{nor}] ,$$

where

$$[\omega_{nor}, \omega_{nor}](X, Y) := [\omega_{nor}(X), \omega_{nor}(Y)] - [\omega_{nor}(Y), \omega_{nor}(X)]$$

for all $X, Y \in TP(M)$. It holds $R_p^*K = \text{Ad}(p^{-1})K$ and K is in vertical direction trivial, i.e., inserting a vertical vector on $P(M)$ into K produces zero. Alternatively, the curvature is uniquely determined by the function

$$\kappa = \kappa_{-1} \oplus \kappa_0 \oplus \kappa_1 : P(M) \rightarrow \mathfrak{m}_{-1}^* \otimes \mathfrak{m}_{-1}^* \otimes \mathfrak{g} ,$$

which is defined by

$$\kappa_p(a, b) = K_p(\omega_{nor}^{-1}(a), \omega_{nor}^{-1}(b))$$

for all $p \in P(M)$ and $a, b \in \mathfrak{m}_{-1}$. The normalisation condition, which makes the connection ω_{nor} unique, is expressed by the curvature properties

- (1) $\kappa_{-1} = 0$, i.e., the connection ω_{nor} is torsion-free, and
- (2) the trace-free condition $\text{tr}\kappa_0 = \sum_{i=1}^n \kappa_0(e_i, a)(b)(e_i^*) = 0$ for all $a, b \in \mathfrak{m}_{-1}$.

Thereby, the e_i 's denote the standard basis of $\mathfrak{m}_{-1} \cong \mathbb{R}^n$ and the e_i^* 's are dual in \mathfrak{m}_1 .

The canonical normal Cartan connection and its curvature can be described with respect to a metric g in the conformal class c as follows. The metric g determines a Weyl connection σ_g , i.e., an invariant lift from $CO(M)$ to $P(M)$. Then the pull-back $\sigma_g^*\omega_{-1}$ of the (-1) -part of ω_{nor} gives the soldering form on $CO(M)$, which in turn identifies (with respect to a base frame) the tangent space of M at a point with \mathfrak{m}_{-1} . The 0-part $\sigma_g^*\omega_0$ is the Levi-Civita connection form to g on $CO(M)$ and $-\sigma_g^*\omega_1$ is tensorial and projects to (M, g) as the Schouten tensor ('rho'-tensor), which is defined by

$$L(X) = \frac{1}{n-2} \left(\frac{\text{scal}}{2(n-1)} X - \text{Ric}(X) \right) ,$$

where Ric denotes the Ricci tensor and scal the scalar curvature to g on M . The (harmonic) 0-part K_0 of the curvature of ω_{nor} corresponds to the Weyl tensor W on (M, g) , which is the trace-free part of the Riemannian curvature tensor R , and can be expressed by

$$W := R - g * L ,$$

where $g * L$ denotes a Kulkarni-Nomizu product. Finally, the negative $-K_1$ of the 1-part of the curvature projects to the Cotton-York tensor C on (M, g) , which is given in terms of g by

$$C(X, Y) := \nabla_X^g L(Y) - \nabla_Y^g L(X) .$$

3. BI-INVARIANT METRICS

We recall here the notion of bi-invariant metrics on compact semisimple Lie groups. They will be the matter of our investigation when we apply in the next paragraph the Cartan calculus to their conformal classes.

Let N be a connected and compact semisimple Lie group of dimension n and let \mathfrak{n} denote its Lie algebra. The Killing form

$$B(X, Y) := \text{tr ad}_X \text{ad}_Y$$

is $\text{Ad}(N)$ -invariant and negative definite on \mathfrak{n} . In particular,

$$B(X, [Y, Z]) = B([X, Y], Z) \quad \text{for all } X, Y, Z \in \mathfrak{n} .$$

The negative $-B$ of the Killing form defines through left translation with the group multiplication an invariant metric $g_{\mathfrak{n}}$ on N . In fact, the metric $g_{\mathfrak{n}}$ is not only left-invariant, but also right-invariant under the group multiplication and the metric $g_{\mathfrak{n}}$ is called a bi-invariant metric on N (cf. [O'N83]). In the following, we will often identify left-invariant vector fields on N with their generators in \mathfrak{n} ,

$$\tilde{X}(n) := \frac{d}{dt}|_{t=0} n \cdot \exp tX \quad \mapsto \quad X \in \mathfrak{n} .$$

For the Levi-Civita connection ∇ of $g_{\mathfrak{n}}$, it holds

$$\nabla_X Y = \frac{1}{2}[X, Y] \quad \text{for all } X, Y \in \mathfrak{n} .$$

The Riemannian curvature tensor is then given by

$$R_{XY}Z = -\frac{1}{4}[[X, Y], Z] \quad \text{for all } X, Y, Z \in \mathfrak{n}$$

and the sectional curvature of a plane spanned by orthonormal elements X, Y in \mathfrak{n} is

$$S(X, Y) := -\frac{1}{4}B([X, Y], [X, Y]) .$$

For the Ricci tensor we find

$$Ric|_{\mathfrak{n}} = -\frac{1}{4}B ,$$

i.e., $Ric = \frac{1}{4}g_{\mathfrak{n}}$ and $g_{\mathfrak{n}}$ is an Einstein metric on N with positive scalar curvature $scal = \frac{n}{4}$. The Schouten tensor is $L = -\frac{1}{8(n-1)}g_{\mathfrak{n}}$ and the Cotton-York tensor C vanishes identically, $C \equiv 0$. For the Weyl tensor, we obtain

$$W = R + \frac{1}{8(n-1)}g_{\mathfrak{n}} * g_{\mathfrak{n}} .$$

4. CONFORMAL CARTAN GEOMETRY AND BI-INVARIANT METRICS

At the end of the last paragraph we calculated already the content of the conformal curvature, the Weyl tensor and the Cotton-York tensor, in terms of the bi-invariant metric on a compact semisimple group. However, we want to establish here a conformal Cartan calculus for bi-invariant metrics. This will help us to get a better understanding of the normal connection and its curvature. Though, our approach will always use a convenient trivialisation, which represents the bi-invariant metric in the conformal class.

Let N be a connected and compact semisimple Lie group of dimension n with Lie algebra \mathfrak{n} and bi-invariant metric $g_{\mathfrak{n}}$. Then there is the conformal structure $c_{\mathfrak{n}} = [g_{\mathfrak{n}}]$ defined on the group N . Let

$$\theta : (\mathfrak{n}, -B) \cong (\mathfrak{m}_{-1}, \langle \cdot, \cdot \rangle_n)$$

be an isometry and (e_1, \dots, e_n) the standard basis on $\mathfrak{m}_{-1} \cong \mathbb{R}^n$. The map θ transfers the Lie bracket $[\cdot, \cdot]_{\mathfrak{n}}$ to \mathfrak{m}_{-1} through the expression

$$\rho_{\mathfrak{n}, \theta}(a, b) := \theta[\theta^{-1}(a), \theta^{-1}(b)]_{\mathfrak{n}} ,$$

where $a, b \in \mathfrak{m}_{-1}$. (Likewise, we will also use the notation $[\cdot, \cdot]_{\mathfrak{n}}$ on \mathfrak{m}_{-1}). Moreover, the map θ induces the orthonormal frame

$$\{E_i := \theta^{-1}(e_i) \mid i = 1, \dots, n\}$$

on $\mathfrak{n} \cong T_e N$. The corresponding left-invariant frame field gives rise to a global trivialisation of the first order conformal frame bundle $CO(N)$ on (N, c) ,

$$\begin{aligned} CO(N) &\cong N \times CO(n) \\ \{\tilde{E}_i(s) \mid i = 1, \dots, n\} &\mapsto (s, e) \end{aligned}$$

(in fact, it is a trivialisation of the orthonormal frame bundle of (N, g_n)). We denote by $P(N)$ the second order conformal frame bundle with parabolic structure group P , which is a subbundle of $GL^2(N)$. The bi-invariant metric g_n induces an invariant lift

$$\sigma_{g_n} : CO(N) \rightarrow P(N)$$

and with the help of the \tilde{E}_i 's also a trivialisation

$$\iota_{\theta, g_n} : P(N) \cong N \times P$$

of the parabolic frame bundle. The left translation $L_s : N \rightarrow N$ preserves for all $s \in N$ the bi-invariant metric, hence the conformal structure c_n . The translation L_s on N induces in a natural way transformations (also denoted by L_s) of the conformal frame bundles $CO(N)$ and $P(N)$. As we have chosen here the trivialisation ι_{θ, g_n} , it holds

$$\begin{aligned} \iota_{\theta, g_n} \circ L_s \circ \iota_{\theta, g_n}^{-1} : N \times P &\rightarrow N \times P . \\ (n, p) &\rightarrow (sn, p) \end{aligned}$$

There exists a canonical Cartan connection on $P(N)$ denoted by

$$\omega_{nor} : TP(N) \rightarrow \mathfrak{g} .$$

This connection is determined by its curvature properties, namely the curvature K is torsion-free and satisfies the trace-free condition on the 0-part K_0 (cf. paragraph 2). By definition, the connection ω_{nor} is right-invariant along the fibres, i.e.,

$$R_p^* \omega_{nor} = \text{Ad}(p^{-1}) \omega_{nor} \quad \text{for all } p \in P ,$$

and it is also left-invariant under the group multiplication on N , i.e.,

$$L_s^* \omega_{nor} = \omega_{nor} .$$

The latter fact is, because, the normal Cartan connection is uniquely determined by the conformal structure and is therefore invariant under the conformal automorphism group. These two facts, the right-invariance by multiplication with P and the left-invariance with respect to N , imply that ω_{nor} on $P(N)$ is uniquely determined by the (vector space) isomorphism induced from ω_{nor} at a single point x_o of $P(N)$. Using the trivialisation ι_{θ, g_n} from above, we can choose this point to $x_o = (e, e)$ and then the isomorphism

$$\omega_{nor}(e, e) : \mathfrak{n} \times \mathfrak{p} \rightarrow \mathfrak{g}$$

determines the canonical Cartan connection. Since the (-1) -part of ω_{nor} corresponds to the soldering form, it holds $\omega_{-1}(e, e)(E_i) = e_i$. This leads us to the definition of the map γ_{nor} (which depends on the choice of θ) through

$$\begin{aligned} \gamma_{nor} : \mathfrak{m}_{-1} &\rightarrow \mathfrak{p} , \\ a &\mapsto \pi_{\mathfrak{p}} \circ \omega_{nor}(e, e) \circ \pi_{\mathfrak{n}} \circ \omega_{nor}^{-1}(e, e)(a) \end{aligned}$$

where $\pi_{\mathfrak{p}}$ and $\pi_{\mathfrak{n}}$ denote the obvious projections (the latter with respect to our trivialisation). This map decomposes to

$$\gamma_{nor} = \gamma_0 + \gamma_1$$

and it still contains the whole information of the canonical Cartan connection ω_{nor} on N , since it holds the relation

$$\omega_{nor}(E_i) = e_i + \gamma_{nor}(e_i) .$$

In fact, ω_{nor} is recovered from γ_{nor} by the latter relation, application of the trivialisation and translation from the left and right.

The curvature K inherits the left- and right-invariance properties from the canonical connection ω_{nor} and K is also determined by its values at the single point (e, e) in $P \times N$. With respect to the trivialisation ι_{θ, g_n} , we find that

$$\begin{aligned} E_i(\omega_{nor}(E_j))(e, e) &= \frac{d}{dt} \Big|_{t=0} \omega_{nor}(E_j)(\exp tE_i, e) = \frac{d}{dt} \Big|_{t=0} L_{\exp tE_i}^* \omega_{nor}(E_j)(e, e) \\ &= \frac{d}{dt} \Big|_{t=0} \omega_{nor}(E_j)(e, e) = 0 \end{aligned}$$

for all $i, j = 1, \dots, n$. This shows for the curvature the identity

$$K(E_i, E_j) = -\omega_{nor}(e, e)([E_i, E_j]_{\mathfrak{n}}) + [e_i + \gamma_{nor}(e_i), e_j + \gamma_{nor}(e_j)]_{\mathfrak{g}}.$$

The curvature function κ of the canonical Cartan connection ω_{nor} can then be expressed by (recall that the curvature is vertically trivial)

$$\kappa(e_i, e_j) = -(id + \gamma_{nor}) \circ \rho_{\mathfrak{n}, \theta}(e_i, e_j) + [e_i + \gamma_{nor}(e_i), e_j + \gamma_{nor}(e_j)]_{\mathfrak{g}}.$$

Thereby, the (-1) -part of κ is given through

$$\kappa_{-1}(e_i, e_j) = -\rho_{\mathfrak{n}, \theta}(e_i, e_j) + [e_i, \gamma_0(e_j)] + [\gamma_0(e_i), e_j].$$

This expression vanishes, since ω_{nor} has no torsion. We see that the Lie bracket of \mathfrak{n} is given on \mathfrak{m}_{-1} by

$$\rho_{\mathfrak{n}, \theta}(e_i, e_j) = -\gamma_0(e_j) \cdot e_i + \gamma_0(e_i) \cdot e_j. \quad (1)$$

The 0 -part of κ is

$$\kappa_0(e_i, e_j) = -\gamma_0 \circ \rho_{\mathfrak{n}, \theta}(e_i, e_j) + [e_i, \gamma_1(e_j)] + [\gamma_1(e_i), e_j] + [\gamma_0(e_i), \gamma_0(e_j)].$$

This part satisfies the trace-free condition

$$\sum_{i=1}^n \gamma_0 \circ \rho_{\mathfrak{n}, \theta}(e_i, a)(b)(e_i^*) = \left\{ \begin{array}{l} \sum_{i=1}^n [e_i, \gamma_1(a)](b)(e_i^*) + [\gamma_1(e_i), a](b)(e_i^*) \\ + \sum_{i=1}^n [\gamma_0(e_i), \gamma_0(a)](b)(e_i^*) \end{array} \right\} \quad (2)$$

for all $a, b \in \mathfrak{m}_{-1}$. The 1 -part κ_1 of the curvature is

$$\kappa_1(e_i, e_j) = -\gamma_1 \circ \rho_{\mathfrak{n}, \theta}(e_i, e_j) + [\gamma_0(e_i), \gamma_1(e_j)] + [\gamma_1(e_i), \gamma_0(e_j)]$$

for all $i, j \in \{1, \dots, n\}$.

The linear map $\gamma_{nor} : \mathfrak{m}_{-1} \rightarrow \mathfrak{p}$ is uniquely determined by the normalisation conditions (1) and (2) with respect to $\rho_{\mathfrak{n}, \theta}$ (which depends on the choice of the frame θ). (Otherwise, we would recover from another γ which shares these properties a further normal connection, which is not possible.) So γ_{nor} depends only on the choice of θ which induces the Lie bracket of \mathfrak{n} on \mathfrak{m}_{-1} . We can introduce the following formal notions.

Definition 1. *Let (G, P) be the flat homogenous model (of conformal geometry) and*

$$\rho : \mathfrak{m}_{-1} \times \mathfrak{m}_{-1} \rightarrow \mathfrak{m}_{-1}$$

a skew-symmetric map, which satisfies the Jacobi identity and defines a Lie algebra bracket (of compact type) on the (-1) -part \mathfrak{m}_{-1} of the grading of \mathfrak{g} .

(1) *We call a linear map*

$$\gamma = \gamma_0 + \gamma_1 : \mathfrak{m}_{-1} \rightarrow \mathfrak{p}$$

(from the (-1) -part to the parabolic) a connection form on the model (G, P) .

(2) *The curvature*

$$\kappa_{\gamma, \rho} = \kappa_{-1} + \kappa_0 + \kappa_1 : \mathfrak{m}_{-1} \times \mathfrak{m}_{-1} \rightarrow \mathfrak{g}$$

of the connection γ with respect to the Lie bracket ρ is defined as

$$\kappa_{\gamma, \rho}(a, b) = -(id + \gamma) \circ \rho(a, b) + [(id + \gamma)(a), (id + \gamma)(b)]_{\mathfrak{g}}$$

for $a, b \in \mathfrak{m}_{-1}$.

- (3) The connection γ is called *torsion-free* with respect to ρ if $\kappa_{-1} = 0$.
(4) The connection γ is called *normal* with respect to ρ if

$$\kappa_{-1} = 0 \quad \text{and} \quad \text{tr } \kappa_0 = 0$$

(cf. equation (2)).

- (5) There exists a unique normal connection with respect to the bracket ρ . We denote it by $\gamma_\rho : \mathfrak{m}_{-1} \rightarrow \mathfrak{p}$ (or γ_{nor} when the bracket is fixed on \mathfrak{m}_{-1}) and call it the *canonical connection form of ρ to the model (G, P)* .

As we can see from formula (1), the Lie bracket of \mathfrak{n} is determined on \mathfrak{m}_{-1} by the 0-part γ_0 of the normal connection γ_{nor} , since it has no torsion. In general, a skew-symmetric map

$$\gamma_0 : \mathfrak{m}_{-1} \rightarrow \mathfrak{p}_0$$

defines a Lie bracket on \mathfrak{m}_{-1} through

$$\rho_{\gamma_0}(a, b) = -\gamma_0(b) \cdot a + \gamma_0(a) \cdot b \quad \text{for all } a, b \in \mathfrak{m}_{-1}$$

if and only if the following sum of even permutations satisfies the relation (Jacobi identity)

$$\sum_{\sigma(i,j,k)} [\gamma_0[e_i, \gamma_0(e_j)] + \gamma_0[\gamma_0(e_i), e_j] - [\gamma_0(e_i), \gamma_0(e_j)], e_k] = 0 \quad (3)$$

for all $i, j, k \in \{1, \dots, n\}$. The map γ_0 can then be extended in an arbitrary manner to a torsion-free connection form γ with respect to ρ_{γ_0} just by adding any linear 1-part γ_1 . In such a situation, the curvature function to γ with respect to ρ_{γ_0} is given by

$$\begin{aligned} \kappa_\gamma(e_i, e_j) &= -\gamma_0([e_i, \gamma_0(e_j)] + [\gamma_0(e_i), e_j]) + [e_i, \gamma_1(e_j)] + [\gamma_1(e_i), e_j] \\ &\quad + [\gamma_0(e_i), \gamma_0(e_j)] \\ &\quad - \gamma_1([e_i, \gamma_0(e_j)] + [\gamma_0(e_i), e_j]) + [\gamma_0(e_i), \gamma_1(e_j)] + [\gamma_1(e_i), \gamma_0(e_j)]. \end{aligned}$$

Of course, not every torsion-free map γ_0 can be extended to the normal connection γ_{nor} with respect to ρ_{γ_0} . The condition on γ_0 for being normally extendible is given by the existence of a γ_1 such that

$$\sum_{i=1}^n ([e_i, \gamma_1(a)] + [\gamma_1(e_i), a])(b)(e_i^*) = \begin{cases} \sum_{i=1}^n \gamma_0([e_i, \gamma_0(a)] + [\gamma_0(e_i), a])(b)(e_i^*) \\ - \sum_{i=1}^n [\gamma_0(e_i), \gamma_0(a)](b)(e_i^*) \end{cases} \quad (4)$$

for all $a, b \in \mathfrak{m}_{-1}$.

Finally, in this paragraph, we want to give explicit expressions for the maps γ_0 and γ_1 of the normal connection form γ_{nor} and also for its curvature function κ when \mathfrak{n} is a semisimple Lie algebra (of compact type). The map γ_0 corresponds in this case to the Levi-Civita connection of the bi-invariant metric $g_{\mathfrak{n}}$ and is given with respect to a reference frame θ by

$$\gamma_0(e_i) = -\theta \circ \nabla_{\theta^{-1}(\cdot)} E_i = \frac{1}{2} \rho_{\mathfrak{n}, \theta}(e_i, \cdot)$$

for all $i = 1, \dots, n$. Obviously, the so-defined map γ_0 , considered as a matrix in $\mathfrak{p}_0 = \mathfrak{co}(n)$ with respect to the basis (e_1, \dots, e_n) , satisfies (1) (also (2)), i.e., γ_0 is torsion-free with respect to \mathfrak{n} (and θ). Then we calculate for the traces on the left hand side in (4):

$$\begin{aligned} \sum_{i=1}^n \gamma_0([e_i, \gamma_0(a)] + [\gamma_0(e_i), a])(b)(e_i^*) &= \frac{1}{2} B_{\mathfrak{n}}(a, b), \\ \sum_{i=1}^n [\gamma_0(e_i), \gamma_0(a)](b)(e_i^*) &= \frac{1}{4} B_{\mathfrak{n}}(a, b). \end{aligned}$$

We set $\lambda = \frac{-1}{8(n-1)}$ and

$$\gamma_1(a) = \lambda a^*$$

for all $a \in \mathfrak{m}_{-1}$. Calculation of the right hand side in (4) gives

$$\sum_{i=1}^n ([e_i, \gamma_1(e_k)] + [\gamma_1(e_i), e_k])(e_l)(e_i^*) = 2\lambda(n-1)\delta_{kl}$$

for all $k, l \in \{1, \dots, n\}$. Comparing both sides of (4) proves that the normal connection form for \mathfrak{n} is determined to

$$\begin{aligned} \gamma_{nor} : \mathfrak{m}_{-1} &\rightarrow \mathfrak{p} . \\ a &\mapsto \frac{1}{2}\rho_{\mathfrak{n},\theta}(a, \cdot) - \frac{1}{8(n-1)}a^* \end{aligned}$$

The curvature functions κ_{-1} and κ_1 vanish identically, since there is no torsion and the Cotton-York tensor C of $g_{\mathfrak{n}}$ is trivial. The 0-part κ_0 is given by the Weyl tensor W of $g_{\mathfrak{n}}$. It is

$$\kappa(a, b) = \kappa_0(a, b) = \theta^*W(\theta^{-1}(a), \theta^{-1}(b)) .$$

5. CANONICAL CARTAN CONNECTION AND HOLONOMY

In this paragraph, we conduct a further discussion of the connection form γ_{nor} . However, one should keep in mind that properties of γ_{nor} depend on the choice of the trivialisation which comes from the bi-invariant metric, and therefore γ_{nor} should be considered as a 'metric object'. Nevertheless, we will derive an explicit formula for its holonomy algebra, which is then a 'purely' conformal invariant.

Let

$$\gamma_{nor} = \gamma_0 + \gamma_1 : \mathfrak{m}_{-1} \rightarrow \mathfrak{p}$$

be the normal connection to some semisimple Lie algebra \mathfrak{n} (of compact type) with induced bracket

$$\rho_{\mathfrak{n}} = [,]_{\mathfrak{n}}$$

on \mathfrak{m}_{-1} (and with respect to $\sigma_{g_{\mathfrak{n}}}$ and some θ). The first observation here is the following. One can show that the image of \mathfrak{m}_{-1} under the 0-part γ_0 of γ_{nor} is the Lie subalgebra $Der(\mathfrak{n})$ in \mathfrak{p}_0 consisting of all derivations of $\mathfrak{n} \cong \mathfrak{m}_{-1}$. Since \mathfrak{n} is semisimple, the Lie algebra \mathfrak{n} itself is naturally isomorphic to $Der(\mathfrak{n})$ by

$$\begin{aligned} \text{ad} : \mathfrak{n} &\rightarrow Der(\mathfrak{n}) . \\ a &\rightarrow \text{ad}_a \end{aligned}$$

However, we will see that the map $\gamma_0 : \mathfrak{m}_{-1} \rightarrow \mathfrak{p}_0$ is not a Lie algebra isomorphism with respect to $\rho_{\mathfrak{n}}$ on \mathfrak{m}_{-1} . In detail, we can verify these statements as follows. Remember that the normal connection form to \mathfrak{n} is given by

$$\gamma_{nor}(a) = \frac{1}{2}\rho_{\mathfrak{n}}(a, \cdot) - \frac{1}{8(n-1)}a^* .$$

With this expression in mind, it is obvious that the Jacobi identity in \mathfrak{n} implies

$$\gamma_0(x) \cdot [a, b]_{\mathfrak{n}} = [\gamma_0(x) \cdot a, b] + [a, \gamma_0(x) \cdot b]$$

for all $x, a, b \in \mathfrak{m}_{-1}$, where the 'dot' denotes matrix multiplication of \mathfrak{p}_0 on \mathfrak{m}_{-1} . This shows that $\gamma_0(x)$ is a derivation on $\mathfrak{n} \cong (\mathfrak{m}_{-1}, \rho_{\mathfrak{n}})$ for all $x \in \mathfrak{m}_{-1}$. Moreover, the kernel of γ_0 is trivial, since $\gamma_0(x) = 0$ implies that x is in the center of $\rho_{\mathfrak{n}}$, which itself is trivial for semisimple \mathfrak{n} . We can conclude that γ_0 is a vector space isomorphism onto $Der(\mathfrak{n})$ and, since $B_{\mathfrak{n}}([a, b]_{\mathfrak{n}}, b) = 0$ for $a, b \in \mathfrak{n}$, every derivation sits in the semisimple part $\mathfrak{so}(\mathfrak{n})$ of \mathfrak{p}_0 . In fact, the following statement is true.

Proposition 1. *Let \mathfrak{n} be a semisimple Lie algebra (of compact type) and $\gamma_{nor} = \gamma_{\mathfrak{n}}$ the corresponding connection form on the model $(\mathfrak{g}, \mathfrak{p})$ (coming from (G, P)). Then the 0-part γ_0 of γ_{nor} is with respect to \mathfrak{n} the only torsion-free map, which sends \mathfrak{m}_{-1} to the derivations $Der(\mathfrak{n})$ sitting in \mathfrak{p}_0 .*

Proof. As we have seen, the map γ_0 of γ_{nor} admits the stated properties. We have to show that any other linear map Q apart from γ_0 sending \mathfrak{m}_{-1} to the derivations $Der(\mathfrak{n})$ has torsion. This can be seen as follows. Since Q is linear and γ_0 onto $Der(\mathfrak{n})$, we can write $\gamma_0 - Q = \gamma_0 \circ A$ for some homomorphism A on \mathfrak{m}_{-1} . The map Q has no torsion if and only if

$$-\gamma_0 \circ A(e_j)e_i + \gamma_0 \circ A(e_i)e_j = 0 \quad \text{for all } i, j .$$

Since $\gamma_0 \circ A(a) \in \mathfrak{so}(n)$ for all $a \in \mathfrak{m}_{-1}$, we find in the torsion-free case the relation $\gamma_0 \circ A(a)a = 0$ for all $a \in \mathfrak{m}_{-1}$. But this also implies

$$\gamma_0 \circ A(e_j)e_i + \gamma_0 \circ A(e_i)e_j = 0 \quad \text{for all } i, j \in 1, \dots, n ,$$

which is only possible if $A = 0$, i.e., $Q = \gamma_0$. \square

Proposition 1 is a characterisation of the Levi-Civita connection to the bi-invariant metric. The stated criteria replace conditions (3) and (4) for a map γ_0 to be the 0-part of the normal connection γ_{nor} for some semisimple \mathfrak{n} .

The 0-part κ_0 of the curvature function to γ_{nor} of \mathfrak{n} satisfies the relation

$$\kappa_0(a, b) - ([a, \gamma_1(b)] + [\gamma_1(a), b]) = -\gamma_0 \circ \rho_{\mathfrak{n}}(a, b) + [\gamma_0(a), \gamma_0(b)] .$$

The expression on the left hand side of this equation is the Riemannian curvature tensor $R|_{\mathfrak{n}}$ for the bi-invariant metric $g_{\mathfrak{n}}$ in the conformal class $c_{\mathfrak{n}}$. Obviously, it measures the deviation of γ_0 from being a Lie algebra isomorphism onto $Der(\mathfrak{n})$ sitting in \mathfrak{p}_0 . Since the Riemannian curvature tensor of a semisimple Lie algebra is not zero, we can conclude that γ_0 is never a Lie algebra isomorphism. We state the following summary about the meaning of the curvature function $\kappa_{\gamma, \mathfrak{n}}$ of an (arbitrary) connection form $\gamma : \mathfrak{m}_{-1} \rightarrow \mathfrak{p}$ with respect to some semisimple Lie algebra \mathfrak{n} :

- (1) The (-1)-part κ_{-1} of $\kappa_{\gamma, \mathfrak{n}}$ measures the deviation of the expression

$$-\gamma_0(b)a + \gamma_0(a)b , \quad a, b \in \mathfrak{m}_{-1}$$

from being a defining map for the Lie bracket of \mathfrak{n} induced on \mathfrak{m}_{-1} .

- (2) The 0-part κ_0 is the deviation of the map γ_0 from being a Lie algebra homomorphism onto $Der(\mathfrak{n})$ up to an expression depending on γ_1 .
- (3) The 1-part κ_1 measures the deviation of the dual map

$$\gamma_1^* : \mathfrak{m}_{-1} \rightarrow \mathfrak{m}_{-1}$$

to γ_1 from being a derivation on $\mathfrak{m}_{-1} \cong \mathfrak{n}$.

- (4) In case that $\gamma = \gamma_{\mathfrak{n}}$ is the normal connection to \mathfrak{n} there is no torsion and γ_1 is a derivation on \mathfrak{m}_{-1} . The part γ_0 always differs from a homomorphism onto $Der(\mathfrak{n})$. The connection γ_{nor} is flat, i.e., $\kappa = 0$, if and only if $\mathfrak{n} \cong \mathfrak{su}(2)$.

We come now to the second part of this paragraph concerning the holonomy of the canonical Cartan connection. Recall that the conformal automorphism group of a compact Riemannian space is always an isometry group with respect to some metric in the conformal class except in the case when the compact space is the sphere S^n with canonical conformally flat structure (Möbius sphere). In particular, the conformal automorphism group of a compact semisimple Lie group with bi-invariant metric consists entirely of the right and left translations with respect to the group multiplication except for the case when $N = \text{SU}(2)$. However, the normal Cartan connection on a conformal space gives rise to another conformal invariant, which can be thought of as an expression for conformal symmetry on a space (and yet seems not to be naturally related to a particular metric in the conformal class), namely the holonomy group $Hol(\omega_{nor})$ of the canonical connection ω_{nor} . To define this holonomy group, we use the natural extension of the normal Cartan connection

ω_{nor} on the parabolic frame bundle $P(M)$ over a space M with conformal structure c to a usual principal fibre bundle connection $\tilde{\omega}_{nor}$ on the extended bundle

$$G(M) = P(M) \times_P G$$

with structure group $G = \text{SO}(1, n + 1)$.

Definition 2. *Let (M, c) be a space with conformal structure. The holonomy group $\text{Hol}(\omega_{nor})$ of the canonical Cartan connection ω_{nor} is defined to be the holonomy group of the naturally extended principal fibre bundle connection $\tilde{\omega}_{nor}$ on $G(M)$, whose group elements arise in the usual way by parallel translation of a fibre in $G(M)$ along closed curves on M . The Lie algebra of $\text{Hol}(\omega_{nor})$ is denoted by $\mathfrak{hol}(\omega_{nor})$.*

We remark that there is a direct way to define the holonomy group of a Cartan connection without using the extended bundle. However, this approach results in general to the same group as in our definition (cf. [Sha97]). Note also that the holonomy group $\text{Hol}(\omega_{nor})$ is always a closed subgroup of the Möbius group $G = \text{SO}(1, n + 1)$. We want to derive here a formula, which can be used for explicit calculations of the conformal holonomy on Lie groups with bi-invariant metric.

Let N be a connected and compact semisimple Lie group. Let \mathfrak{n} denote its Lie algebra with bi-invariant Riemannian metric $g_{\mathfrak{n}}$. The canonical Cartan connection ω_{nor} induces in a natural manner (with respect to the trivialisation coming from $g_{\mathfrak{n}}$) the map

$$\gamma_{nor} : \mathfrak{m}_{-1} \cong \mathfrak{n} \rightarrow \mathfrak{p} ,$$

which possesses all the informations of ω_{nor} and, in fact, can be used to recover the canonical Cartan connection on $P(N)$. We denote by

$$\Lambda(\mathfrak{m}_{-1}) := \text{span}\{(id + \gamma_{nor})(a) \mid a \in \mathfrak{m}_{-1}\} \subset \mathfrak{g}$$

the images of the normal connection and by

$$\mathfrak{q} := \text{span}\{\kappa_{\mathfrak{n}}(a, b) \mid a, b \in \mathfrak{m}_{-1}\} \subset \mathfrak{p}$$

the vector space of curvature values to the connection γ_{nor} . There is a classical formula for the holonomy algebra of an invariant connection on a homogenous space with arbitrary structure group G (cf. [KN63]). We use this result to derive here easily the following formula for the conformal holonomy algebra of a bi-invariant metric.

Theorem 1. *(cf. [KN63]) Let N be a connected and compact semisimple Lie group with conformal structure $[g_{\mathfrak{n}}]$. Then the holonomy algebra of the normal Cartan connection ω_{nor} on $(N, [g_{\mathfrak{n}}])$ is given by the expression*

$$\mathfrak{hol}(\omega_{nor}) := \mathfrak{q} + [\Lambda(\mathfrak{n}), \mathfrak{q}] + [\Lambda(\mathfrak{n}), [\Lambda(\mathfrak{n}), \mathfrak{q}]] + \dots ,$$

which is a subalgebra of $\mathfrak{g} = \mathfrak{so}(1, n + 1)$. The reduced holonomy group $\text{Hol}_o(\omega_{nor})$ is the connected subgroup of the identity component $\text{SO}_o(1, n + 1)$ belonging to $\mathfrak{hol}(\omega_{nor})$.

Usually, we do all our calculations with respect to the bi-invariant metric $g_{\mathfrak{n}}$. According to this, we want to present the above formula for the conformal holonomy algebra in a more 'suggestive' form. Let us denote

$$\mathfrak{L}\mathfrak{C} := \text{span}\{\gamma_0(a) \mid a \in \mathfrak{m}_{-1}\} \subset \mathfrak{p}_0 .$$

This space is the span of the values in $\mathfrak{so}(n)$ generated by the Levi-Civita connection of $g_{\mathfrak{n}}$ and is isomorphic as Lie algebra to the derivations $\text{Der}(\mathfrak{n})$. Then it is

$$\Lambda(\mathfrak{m}_{-1}) = \text{span}\{\gamma_0^{-1}(l) + l + \gamma_1 \circ \gamma_0^{-1}(l) \mid l \in \mathfrak{L}\mathfrak{C}\} .$$

In short, we use the notation $\Lambda(\mathfrak{m}_{-1}) = \gamma_{nor}^{-1}(\mathfrak{L}\mathfrak{C})$. This space is isomorphic to $\mathfrak{L}\mathfrak{C}$ as vector space, but it is not a subalgebra in \mathfrak{g} . Moreover, we set $\mathfrak{W} := \mathfrak{q}$, which just shall remember to the fact that \mathfrak{q} is generated from the values of the Weyl tensor W of \mathfrak{g}_n . Then our formula for the holonomy algebra takes the form

$$\mathfrak{hol}(\omega_{nor}) = \mathfrak{W} + [\gamma_{nor}^{-1}(\mathfrak{L}\mathfrak{C}), \mathfrak{W}] + [\gamma_{nor}^{-1}(\mathfrak{L}\mathfrak{C}), [\gamma_{nor}^{-1}(\mathfrak{L}\mathfrak{C}), \mathfrak{W}]] + \dots .$$

6. EXAMPLES

We want to make some explicit use of the developed calculus for conformal geometry on bi-invariant metrics.

Example 1. Let $N = \text{SO}(3)$ be the special orthogonal group in dimension 3, which is a 3-dimensional compact and semisimple Lie group. Let $\mathfrak{so}(3)$ denote its Lie algebra. We use for $\mathfrak{so}(3)$ the standard basis $\{E_{ij} \mid 1 \leq i < j \leq 3\}$, where the E_{ij} 's are defined by matrix operations as

$$E_{ij} := e_i \cdot {}^t e_j - e_j \cdot {}^t e_i$$

with respect to the standard basis (e_1, e_2, e_3) of \mathbb{R}^3 .

The Lie algebra $\mathfrak{so}(3)$ is isomorphic to $\mathfrak{su}(2)$ and the universal covering of the group $\text{SO}(3)$ is

$$S^3 = \text{Spin}(3) \cong \text{SU}(2) .$$

The bi-invariant metric on $\text{SO}(3)$ is conformally flat, since the Weyl tensor W vanishes in dimension 3 in general, and C vanishes for any bi-invariant metric. Of course, this is also clear from the fact that the bi-invariant metric on the universal covering group $\text{SU}(2)$ is the standard metric on S^3 . For that reason, the calculations of conformal curvature and holonomy should produce trivial results here. The connection form $\gamma_{nor} = \gamma_0 + \gamma_1$ can be presented in the following form.

The chosen basis $\{E_{ij}\}$ in $\mathfrak{so}(3)$ is orthogonal and $-B(E_{ij}, E_{ij}) = 2$ for all its elements. We set the frame

$$\theta\left(\frac{1}{\sqrt{2}}E_{12}\right) = e_1, \quad \theta\left(\frac{1}{\sqrt{2}}E_{13}\right) = e_2, \quad \theta\left(\frac{1}{\sqrt{2}}E_{23}\right) = e_3 .$$

Then we find

$$\begin{aligned} \gamma_0(e_1) &= -\frac{1}{\sqrt{2}}\nabla E_{12} = \frac{1}{2\sqrt{2}}[E_{12}, \cdot] = \frac{1}{2\sqrt{2}}E_{23} , \\ \gamma_0(e_2) &= -\frac{1}{\sqrt{2}}\nabla E_{13} = \frac{1}{2\sqrt{2}}[E_{13}, \cdot] = -\frac{1}{2\sqrt{2}}E_{13} , \\ \gamma_0(e_3) &= -\frac{1}{\sqrt{2}}\nabla E_{23} = \frac{1}{2\sqrt{2}}[E_{23}, \cdot] = \frac{1}{2\sqrt{2}}E_{12} . \end{aligned}$$

For γ_1 we have

$$\gamma_1(e_i) = -\frac{1}{16}e_i^* \quad i = 1, 2, 3 .$$

These identities express the normal Cartan connection on $\text{SO}(3)$ in the trivialisation of the bi-invariant metric. In fact, one can see from this γ_{nor} that the curvature κ_0 vanishes identically. In particular, the conformal holonomy group is trivial, which results from the formula in Theorem 1 with $\mathfrak{q} = 0$.

To complete the discussion, we state that the conformal automorphism group of the covering space $\text{SU}(2)$ is the Möbius group $\text{SO}(1, 4)$. However, the conformal automorphism group of $\text{SO}(3)$ consists only of the isometry group

$$\text{SO}(4)/\mathbb{Z}_2 = \text{SO}(3) \times \text{SO}(3)$$

of the bi-invariant metric.

Example 2. We apply the conformal Cartan calculus now to the 6-dimensional compact and semisimple Lie group $\text{SO}(4)$ with bi-invariant metric induced by the Killing form B . The conformal automorphism group of $\text{SO}(4)$ consists entirely of

the isometries, which are the left and right translations, i.e. $SO(4) \times SO(4)$. The Lie algebra $\mathfrak{so}(4)$ is isomorphic to

$$\mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

and as its basis we use two copies of the basis $\{E_{ij}\}$ of $\mathfrak{so}(3)$, namely

$$\{E_{ij}, E_{kl} \mid 1 \leq i < j \leq 3 \text{ and } 4 \leq k < l \leq 6\} .$$

This basis is orthogonal with $-B(E_{ij}, E_{ij}) = 2$ and it provides an embedding of $\mathfrak{so}(4)$ into $\mathfrak{so}(6)$.

The bi-invariant metric $g_{\mathfrak{so}(4)}$ induced by the Killing form on $SO(4)$ is Einstein with positive scalar curvature. Obviously, it has non-constant sectional curvature S (cf. paragraph 3). Hence, it is not conformally flat. For that reason, we expect in our calculation non-trivial curvature and holonomy for the conformal structure $c_{\mathfrak{so}(4)}$ on $SO(4)$.

First, we calculate the normal connection form $\gamma_{mor} = \gamma_0 + \gamma_1$. We use the frame

$$\begin{aligned} \theta\left(\frac{1}{\sqrt{2}}E_{12}\right) &= e_1, & \theta\left(\frac{1}{\sqrt{2}}E_{13}\right) &= e_2, & \theta\left(\frac{1}{\sqrt{2}}E_{23}\right) &= e_3, \\ \theta\left(\frac{1}{\sqrt{2}}E_{45}\right) &= e_4, & \theta\left(\frac{1}{\sqrt{2}}E_{46}\right) &= e_5, & \theta\left(\frac{1}{\sqrt{2}}E_{56}\right) &= e_6. \end{aligned}$$

From the calculations for the case of $\mathfrak{so}(3)$ we get

$$\begin{aligned} \gamma_0(e_1) &= \frac{1}{2\sqrt{2}}E_{23}, & \gamma_0(e_2) &= -\frac{1}{2\sqrt{2}}E_{13}, & \gamma_0(e_3) &= \frac{1}{2\sqrt{2}}E_{12}, \\ \gamma_0(e_4) &= -\frac{1}{2\sqrt{2}}E_{56}, & \gamma_0(e_5) &= -\frac{1}{2\sqrt{2}}E_{46}, & \gamma_0(e_6) &= \frac{1}{2\sqrt{2}}E_{45}. \end{aligned}$$

The 1-part γ_1 is given by

$$\gamma_1(e_i) = -\frac{1}{40}e_i^* \quad i = 1, \dots, 6 .$$

The (-1) -part κ_{-1} of the curvature vanishes, since γ_{mor} is torsion-free. The 1-part κ_1 also vanishes, since the Cotton-York tensor C of $g_{\mathfrak{so}(4)}$ is zero. The 0-part κ_0 consists of the Weyl tensor W . As next we have to calculate the images of κ_0 , which will give us the space \mathfrak{q} . It is

$$W = R + \frac{1}{8(n-1)}g_n * g_n .$$

For the Kulkarni-Nomizu product in this sum it is easy to see that

$$\theta^*(B * B)(\theta^{-1}(e_i), \theta^{-1}(e_j)) = \frac{1}{20}E_{ij} \quad \text{for all } i, j = 1, \dots, 6 .$$

For the Riemannian curvature tensor we find

$$\theta^*R_n(\theta^{-1}(e_i), \theta^{-1}(e_j)) = -\frac{1}{8}E_{ij}$$

for all $i, j \in \{1, \dots, 3\}$ and $i, j \in \{4, \dots, 6\}$. The remaining curvature expressions for R_n are zero. This shows that the span of the 0-part κ_0 of the Cartan curvature is equal to $\mathfrak{so}(6)$, which is the semisimple part of \mathfrak{p}_0 in the Möbius algebra $\mathfrak{so}(1, 7)$. Hence, we get for the span of the curvature values

$$\mathfrak{q} = \mathfrak{so}(6) \subset \mathfrak{p}_0 .$$

Obviously, the span of the normal connection $id + \gamma_{mor}$ is given by

$$\Lambda(\mathfrak{m}_{-1}) = \{e_i + \gamma_0(e_i) - \frac{1}{40}e_i^* \mid i = 1, \dots, 6\} .$$

Then it is a straightforward calculation to see that the space $[\Lambda(\mathfrak{m}_{-1}), \mathfrak{q}]$ of commutators is equal to

$$\text{span}\{e_i - \frac{1}{40}e_i^* \mid i = 1, \dots, 6\} \oplus \mathfrak{so}(6) .$$

We denote

$$\mathfrak{l} := \text{span}\{x + \gamma_1(x) \mid x \in \mathfrak{m}_{-1}\} .$$

The space \mathfrak{l} is stable under the action of $\mathfrak{so}(6)$ sitting in \mathfrak{p}_0 . This shows that all the spaces

$$[\Lambda(\mathfrak{m}_{-1}), \dots, [\Lambda(\mathfrak{m}_{-1}), \mathfrak{q}] \dots]$$

of commutators are equal to $\mathfrak{l} \oplus \mathfrak{q}$, which is seen to be isomorphic to the Lie algebra $\mathfrak{so}(7)$ embedded into $\mathfrak{so}(1, 7)$. We conclude for the holonomy algebra that

$$\mathfrak{hol}(\omega_{nor}) = \mathfrak{l} \oplus \mathfrak{q} \cong \mathfrak{so}(7) .$$

The holonomy group of ω_{nor} on $SO(4)$ is then given by

$$Hol(\omega_{nor}) = SO(7).$$

This result has the following interpretation. The bi-invariant metric g_n in the conformal class c_n on $N = SO(4)$ is Einstein. It is well known in general that the conformal Einstein condition implies that the holonomy group $Hol(\omega_{nor})$ of the normal Cartan connection stabilises a standard 'tractor', that is a vector in the standard representation $\mathbb{R}^{1,7}$ of the Möbius group $SO(1, 7)$ (cf. e.g. [Lei04a]). In case that the scalar curvature of the Einstein metric is positive, this vector is timelike, which explains that for our case the holonomy group $Hol(\omega_{nor})$ of $SO(4)$ is automatically reduced to $SO(7)$. Our calculation then shows that the holonomy is not further reduced and we can say that $SO(4)$ has generic conformal holonomy up to the fact that it is (conformally) Einstein. In particular, we can read off from the holonomy result that $SO(4)$ does not admit any conformal Killing spinors nor normal conformal Killing forms. Also, there is no further Einstein metric in the conformal class c_n beside the bi-invariant metric g_n (cf. [Lei04a]).

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