A NOTE ON TWISTOR SPINORS WITH ZEROS IN LORENTZIAN GEOMETRY

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Abstract. We show in this note that if a twistor spinor has a zero on a Lorentzian spin manifold M of arbitrary dimension then the twistor is almost everywhere on M locally conformally equivalent to a parallel spinor.

Contents

1. Introduction

Let (M^n, g) be a spin manifold with spinor bundle S. The twistor equation for a spinor field $\varphi \in \Gamma(S)$ given by

$$\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \qquad \text{for all } X \in TM,$$

where ∇^S denotes the spinor derivative, D is the Dirac operator and the dot '$\cdot$' denotes the Clifford multiplication, is a conformally covariant differential equation of first order. Its solutions are called twistor spinors (cf. [Pen67], [BFGK91]).

There are several structure results for solutions of the twistor equation on spinors in the case of Riemannian spin manifolds (cf. [Wang89], [BFGK91], [Bär93], [KR94], [KR98]) and also for the Lorentzian setting (cf. [Baum00], [BL04], [Lei04]). In many situations, solutions of the twistor equation can be ascribed to Killing spinors, which by definition solve additionally to the twistor equation also the Dirac equation $D\varphi = \lambda\varphi$ for some constant $\lambda \in \mathbb{C}$ and which represent the existence of a metrical (super)symmetry.

Of special interest are twistor spinors admitting a non-trival zero set, since they indicate an 'essential' conformal (super)symmetry. (Killing spinors never have zeros.) Trivial examples with zeros appear on the (pseudo)-Euclidean spaces. In the Riemannian case there is a very nice construction of a twistor spinor with a zero on the conformal completion of the Eguchi-Hanson metric to infinity (cf. [KR96]). Thereby, the Eguchi-Hanson metric admits parallel spinors which are rescaled and extended to twistors with zeros in the point at infinity. The Eguchi-Hanson metric is self-dual, but not conformally flat.

In the Lorentzian case each twistor spinor φ can be squared to produce a not identically vanishing conformal vector field V_φ. Thereby, the zero sets of the spinor

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φ and the vector field V_φ always coincide. In a zero $p \in M$ of φ, the conformal field V_φ has the properties

$$V_\varphi(p) = 0, \qquad \nabla V_\varphi(p) = 0, \qquad divV(p) = 0 \qquad \text{and} \qquad grad(divV)(p) \neq 0 \ .$$

These properties show that the conformal field is essential, that means there exists no metric $\tilde{g}$ in the conformal class $[g]$ with respect to which V_φ is a Killing vector field. Standard examples for twistors with zeros are known on the n-dimensional Minkowski spaces. There exist also examples of twistors with zeros on non-compact spaces in dimension 4, which have locally the form of a pp-wave and which are not any longer everywhere conformally flat, but are still conformally flat in a neighborhood of the zero set (cf. [Lei01]). Such examples on compact Lorentzian spaces are not known. It is also not known whether a zero can occur where the space is not conformally flat. We remark that in both these cases, the existence of twistor spinors with zeros would give rise to counter examples of the Lichnerowicz conjecture, which states that a compact Lorentzian space with essential conformal transformation group is necessarily conformally flat (cf. [D'AG91]).

An idea for constructing such twistor spinors with zero on a Lorentzian space would be to try to make the process of a conformal completion to some set at infinity (as it was successfully examined in the Riemannian case). However, to start with such an attempt, it was not clear until now, which geometric structure the underlying Lorentzian space should possess away from the set at infinity. To compare, in the Riemannian case one can show by rescaling with respect to the length function $\langle\varphi,\varphi\rangle$ of the twistor φ that the twistor must be conformally equivalent to a parallel spinor outside of its (isolated) zero points. In the Lorentzian case this exercise does not work, since non-trivial twistors may have vanishing square length $\langle\varphi,\varphi\rangle \equiv 0$. Nevertheless, we will show in this note that a similar result as in the Riemannian case is true: Every twistor with zero on a Lorentzian space is (at least) locally conformally equivalent to a parallel spinor outside of a certain singularity set.

2. Twistors as covariant constant objects

We collect here some standard facts concerning twistor spinors on Lorentzian spin manifolds (cf. [Baum00], [BL04]).

Let (M^n, g) be a connected and time-oriented Lorentzian spin manifold of dimension $n \geq 3$ with (complex) spinor bundle S. We then have an indefinite Hermitian product $\langle\cdot,\cdot\rangle$ on S and the spinor connection ∇^S such that

$$\begin{aligned} \langle X\cdot\varphi,\psi\rangle &= \langle\varphi, X\cdot\psi\rangle \qquad \text{and} \\ X(\langle\varphi,\psi\rangle) &= \langle\nabla^S_X\varphi,\psi\rangle + \langle\varphi,\nabla^S_X\psi\rangle \end{aligned}$$

for all vector fields X and all spinor fields φ,ψ. Each spinor field $\varphi \in \Gamma(S)$ defines a vector field V_φ on M, the so-called Dirac current, through the relation

$$g(V_\varphi, X) := -\langle X\cdot\varphi,\varphi\rangle$$

for all vector fields X. The Dirac current is always causal, i.e. lightlike or timelike, and it has the nice property that its zero set $zero(V_\varphi)$ coincides with the zero set $zero(\varphi)$ (cf. [Baum00]).

The twistor equation for a spinor field is given by

$$\nabla^S_X\varphi + \frac{1}{n}X\cdot D\varphi = 0 \qquad \text{for all } X \in TM \ .$$

This equation is conformally covariant, i.e. if we change the metric g to $\tilde{g} = e^{-2\phi}\cdot g$, where ϕ is some smooth function on M, then the spinor bundles S and $\tilde{S}$ are naturally identified and introduction of a scaling factor gives rise to a twistor spinor

$$\tilde{\varphi} := e^{-1/2\phi} \cdot \varphi \in \Gamma(\tilde{S})$$

with respect to the metric $\tilde{g}$. The twistor equation expresses the kernel of the Penrose (twistor) operator, which is defined as the superposition

$$pr_W \circ \nabla^S : \Gamma(S) \to \Gamma(W) \ ,$$

where the bundle $T^*M \otimes S$ of 1-forms with values in the spinors decomposes to $S \oplus W$ and pr_W denotes the projection to the component W, which is sometimes called the twistor part of the bundle $T^*M \otimes S$. Obviously, the parallel spinors satisfy the twistor equation. However, other kinds of solutions for twistors are known (cf. [Baum00]).

The Dirac current to a twistor spinor is a conformal vector field, i.e. it holds

$$\mathfrak{L}_{V_\varphi} g = \lambda g$$

for some function λ on M, where $\mathfrak{L}$ denotes the Lie derivative. Equivalently, the dual 1-form α_φ to V_φ satisfies the conformally covariant equation

$$\nabla_X \alpha_\varphi - \frac{1}{2} X \lrcorner\ d\alpha_\varphi + \frac{1}{n} X^\flat \wedge d^* \alpha_\varphi = 0$$

for all $X \in TM$, where ∇ is the Levi-Civita connection and d^* denotes the codifferential.

In [Lei99] it is shown that the zero set $zero(\varphi)$ of a twistor spinor consists of isolated points and/or isolated images of maximal lightlike geodesics. At a zero $p \in M$ of the twistor spinor φ, it is $D\varphi(p) \neq 0$, and moreover, for the Dirac current we have

$$V_\varphi(p) = 0, \qquad \nabla V_\varphi(p) = 0, \qquad divV(p) = 0 \quad \text{and} \quad grad(divV)(p) \neq 0\ .$$

The length function $\langle \varphi, \varphi \rangle$ of a (non-trivial) twistor spinor φ may be the trivial function, in general. Furthermore, the zero set $zero(\|V_\varphi\|^2)$ of the length function of V_φ is either identical to M or singular in M. In case that $zero(\|V_\varphi\|^2)$ is singular it is a smooth hypersurface in $M \smallsetminus zero(\varphi)$. Whereas around $p \in zero(\varphi)$, it looks like the lightcone emerging from the apex p. We define the singularity set $sing(\varphi)$ of a twistor spinor φ by

$$sing(\varphi) := zero(\|V_\varphi\|^2)$$

when $zero(\|V_\varphi\|^2)$ is singular and otherwise we set

$$sing(\varphi) := zero(\varphi)\ .$$

The latter case is when V_φ is either null or zero in every point on M.

Differentiating the twistor equation shows that a twistor φ always satisfies

$$\nabla_X D\varphi - \frac{n}{2} K(X) \cdot \varphi = 0$$

for all $X \in TM$, where K is the Schouten tensor of conformal geometry, i.e. it is

$$K(X) = \frac{1}{n-2}\Big(\frac{scal}{2(n-1)} X - Ric(X)\Big)$$

with Ric denoting the Ricci tensor and $scal$ the scalar curvature to the metric g. It is a well-known fact that

$$\begin{pmatrix} \nabla^S_X & \frac{1}{n} X \cdot \\ -\frac{n}{2} K(X) \cdot & \nabla^S_X \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0 \tag{1}$$

holds for all $X \in TM$ if and only if φ is a twistor spinor and $\psi = D\varphi$ (cf. [Fri76], [BFGK91]). This is an expression of a twistor as a constant section with respect to the covariant derivative defined by the above 2×2-matrix acting on the bundle $S \oplus S$.

We can make a similar statement for the dual 1-form α_φ to the Dirac current of a twistor spinor φ. It is

$$\begin{pmatrix} \nabla^{LC}_X & -X\,\lrcorner & X^\flat\wedge & 0 \\ -K(X)^\flat\wedge & \nabla^{LC}_X & 0 & X^\flat\wedge \\ K(X)\,\lrcorner & 0 & \nabla^{LC}_X & X\,\lrcorner \\ 0 & K(X)\,\lrcorner & K(X)^\flat\wedge & \nabla^{LC}_X \end{pmatrix}\begin{pmatrix} \alpha_\varphi \\ \beta_\varphi \\ \gamma_\varphi \\ \delta_\varphi \end{pmatrix} = 0 \tag{2}$$

for all $X \in TM$, where $\beta_\varphi = \frac{1}{2}d\alpha_\varphi$, $\gamma_\varphi = \frac{1}{n}d^*\alpha_\varphi$ and $\delta_\varphi = \frac{1}{n-2}(\nabla^*\nabla - \frac{scal}{2(n-1)})\alpha_\varphi$. Thereby, $\nabla^*\nabla$ denotes the Bochner-Laplacian on 1-forms. That means the quadruple $(\alpha_\varphi, \beta_\varphi, \gamma_\varphi, \delta_\varphi)$ depends only on the first component α_φ and the whole object is constant with respect to the covariant derivative defined in the 4×4-matrix.

3. Twistors and the normal connection

There is a reason behind the existence of the covariant constant expressions that we presented at the end of the last section. We briefly explain this here in terms of 'tractors' and the canonical conformal connection (cf. [Kob72], [CSS97], [Lei04]). In particular, it is crucial for our purpose here to understand that the covariant derivatives in (1) and (2) have structure group $Spin(2, n)$ resp. $\mathrm{SO}(2, n)$.

A conformal structure can be seen as equivalence class of metrics $[g]$ on a manifold M, which determines a reduction of the general linear frame bundle $GL(M)$ to a principal fibre bundle $CO(M)$ with structure group

$$CO(1, n) = \mathrm{SO}(1, n) \times \mathbb{R}_+ \ .$$

There is no choice of a connection form on $CO(M)$ solely depending on the conformal structure. However, there is a 'prolonged' principal fibre bundle $G(M)$ with structure group $\mathrm{SO}(2, n)$, which is the Möbius group for the conformally flat standard space with Lorentzian signature. The prolongation to the bundle $G(M)$ depends only on the conformal structure and yet on the bundel $G(M)$, there exists a canonical principal fibre bundle connection ω, which depends solely on the conformal structure. This is the normal connection of conformal geometry. In case of a conformal spin manifold there exists also a spin frame bundle $\tilde{G}(M)$ with structure group $Spin(2, n)$, with a natural projection to $G(M)$ and with canonical connection, which we also denote by ω.

Representations of the Möbius group and $Spin(2, n)$ give rise to 'tractor' bundles over $(M, [g])$. We consider here the form tractor and spinor tractor bundles. For this let $\Lambda^{p+1}_{2,n}$ denote the $(p+1)$-forms on the pseudo-Euclidean space $\mathbb{R}^{2,n}$ and let $\Delta_{2,n}$ denote the spinor module in signature $(2, n)$. The Möbius group acts naturally by ι on the forms. The spinor representation is denoted by κ. We obtain the associated vector bundles

$$\Lambda^{p+1}_C(M) := G(M) \times_\iota \Lambda^{p+1}_{2,n} \ , \qquad p = 0, \dots n \ ,$$

which we call form tractor bundles. And there is also the spinor tractor bundle

$$S_C := \tilde{G}(M) \times_\kappa \Delta_{2,n} \ .$$

Both bundles admit invariant inner products. Moreover, the canonical connection ω induces covariant derivatives on the form tractors and spinor tractors, which we denote both by ∇^{NC}.

Choosing the metric g gives rise to an identification of the form tractors of degree $p + 1$ with a quadruple of usual differential form bundles:

$$\Lambda^{p+1}_C(M) \cong_g \Lambda^p(M) \oplus \Lambda^{p+1}(M) \oplus \Lambda^{p-1}(M) \oplus \Lambda^p(M) \ .$$

Moreover, for the spinors we have an identification

$$S_C \cong_g S \oplus S$$

with the doubled spinor bundle. We want to make the identification in case of the form tractors explicit. Let $e^\flat = (e_1^\flat, \dots, e_n^\flat)$ be a local orthonormal coframe with respect to the metric g on M. The coframe $e^\flat$ can be extended to a local orthonormal coframe of the form tractor bundle $\Lambda_C^1(M) \cong \Lambda^0(M) \oplus \Lambda^1(M) \oplus \Lambda^0(M)$ of degree 1 by completing the basis $e^\flat$. We denote this coframe by

$$e_c^\flat = (e_t^\flat, e_s^\flat, e_1^\flat, \dots, e_n^\flat) \ ,$$

where $e_t^\flat$ is a timelike form tractor and $e_s^\flat$ is a spacelike form tractor of degree 1, and we define

$$e_-^\flat := \frac{1}{\sqrt{2}}(e_s^\flat - e_t^\flat) \qquad \text{and} \qquad e_+^\flat := \frac{1}{\sqrt{2}}(e_s^\flat + e_t^\flat) \ .$$

With these notations we find that a form tractor Ψ of degree $p+1$ can be written locally as

$$\Psi = e_-^\flat \wedge \alpha + \beta + e_-^\flat \wedge e_+^\flat \wedge \gamma + e_+^\flat \wedge \delta \ , \tag{3}$$

where $\alpha \in \Lambda^p(M)$, $\beta \in \Lambda^{p+1}(M)$, $\gamma \in \Lambda^{p-1}(M)$ and $\delta \in \Lambda^p(M)$ (cf. [Lei04]).

Now it can be shown that the canonical connection ∇^{NC} acting on sections of S_C resp. $\Lambda_C^{p+1}(M)$ reproduces (with respect to the splittings of the tractor bundles coming from the metric g) the covariant derivatives expressed in (1) and (2). And it is true that the equations

$$\nabla^{NC}\Phi = 0 \qquad \text{and} \qquad \nabla^{NC}\Psi = 0$$

for spinor tractors $\Phi \in \Gamma(S_C)$ resp. form tractors $\Psi \in \Omega_C^{p+1}(M)$ are equivalent to the sets of equations expressed in (1) resp. (2). In particular, twistor spinors are in 1-to-1-correspondence with parallel sections in the spinor tractor bundle S_C. For a parallel section $\Psi \in \Omega_C^2(M)$ we find with the identification from above that α is dual to a conformal vector field, and moreover, $\beta = \frac{1}{2}d\alpha$, $\gamma = \frac{1}{n}d^*\alpha$ and $\delta = \frac{1}{n-2}(\nabla^*\nabla - \frac{scal}{2(n-1)})\alpha$ (cf. [Lei04]).

4. Normal forms of the Dirac current

Let φ be a twistor spinor on a Lorentzian spin manifold (M,g) and α_φ the dual 1-form to the corresponding Dirac current. We have seen that there are unique $\Phi \in \Gamma(S_C)$ and $\Psi \in \Omega_C^2(M)$ to φ resp. α_φ, which are parallel with respect to ∇^{NC}. In the same manner as the 1-form α_φ is attached to the spinor φ by the relation

$$g(\alpha_\varphi, X) := -\langle X \cdot \varphi, \varphi \rangle$$

for all 1-forms X, the form tractor Ψ of degree 2 is attached to Φ by the relation

$$c(\Psi, X^2) := -i\langle X^2 \cdot \Phi, \Phi \rangle_c$$

for all $X^2 \in \Lambda_C^2(M)$, where c and $\langle \cdot, \cdot \rangle_c$ denote the invariant inner products on the tractor bundles.

$$\begin{array}{ccc} \begin{matrix}\varphi \\ \text{twistor}\end{matrix} & \longleftrightarrow & \begin{matrix}\Phi \in \Gamma(S_C) \\ \text{parallel}\end{matrix} \\ \downarrow & & \downarrow \\ \begin{matrix}\alpha_\varphi \\ \text{1-form}\end{matrix} & \longleftrightarrow & \begin{matrix}\Psi \in \Omega_C^2(M) \\ \text{parallel}\end{matrix} \end{array}$$

We will attach in the following a 2-form $N_\varphi \in \Lambda^2(\mathbb{R}^{2,n})$ to every twistor spinor φ. This 2-form is a certain distinguished normal form.

In general, a 2-form on $\mathbb{R}^{2,n}$ can be brought into a certain normal form with respect to some suitable basis of $\mathbb{R}^{2,n*}$. One can think of these normal forms as distinguished points in the orbits of $\Lambda^2(\mathbb{R}^{2,n})$ with respect to the action of $\mathrm{SO}(2,n)$. For example, there are 3 types of normal forms for decomposable 2-forms on $\Lambda^2(\mathbb{R}^{2,n})$ with vanishing length, that means 2-forms a with $a \wedge a = 0$ and $\langle a, a\rangle = 0$. These normal forms are expressed by:

(1) $l_1^\flat \wedge l_2^\flat$, where l_1 and l_2 are null vectors, orthogonal to each other,
(2) $l_1^\flat \wedge t_1^\flat$, where l_1 is null and t_1 is timelike and orthogonal to l_1,
(3) $l_1^\flat \wedge s_1^\flat$, where l_1 is null and s_1 is spacelike and orthogonal to l_1.

Distinguished normal forms for all orbits in $\Lambda^2(\mathbb{R}^{2,n})$ are established in [Bou00].

Consider now the normal form of a parallel form tractor Ψ of degree 2 in a point $p \in M$ and choose a smooth path ϱ to any other point q on the connected manifold M. Let $e_c^\flat(p)$ be an orthonormal basis in $p \in M$ of tractors of degree 1 for which the 2-form $\Psi(p)$ takes its distinguished normal form. Then we can translate the coframe $e_c^\flat(p)$ along the path ϱ parallel with respect to ∇^{NC}. Since Ψ is parallel itself, the normal form of Ψ on the image of the path ϱ is constant with respect to the parallel translated coframe $e_c^\flat$. Since the end point q can be choosen arbitrarily, we have shown that Ψ takes the same distinguished normal form in every point of the manifold M and we map a twistor spinor to this normal form:

$$\varphi \in Ker(pr_W \circ \nabla^S) \qquad \mapsto \qquad N_\varphi \in \Lambda^2(\mathbb{R}^{2,n}) \quad \text{(normal form)} .$$

5. The conclusions

We are now prepared to discuss the geometry of Lorentzian metrics outside of the singularity set $sing(\varphi)$ of a twistor spinor φ with zeros (notice that $sing(\varphi)$ is in general not the zero set of φ; cf. paragraph 2).

Let (M,g) be a Lorentzian spin manifold with twistor spinor $\varphi \not\equiv 0$ and non-vanishing zero set $zero(\varphi) = zero(V_\varphi)$. In a point $p \in zero(\varphi)$, it is

$$\alpha_\varphi(p) = 0, \qquad d\alpha_\varphi(p) = 0 \qquad \text{and} \qquad d^*\alpha_\varphi = 0 .$$

This shows that we can write in $p \in M$ the parallel form tractor Ψ belonging to φ with respect to any local coframe as

$$\Psi(p) = e_+^\flat \wedge \delta(p) .$$

Thereby, $\delta(p)$ is causal, since it is the Dirac current of $D\varphi$ in p. Obviously, the corresponding normal form of Ψ in p is either

$$l_1^\flat \wedge l_2^\flat \qquad \text{or} \qquad l_1^\flat \wedge t_1^\flat ,$$

where l_1 and l_2 are orthogonal and lightlike or l_1 and t_1 are orthogonal and t_1 is timelike (cf. paragraph 4). Now we can argue that the normal form is constant over the whole manifold M and we find that either $N_\varphi = l_1^\flat \wedge l_2^\flat$ or $N_\varphi = l_1^\flat \wedge t_1^\flat$. In case that $N_\varphi = l_1^\flat \wedge l_2^\flat$ the Dirac current is in every point of M either null or zero, i.e. the singularity set is defined as

$$sing(\varphi) = zero(\varphi) .$$

For the case $N_\varphi = l_1^\flat \wedge t_1^\flat$ the Dirac current V_φ is almost everywhere timelike and on a singular set it is null or zero. We have

$$sing(\varphi) = zero(\|V_\varphi\|^2) .$$

We consider now an arbitrary point q outside of the singularity set $sing(\varphi)$ of a twistor spinor φ. Then there is a neigborhood

$$U(q) \subset M \setminus sing(\varphi)$$

of q, where $\alpha_\varphi \neq 0$. Since $\Psi^2 = 0$, we can conclude that $\alpha_\varphi \wedge d\alpha_\varphi = 0$. With the Frobenius' Theorem it follows that V_φ is a hypersurface orthogonal conformal vector field on $U(q)$, which does not change the causal type. It is well known that such conformal vector fields are locally conformally equivalent to a parallel vector field with respect to some conformally changed metric $\tilde{g}$. The corresponding twistor spinor is then parallel, too (cf. [BL04]). Moreover, we can choose the neighborhood $U(q)$ in such a way that there exists a scaling function $e^{2\phi}$ on the whole of $U(q)$ and the rescaled twistor spinor $\tilde{\varphi} := e^{-1/2\phi} \cdot \varphi$ is parallel on $U(q)$ with respect to $\tilde{g} = e^{-2\phi} \cdot g$. Then there are two cases. Either V_φ is lightlike on $U(q)$ and we call $\tilde{g}$ a Brinkmann metric, since it admits a lightlike parallel vector. In the other case V_φ is timelike and $U(q)$ can be choosen such that $\tilde{g}$ is the product of $-ds^2$ with some Ricci-flat Riemannian metric h admitting a parallel spinor (cf. [BL04]).

Proposition 1. *Let φ be a twistor spinor with zero on (M, g). Then there exists for all $q \notin sing(\varphi)$ a neigborhood $U(q)$ and a scaling function $e^{-2\phi}$ such that the rescaled twistor spinor $\tilde{\varphi} = e^{-1/2\phi} \cdot \varphi$ is parallel with respect to $\tilde{g} = e^{-2\phi} g$ on $U(q)$. There are two cases:*

(1) *The Dirac current V_φ is timelike in some point of M and then for all $q \notin sing(\varphi)$ the rescaled metric takes locally the form*
$$\tilde{g} = -ds^2 + h \ ,$$
where h is a Riemannian metric with parallel spinor or

(2) *V_φ is lightlike on $M \setminus zero(\varphi)$ and $\tilde{g}$ is locally on $U(q)$ a Brinkmann metric with parallel spinor.*

The result is trivially true in dimension $n = 2$.

In [Lei99] we discussed the shape of the zero set $zero(\varphi)$ of a twistor spinor on a Lorentzian space. The result stated there says that the zero set consists of isolated points and/or isolated images of maximal ligthlike geodesics. In particular, if the Dirac current $V_{D\varphi}(p)$ of $D\varphi$ in a zero p of φ is lightlike then the maximal lightlike geodesic $\varrho : \mathbb{R} \to M$ with $\dot{\varrho}(0) = V_{D\varphi}(p)$ is entirely included in the zero set of φ. On the other side, if $V_{D\varphi}(p)$ is timelike in a zero p of φ then this zero must be an isolated point. From our normal form classification for twistor spinors with zeros we can now easily conclude the following result.

Theorem 1. *Let $\varphi \not\equiv 0$ be a twistor spinor on a Lorentzian spin manifold (M, g) with $zero(\varphi) \neq \emptyset$. Then $zero(\varphi)$ consists either of*

(1) *isolated images of lighlike geodesics and outside of the zero set the metric g is locally conformally equivalent to a Brinkmann metric with parallel spinor or*

(2) *isolated points and outside of the singularity set $sing(\varphi)$ the metric g is locally conformally equivalent to a product $-ds^2 + h$, where h is a Riemannian metric admitting a parallel spinor.*

We remember that in the timelike case the set $sing(\varphi)$ looks around a zero p of φ like a lightcone with apex p. And we see that on $M \setminus zero(\varphi)$ the function $||V_\varphi||^2 < 0$ serves as a global scaling function for which the twistor spinor φ becomes parallel with respect to the conformally changed metric. This statement determines the topological nature of a completion to infinity for the case that the twistor spinor has timelike Dirac current. The answer is that it looks (locally) like the standard completion for the Minkowski space. We conjecture that the topological nature of a completion in the lightlike case also behaves like the standard completion of the Minkowski space.

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