Dynamical Stability of Compressible Drops and Stars

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Abstract

Interest is directed to linearized free boundary motion of a compressible liquid subject to surface tension and self-gravitation repectively. Linearization relative to an a-priori given solution to the non-linear equations leads to a non-local second order evolution problem to be posed in a space-time cylinder whith variable cross section subject to Fréchet boundary conditions along the lateral boundary part. Well-posedness of the corresponding initial value problem in a natural weak formulation is proved.

Keywords: *Free boundary motion, compressible capillary drop, stellar pulsation*

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1. Introduction

In a recent paper [5] we formulated a geometric approach to linearized stability of free boundary motion of an incompressible perfect fluid under various forces. Here we extend our studies to the linearization of potential flows of compressible fluids with free boundaries. Our analysis applies Hamilton's principle to compressible free boundary motion cf. [2], [3]. Attention is directed to adiabatic flows driven by surface tension and, in a cosmic scale, by self-gravitation respectively. In Section 2, after reviewing the nonlinear Euler-Lagrange equations to the action integral, we determine the second variation of the action along a critical path, the so called accessory variational integral. Linearized motion, by definition, calls for a solution of the Euler-Lagrange equation to the accessory variational integral, commonly referred to as the Jacobi equation at the reference flow. We speak of linearized stability of the reference flow if the initial value problem for the Jacobi equation is well-posed. Proposition 2.6 shows that the linearized velocity potential satisfies a non-local wave-type equation with Fréchet boundary conditions in a space-time cylinder the variable cross section of which is determined by the reference flow. Stressing the geometric point of view we abstain from transforming the Jacobi equation to a vertical cylinder (for instance via particle paths) and, afterwards, from applying standard existence theory for linear evolution equations. Instead, following J. L. Lions [18], we solve the initial value problem in a natural weak sense. In Section 3 this is explained in detail for flows driven by surface tension, cf. Proposition 3.3. Section 4 summarizes the necessary modifications in the setting of the initial valueproblem if self-gravitation plays the predominant role. Whereas surface tension causes a flow always stable, stability of a self-gravitating gaseous star requires - as the Taylor sign criterion in incompressible flows - negativity of the outer normal derivative of the reference pressure at the boundary. As to a detailed discussion of adiabatic stellar oscillations, the mathematical theory of which was originated by Eddington [10], and for an extensive bibliography on that subject the reader is referred to the monograph [9] and to the surveys [15], [16]. Astrophysical literature, as a rule, analyses small oscillations around static equilibria, typically with spherical symmetry, via Fourier expansion. In contrast, our approach covers any time-dependent reference flow, possibly far from equilibrium.

Concerning existence and uniqueness results for the initial value problem to incompressible flows with free boundaries we refer e.g. to [1], [4], [17], [20] and to the references cited there.

2. The Jacobi equation

In the following we formulate the physical problem relative to the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, ..., x_n)$. Let $t \in [0, t_0]$ denote the time and let $\Omega_t \subseteq \mathbb{R}^n$ be the drop domain at time t. We assume Ω_t to be bounded with smooth boundary $\Gamma_t = \partial\Omega_t$. In the present Section our analysis is restricted to compressible potential flows driven by surface tension. Let $\phi = \phi(t,x)$ be the velocity potential of the flow and $\varrho = \varrho(t,x) > 0$ its density, both functions defined on Ω_t. If $W = W(\varrho)$ denotes the inner energy density then the motion is governed by the action principle

$$A = \int_0^{t_0} \int_{\Omega_t} \left(\tfrac{1}{2}\varrho|\nabla\phi|^2 - \varrho W - p_0\right) dx\, dt - \beta \int_0^{t_0} |\Gamma_t|\, dt \to \text{stationary} \tag{2.1}$$

under side:

$$\int_{\Omega_t} \varrho\, dx = m > 0 \tag{2.2}$$

and endpoint conditions

$$\delta\varrho = 0,\ \delta\Gamma = 0 \quad \text{at } t = 0 \text{ and } t = t_0 \tag{2.3}$$

respectively. In (2.1), $p_0 > 0$ denotes the outer pressure, $|\Omega|$ and $|\Gamma|$ the n- and $(n-1)$-dimensional Euclidean measures of Ω and Γ, respectively; $\beta > 0$ is the coefficient of surface tension. The integral

$$T = \frac{1}{2}\int_{\Omega_t} \varrho|\nabla\phi|^2\,dx \tag{2.4}$$

in (2.1) represents the kinetic energy at time t. If $\vec{\nu}_t$ denotes the unit outer normal to Γ_t and, correspondingly, $\dot{\Gamma}_t$ the normal velocity of the family Γ_t, measured in the direction of $\vec{\nu}_t$, then the velocity potential ϕ is to evaluate according to

$$-\operatorname{div}(\varrho\nabla\phi) = \dot{\varrho} \ \text{ in } \ \Omega_t, \quad \varrho\partial_\nu\phi = \varrho\dot{\Gamma}_t \ \text{ on } \ \Gamma_t. \tag{2.5}$$

The latter Neumann problem expresses mass conservation and, on Γ_t, the equality of the normal velocities of the flow and of Γ_t (kinematic boundary condition). Clearly, (2.5) takes the solvability condition

$$\int_{\Omega_t} \dot{\varrho}\,dx + \int_{\Gamma_t} \varrho\dot{\Gamma}\,d\Gamma = 0, \tag{2.6}$$

which expresses mass conservation differentially, for granted.

We list some notation: Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded domain with C^∞-boundary Γ_0, and let $\vec{\nu}$ denote the outer unit normal to Γ_0. For $\xi \in C^\infty(\Gamma_0)$ and $t \in (-\delta, \delta)$ let

$$\Gamma_t = \{P_t = P + t\xi(P)\vec{\nu}(P) : P \in \Gamma_0\} \tag{2.7}$$

be a deformation family of Γ_0. For sufficiently small t, Γ_t bounds a neighbouring domain Ω_t. The real-valued function $\xi : \Gamma_0 \to \mathbb{R}$ coincides with the normal velocity $\dot{\Gamma}$ of the family Γ_t at $\Gamma_0 : \xi = \dot{\Gamma}$. We remind the reader of the differentiation rule for surface integrals over varying domains: Let $\Gamma \mapsto \eta_\Gamma \in C^\infty(\Gamma)$ be a smooth map defined in a neighbourhood of Γ_0. In differential geometric terms the map constitutes a local section in the Fréchet vector bundle $\cup_\Gamma C^\infty(\Gamma)$. Setting

$$\bar{\eta}(t, P) = \eta_{\Gamma_t}(P_t) \ \text{ for } \ (t, P) \in (-\delta, \delta) \times \Gamma_0,$$

we define the covariant derivative:

$$(D_{\dot{\Gamma}}\eta)(P) = (\partial_t\bar{\eta})(0, P) \ \text{ for } \ P \in \Gamma_0$$

of the section η in direction $\dot{\Gamma}$, evaluated at Γ_0. Then we have the differentiation rule

$$\partial_t \int_{\Gamma_t} \eta_{\Gamma_t}\,d\Gamma_t\Big|_{t=0} = \int_{\Gamma_0} \big(D_{\dot{\Gamma}}\eta - H\dot{\Gamma}\eta\big)\,d\Gamma_0, \tag{2.8}$$

where H equals $(n-1)$ times the mean curvature of Γ_0; we agree on its sign according to (2.8). Concerning the relationship to covariant differentiation in $\cup_\Gamma C^\infty(\Gamma)$ cf. [5], [14]. It is important to recognize that $D_{\dot{\Gamma}}$ is torsion free. In fact, we need only

Lemma 2.1. *For a smooth deformation family* $(\varepsilon, t) \mapsto \Omega_{\varepsilon,t}$, $(\varepsilon, t) \in (-\varepsilon_0, \varepsilon_0) \times (-\delta, \delta)$, *let* Γ', $\dot\Gamma$ *denote the normal velocities of the family* $\Gamma_{\varepsilon,t} = \partial\Omega_{\varepsilon,t}$ *relative to* ε *and* t *respectively. Then we have*

$$D_{\Gamma'}\dot\Gamma - D_{\dot\Gamma}\Gamma' = 0. \tag{2.9}$$

Proof. For $\varphi \in C^\infty(\mathbb{R}^n)$, differentiate $\int_{\Omega_{\varepsilon,t}} \varphi\, dx$. In view of (2.8) this leads to

$$\begin{aligned}\partial_t\partial_\varepsilon \int_{\Omega_{\varepsilon,t}} \varphi\, dx &= \partial_t \int_{\Gamma_{\varepsilon,t}} \varphi\Gamma'\, d\Gamma_{\varepsilon,t} \\ &= \int_{\Gamma_{\varepsilon,t}} \left(\Gamma' D_{\dot\Gamma}\varphi + \varphi D_{\dot\Gamma}\Gamma' - \varphi H \Gamma'\dot\Gamma\right) d\Gamma_{\varepsilon,t}. \end{aligned} \tag{2.10}$$

As is easily checked, we have $D_{\dot\Gamma}\varphi = \dot\varphi + \dot\Gamma\partial_\nu\varphi = \dot\Gamma\partial_\nu\varphi$. Hence, after interchanging the derivatives in (2.10) and subtraction we get at $(\varepsilon, t) = (0, 0)$:

$$\int_\Gamma (D_{\dot\Gamma}\Gamma' - D_{\Gamma'}\dot\Gamma)\varphi\, d\Gamma = 0.$$

This proves the lemma since every $\varphi \in C^\infty(\Gamma)$ can be smoothly extended to $\mathbb{R}^n$. $\square$

For later reference we quote the first variation formula of mean curvature: Let $t \mapsto \Gamma_t$ be a smooth deformation family of hypersurfaces satisfying the initial conditions $\dot\Gamma = \xi$ and $D_{\dot\Gamma}\dot\Gamma = 0$ at $t = 0$. Then we have

$$\int_{\Gamma_0} \dot\Gamma D_{\dot\Gamma} H\, d\Gamma_0 = \int_{\Gamma_0} \left(k^2\xi^2 - |\nabla_{\Gamma_0}\xi|^2\right) d\Gamma_0, \tag{2.11}$$

where k^2 denotes the sum of the squares of the principal curvatures of Γ_0. In fact, we may assume the family to have the form (2.7). Considering the initial conditions the assertion follows immediately via comparing the formulae

$$\partial_t^2|\Gamma_t|\,|_{t=0} = \int_{\Gamma_0} \left(-\dot\Gamma D_{\dot\Gamma} H - H D_{\dot\Gamma}\dot\Gamma + H^2\dot\Gamma^2\right) d\Gamma_0$$

and

$$\partial_t^2|\Gamma_t|\,|_{t=0} = \int_{\Gamma_0} \left(|\nabla_\Gamma\xi|^2 - (k^2 - H^2)\xi^2\right) d\Gamma_0,$$

the first of which is a consequence of (2.8); for the second cf. [6], [13].

First of all, as a necessary prerequisite, we review the Euler-Lagrange equation of (2.1), cf. [2], [3], [12]. For any critical path $c(t) = \big(\varrho(t, .), \Gamma_t\big)$ in $(0, t_0)$ let $c(\varepsilon, t) = \big(\varrho(\varepsilon, t, .), \Gamma_{\varepsilon,t}\big)$ for $(\varepsilon, t) \in (-\varepsilon_0, \varepsilon_0) \times (0, t_0)$ be a smooth deformation embedding c, i.e. $c(0,t) = c(t)$ and respecting condition (2.6). In view of (2.4) we get

$$\partial_\varepsilon (T \circ \dot{c}) = \int_{\Omega_{\varepsilon,t}} \varrho \nabla\phi \nabla\phi' + \frac{1}{2} \int_{\Omega_{\varepsilon,t}} \varrho' \left|\nabla\phi\right|^2 + \frac{1}{2} \int_{\Gamma_{\varepsilon,t}} \varrho \left|\nabla\phi\right|^2 \Gamma'. \tag{2.12}$$

for the variation of T. For shortness, here and in the following we drop constantly differentials dx, $d\Gamma$ and similarly $dx\,dt$, $d\Gamma\,dt$ in integrals. In (2.12) a prime denotes partial differentiation in ε, which means $\varrho'(\varepsilon,t,x) = \partial_\varepsilon \varrho(\varepsilon,t,x)$, etc., for x in any compact subset of $\Omega_{\varepsilon,t}$. Along the deformation the velocity potential $\phi = \phi(\varepsilon,t,.)$ has to satisfy (2.5) relative to $\Omega_{\varepsilon,t}$. In particular,

$$\int_{\Omega_{\varepsilon,t}} \varrho \left|\nabla\phi\right|^2 = \partial_t \int_{\Omega_{\varepsilon,t}} \varrho\phi - \int_{\Omega_{\varepsilon,t}} \varrho\dot{\phi}.$$

Differentiation in ε gives

$$\begin{aligned}\partial_\varepsilon \int_{\Omega_{\varepsilon,t}} \varrho \left|\nabla\phi\right|^2 &= \partial_\varepsilon\partial_t \int_\Omega \varrho\phi - \int_\Omega (\varrho'\dot{\phi} + \varrho\dot{\phi}') - \int_\Gamma \varrho\dot{\phi}\Gamma' \\ &= -\partial_t \int_\Omega \varrho\phi' + \int_\Omega (\dot{\varrho}\phi' - \varrho'\dot{\phi}) + \int_\Gamma \varrho(\dot{\Gamma}\phi' - \Gamma\dot{\phi}') \\ &= \int_\Omega \varrho\nabla\phi\nabla\phi' - \int_\Omega \varrho'\dot{\phi} - \int_\Gamma \varrho\dot{\phi}\Gamma',\end{aligned} \tag{2.13}$$

the equations of this chain to be understood modulo time derivatives. Now, comparison with (2.12) shows

$$(T \circ \dot{c})' \big|_{\varepsilon=0} = -\int_\Omega \big(\dot{\phi} + \tfrac{1}{2}|\nabla\phi|^2\big)\varrho' - \int_\Gamma \varrho\Gamma'\big(\dot{\phi} + \tfrac{1}{2}|\nabla\phi|^2\big) \tag{2.14}$$

modulo time derivatives. Including the remaining terms of the action leads to

Proposition 2.2. *The first variation* $A' = \partial_\varepsilon A\big(c(\varepsilon,t)\big)\big|_{\varepsilon=0}$ *of the action integral* (2.1) *reads*

$$\begin{aligned}A' = &-\int_Q \big(\dot{\phi} + \tfrac{1}{2}|\nabla\phi|^2 + \partial_\varrho(\varrho W)\big)\,\varrho' \\ &-\int_Z \big(\dot{\phi} + \tfrac{1}{2}|\nabla\phi|^2 + W + \varrho^{-1}(p_0 - \beta H)\big)\,\varrho\Gamma',\end{aligned} \tag{2.15}$$

for fields consistent with the mass constraint (2.2). *Here we have set* $Q = \big\{(t,x) \in \mathbb{R}^{n+1} : t \in (0,t_0),\, x \in \Omega_t\big\}$ *and* $Z = \big\{(t,x) \in \mathbb{R}^{n+1} : t \in (0,t_0),\, x \in \Gamma_t\big\}$.

In view of the Lagrange multiplier rule (2.15) implies, apart from (2.5),

$$\dot{\phi} + \tfrac{1}{2}|\nabla\phi|^2 + \partial_\varrho(\varrho W) = \lambda \;\text{ in }\; Q \tag{2.16}$$

with boundary conditions

$$\dot{\phi} + \tfrac{1}{2}|\nabla\phi|^2 + W + \varrho^{-1}(p_0 - \beta H) = \lambda \ \text{ on } \ Z \tag{2.17}$$

to be the Euler-Lagrange equations for the motion. Here λ denotes a time-dependent multiplier which, after normalizing the velocity potential appropriately, may be assumed to vanish identically. If we introduce the hydrodynamic pressure

$$p = -\varrho\big(\dot{\phi} + \tfrac{1}{2}|\nabla\phi|^2 + W\big), \tag{2.18}$$

(2.15) takes the form

$$A' = \int_Q \big(\varrho^{-1}p - \varrho\partial_\varrho W\big)\varrho' + \int_Z (p - p_0 + \beta H)\Gamma'. \tag{2.19}$$

Vanishing of the boundary term reflects proportionality of pressure difference and mean curvature on the boundary of any critical trajectory.

For later reference we draw attention to the differential equation and boundary conditions satisfied by ϕ' in (2.12):

Lemma 2.3. *ϕ' solves the Neumann problem*

$$-\operatorname{div}(\varrho\nabla\phi') = M\varrho' \ \text{ in } \ Q, \quad \varrho\partial_\nu\phi' = N(\varrho\Gamma') \ \text{ on } \ Z. \tag{2.20}$$

Here we have introduced the first-order differential operators

$$\sigma \mapsto M\sigma := \dot{\sigma} + \operatorname{div}(\sigma\nabla\phi), \tag{2.21}$$

$$\eta \mapsto N\eta := D_{\dot{\Gamma}}\eta + \operatorname{div}_\Gamma(\eta\nabla_\Gamma\phi) - H\dot{\Gamma}\eta \tag{2.22}$$

acting on functions σ defined in Q and η on Z respectively.

It should be noted that N is correctly defined on Z. In fact, if η is a section along the solution path Γ_t then we have $D_{\dot{\Gamma}}\eta = \dot{\eta} + \dot{\Gamma}\partial_\nu\eta$ where the right-hand side operators act tangentially on Z. The remaining parts of N obviously share this property.

Proof. Along our deformation family $(\varepsilon, t) \mapsto \Omega_{\varepsilon,t}$ we have in view of (2.5)

$$\int_{\Omega_{\varepsilon,t}} \varrho\nabla\phi\nabla\psi = \int_{\Omega_{\varepsilon,t}} \dot{\varrho}\psi + \int_{\Gamma_{\varepsilon,t}} \varrho\dot{\Gamma}\psi$$

for every time- and parameter-independent test function $\psi \in C^\infty(\mathbb{R}^n)$. Having (2.8) in mind we get via differentiation w.r. to ε

$$\begin{aligned}\int_\Omega (\varrho'\nabla\phi + \varrho\nabla\phi')\nabla\psi + \int_\Gamma \varrho\Gamma'\nabla\phi\nabla\psi = \\ \int_\Omega \dot{\varrho}'\psi + \int_\Gamma \big(\dot{\varrho}\Gamma'\psi + \psi D_{\Gamma'}(\varrho\dot{\Gamma}) + \varrho\dot{\Gamma}\Gamma'\partial_\nu\psi - \psi\varrho H\dot{\Gamma}\Gamma'\big)\end{aligned}$$

and, considering (2.9),

$$\int_\Omega \varrho\nabla\phi'\nabla\psi = \int_\Omega (\dot\varrho'\psi - \varrho'\nabla\phi\nabla\psi) - \int_\Gamma \varrho\Gamma'\nabla_\Gamma\phi\nabla_\Gamma\psi$$
$$+ \int_\Gamma \psi\big(\dot\varrho\Gamma' + (\varrho' + \Gamma'\partial_\nu\varrho)\dot\Gamma + \varrho D_{\dot\Gamma}\Gamma' - \varrho H\dot\Gamma\Gamma'\big) \tag{2.23}$$

since $\nabla\phi\nabla\psi = \dot\Gamma\partial_\nu\psi + \nabla_\Gamma\phi\nabla\psi$ on Γ. Obviously, (2.23) are the variational equations to (2.20). □

In the following, integration by parts relative to the operators M, N plays an important role. For $t \in [0, t_0]$ let $c : t \mapsto (\varrho, \Gamma; \phi)$ be a path which satisfies (2.5). Presupposing c to be sufficiently regular, we have

Lemma 2.4. *Let $\vec V = \nabla\phi$ be the velocity field of the path and, as above, Q and Z the space-time trajectories of Ω and Γ respectively. If*

$$\psi \mapsto D\psi = \dot\psi + \vec V\nabla\psi \tag{2.24}$$

denotes differentiation along a particle path we have

$$\int_Q \sigma D\psi = -\int_Q \psi M\sigma + \int_\Omega \psi\sigma\,\Big|_0^{t_0}$$

and

$$\int_Z \eta D\psi = -\int_Z \psi N\eta + \int_\Gamma \eta\psi\,\Big|_0^{t_0}$$

for functions $\psi, \sigma \in C^1(\bar Q)$ and $\eta \in C^1(\bar Z)$ respectively.

Proof. In view of (2.8) there holds

$$\partial_t \int_\Omega \sigma\psi = \int_\Omega (\sigma\dot\psi + \psi\dot\sigma) + \int_\Gamma \psi\sigma\dot\Gamma = \int_\Omega \sigma D\psi + \psi M\sigma, \tag{2.25}$$

and

$$\partial_t \int_\Gamma \eta\psi = \int_\Gamma \eta D_{\dot\Gamma}\psi + \psi D_{\dot\Gamma}\eta - \psi H\dot\Gamma = \int_\Gamma \eta D\psi + \psi N\eta \tag{2.26}$$

which gives the assertions. □

The above Lemma shows the mappings M, N - if restricted by (2.5) - to be the L^2-adjoint of D relative to the space-time domain Q and ist lateral boundary Z respectively and for test functions vanishing at top and bottom. For later use we point to the relations

$$D\sigma = M\sigma - \sigma\Delta\phi, \quad D\eta = \rho^{-1}N(\rho\eta) + \eta\partial_\nu^2\phi \tag{2.27}$$

which are immediate consequences of the definitions of D, N, M and of the splitting

$$\Delta = \partial_\nu^2 - H\partial_\nu + \Delta_\Gamma, \quad \partial_\nu^2 = \nu_i\nu_j\partial_i\partial_j$$

of the Laplacian relative to normal and tangential derivatives along Γ.

The next proposition describes the second variation

$$A''(c)(u,u) = \partial_\varepsilon^2 A\big(c(\varepsilon,t)\big)\,\big|_{\varepsilon=0} \tag{2.28}$$

at a critical path $c = (\varrho, \Gamma)$ relative to a variation $u = (\varrho', \Gamma')$. In the following text the latter variation is relabelled $(\varrho', \Gamma') = (\sigma, \eta)$. (2.28) is usually referred to as the accessory variational integral to (2.1).

Proposition 2.5. *At a critical path $c = (\varrho, \Gamma)$ there holds*

$$A'' = \int_Q \varrho\big(|\nabla\phi'|^2 - \sigma^2 W''(\varrho)\big) + \int_Z \big((\beta k^2 + \partial_\nu p)\eta^2 - \beta|\nabla\eta|^2\big) \tag{2.29}$$

for every variation $u = (\sigma, \eta)$ vanishing at the endpoints Γ_0, Γ_{t_0} and satisfying the linearized volume constraint

$$\int_{\Omega_t} \sigma + \int_{\Gamma_t} \varrho\eta = 0 \;\; for \;\; t \in [0, t_0]. \tag{2.30}$$

As above, ϕ' is the solution of (2.20), i.e.

$$-\operatorname{div}(\varrho\nabla\phi') = M\sigma \;\; in \;\; Q, \quad \varrho\partial_\nu\phi' = N(\varrho\eta) \;\; on \;\; Z. \tag{2.31}$$

As already introduced, in Lemma 2.3, $k^2 = k^2(t,x)$ denotes the sum of the squares of the principal curvatures of Γ_t.

Proof. Differentiation of (2.19) in ε yields

$$\begin{aligned} A'' &= \int_Q \big(\varrho^{-1}p' - \varrho^{-2}p\sigma - \sigma\partial_\varrho(\varrho\partial_\varrho W(\varrho))\big)\sigma + \int_Z \big(p' + \eta\partial_\nu p + \beta D_{\Gamma'}H\big)\eta \\ &= -\int_Q \big(D\phi' + \varrho\sigma\partial_\varrho^2 W(\varrho)\big)\sigma + \int_Z \big(p' + \eta\partial_\nu p + \beta D_{\Gamma'}H\big)\eta \end{aligned}$$

at a critical path (2.16), (2.17). Inserting

$$-\int_Q \sigma D\phi' = \int_Q \phi' M\sigma = -\int_Q \phi' \operatorname{div}(\varrho\nabla\phi') = \int_Q \varrho|\nabla\phi'|^2 - \int_Z \varrho\phi'\partial_\nu\phi'$$

and

$$\begin{aligned} \int_Z p'\eta &= \int_Z \varrho^{-1}p\sigma\eta - \int_Z \varrho\eta\big(D\phi' + \sigma\partial_\varrho W(\varrho)\big) \\ &= -\int_Z \varrho\eta D\phi' = \int_Z \phi' N(\varrho\eta) = \int_Z \varrho\phi'\partial_\nu\phi' \end{aligned}$$

as well as (2.11) * into the latter result gives the claim. □

*we may assume $D_{\Gamma'}\Gamma'|_{\varepsilon=0} = 0$ without loss of generality

By definition $u = (\sigma, \eta)$ is a solution of the Jacobi equation at c if it solves the variational equations

$$A''(c)(u, \delta u) = 0 \tag{2.32}$$

for every section $\delta u = (\delta\sigma, \delta\eta)$ along c which satisfies the constraints (2.30). If we introduce the solution $\delta\phi'$ of

$$-\operatorname{div}(\varrho\nabla\delta\phi') = M(\delta\sigma) \text{ in } Q, \quad \varrho\partial_\nu\delta\phi' = N(\varrho\delta\eta) \text{ on } Z$$

then we have

$$\int_Q \varrho\nabla\phi'\nabla\delta\phi' = \int_Q \phi' M(\delta\sigma) + \int_Z \phi' N(\rho\delta\eta) = -\int_Q \delta\sigma D\phi' - \int_Z \varrho\delta\eta D\phi'$$

modulo an integration on Γ_0 and Γ_{t_0}. Hence, due to (2.29),

$$A''(c)(u, \delta u) = -\int_Q (D\phi' + \sigma W'')\sigma' - \int_Z (\varrho D\phi' + b\eta)\eta' + \beta\nabla_\Gamma\eta\nabla_\Gamma\eta', \tag{2.33}$$

where we have set $b = -\beta k^2 - \partial_\nu p$. This establishes

Proposition 2.6. *Any regular solution $u = (\sigma, \eta)$ of (2.32) satisfies, apart from (2.20) and (2.30), the Euler-Lagrange equations*

$$D\phi' + \varrho W''(\varrho)\sigma = 0 \;\; \textit{in} \;\; Q \tag{2.34}$$

with boundary conditions

$$\varrho D\phi' + b\eta - \beta\Delta_\Gamma\eta = 0 \;\; \textit{on} \;\; Z. \tag{2.35}$$

Assuming $\varrho W''(\varrho) > 0$ in $\bar{Q}$ at the reference flow and, additionally, the existence of $(-\beta\Delta_\Gamma + b)^{-1}$, then σ and η may be eliminated in (2.34), (2.35). Remebering (2.20) this leads to

$$\operatorname{div}(\varrho\nabla\phi') = M\big(D\phi'/\varrho W''\big) \text{ in } Q, \quad \varrho\partial_\nu\phi' = N\big(\varrho(\beta\Delta_\Gamma - b)^{-1}(\varrho D\phi')\big) \text{ on } Z$$

for the linearized velocity potential. In view of $M = -D^\star$ and $N = -D|_Z^\star$ this shows the linearized potential to satisfy a wave-type equation relative to the derivative $D = \partial_t + \vec{V}\nabla$.

3. The initial value problem for the Jacobi equation

In this section we consider the initial value problem for the Jacobi equation (2.32). With the previous notation let $c = (\varrho, \Gamma)$ be a smooth critical path of (2.1) which, in our context, means $\bar{Z} \in C^\infty$ and $\varrho \in C^\infty(\bar{Q})$. In addition, we assume $\varrho \geq \varrho_0 > 0$ in Q. We introduce the space

$$X = \Big\{ u = (\sigma, \eta) \in L^2(Q) \times \int^{\oplus} H^1(\Gamma_t)\, dt : \int_{\Omega_t} \sigma + \int_{\Gamma_t} \varrho\eta = 0 \text{ a.e. in } (0, t_0) \Big\},$$

where the direct integral extends over $(0, t_0)$. In customary notation, the integral $\int^{\oplus} H^1(\Gamma_t)\, dt$ denotes the space of (equivalence classes of) mappings $t \mapsto \eta(t) = \eta(t, \cdot) \in H^1(\Gamma_t)$ such that

$$t \mapsto \|\eta(t)\|^2_{H^1(\Gamma_t)} = \|\eta(t)\|^2_{L^2(\Gamma_t)} + \|\nabla_{\Gamma_t}\eta(t)\|^2_{L^2(\Gamma_t)}$$

belongs to $L^1(0, t_0)$. Correspondingly, we equip X with the norm

$$\|u\|^2_X = \|\sigma\|^2_{L^2(Q)} + \int_0^{t_0} \|\eta(t)\|^2_{H^1(\Gamma_t)}\, dt.$$

Similarly, we set

$$Y = \Big\{ \varphi \in \int^{\oplus} H^1(\Omega_t)\, dt : \int_{\Omega_t} \varphi = 0 \text{ a.e. in } (0, t_0) \Big\}$$

and

$$\|\varphi\|^2_Y = \int_Q \varrho |\nabla\varphi|^2.$$

In the following we agree on the notation $X(t)$, $Y(t)$ for the integrands of the direct integrals, i.e.

$$X(t) = \Big\{ u \in L^2(\Omega_t) \times H^1(\Gamma_t) : \int_{\Omega_t} \sigma + \int_{\Gamma_t} \varrho\eta = 0 \Big\}, \text{ etc.}$$

The solvability of the Neumann problem being guaranteed for data $u = (\sigma, \eta) \in X$, there exists a unique solution $\psi\{u\} \in Y$ of the variational equations

$$\int_Q \varrho \nabla\psi\{u\}\nabla\varphi = \int_Q \sigma\varphi + \int_Z \varrho\eta\varphi \quad \text{for all } \varphi \in Y. \tag{3.1}$$

Clearly: $\psi\{u\}(t) = \psi\{u(t)\}$ a. e. where $\psi\{u(t)\} \in Y(t)$ denotes the solution of the corresponding Neumann problem at time t. Due to the trace mapping theorem we have $\varphi|_{\Gamma_t} \in H^{1/2}(\Gamma_t)$ for $\varphi \in Y(t)$, hence

$$\|\psi\{u\}\|_Y \leq C\|u\|_X, \quad \|\psi\{u(t)\}\|_{Y(t)} \leq C\|u(t)\|_{X(t)} \tag{3.2}$$

for $u \in X$ and $u(t) \in X(t)$ respectively.

Since, as a consequence of the differentiation rule

$$0 = \partial_t \left(\int_{\Omega_t} \sigma + \int_{\Gamma_t} \varrho\eta \right) = \int_{\Omega_t} M\sigma + \int_{\Gamma_t} N(\varrho\eta),$$

the image of the differential operator

$$u \mapsto Lu = \big(M\sigma, \varrho^{-1}N(\varrho\eta)\big), \tag{3.3}$$

- defined on smooth $u \in X$ - belongs to X, the above rule applies to the right-hand side Lu as well to get $\psi\{Lu\}$ which was originally characterized by (2.20). Accordingly we define the Hilbert space W to be the completion of the set of functions $u \in C^\infty(\bar{\Omega}) \times C^\infty(\bar{Z})$ satisfying the linearized volume constraint (2.30) w.r. to the norm

$$\|u\|_W^2 = \|u\|_X^2 + \|\psi\{Lu\}\|_Y^2, \tag{3.4}$$

the second term of which measures the dynamic part of the Jacobi action. Closability of the map $u \mapsto \psi\{Lu\}$ (with the above domain of definition) relative to the spaces X and Y generates a natural embedding of W into X.

The solution $\psi\{u\}$ of (3.1) has less physical significance. The next Lemma, however, shows $D\psi\{u\}$ to be the principal part of the linearized velocity potential $\psi\{Lu\}$.

Lemma 3.1. *$u \in W$ implies $D\psi\{u\} \in Y$ and there holds*

$$\|D\psi\{u\}\|_Y \leq C\big(\|\psi\{Lu\}\|_Y + \|\psi\{u\}\|_Y\big) \quad \textit{for all} \ \ u \in W \tag{3.5}$$

with a constant $C > 0$ independent of u. Moreover, for $\varphi \in Y$ we have

$$\int_Q \varrho\nabla\big(D\psi\{u\} - \psi\{Lu\}\big)\nabla\varphi = 2\int_Q \varrho V_{ij}\partial_i\psi\{u\}\partial_j\varphi, \tag{3.6}$$

where $V_{ij} = \partial_i\partial_j\phi$ are the second derivatives of the reference velocity potential.

Proof. The estimate (3.5) follows immediately from (3.6). It suffices to prove (3.6) for smooth u and $\varphi \in C_0^\infty(Q \cup Z)$. Because of $\partial_i D\varphi = D\partial_i\varphi + V_{ij}\partial_j\varphi$ the definition (3.1) of $\psi\{u\}$ implies

$$\int_Q \varrho\nabla\psi\{u\}D\nabla\varphi + \int_Q \varrho V_{ij}\partial_i\psi\{u\}\partial_j\varphi = \int_Q \sigma D\varphi + \int_Z \varrho\eta D\varphi.$$

After an integration by parts, this leads to

$$\begin{aligned} -\int_Q M\big(\varrho\nabla\psi\{u\}\big)\nabla\varphi + \int_Q \varrho V_{ij}\partial_i\psi\{u\}\partial_j\varphi &= -\int_Q M\sigma\varphi - \int_Z N(\varrho\eta)\varphi \\ &= -\int_Q \varrho\nabla\psi\{Lu\}\nabla\varphi \end{aligned}$$

since integrals over top and bottom drop out. Inserting of

$$M\big(\varrho\nabla\psi\{u\}\big)\nabla\varphi = \varrho\nabla\big(D\psi\{u\}\big)\nabla\varphi - \varrho V_{ij}\partial_i\psi\{u\}\partial_j\varphi$$

completes the proof. □

Lemma 3.2. *For $u \in W$ the map $t \mapsto \|\psi\{u(t)\}\|^2_{Y(t)}$ (after correction on a set of measure zero) is absolutely continuous in $[0, t_0]$ and there holds*

$$\|\psi\{u(t)\}\|_{Y(t)} \le C\|u\|_W \quad \textit{for} \quad t \in [0, t_0] \tag{3.7}$$

with a constant independent of t.

Proof. For $\psi = \psi\{u\}$ we get, via differentiation in t,

$$\partial_t \int_{\Omega_t} \varrho|\nabla\psi|^2 = \int_{\Omega_t} \varrho D|\nabla\psi|^2 = 2\int_{\Omega_t} \varrho\big(\nabla\psi\nabla D\psi - V_{ij}\partial_i\psi\partial_j\psi\big)$$

in view of (2.25) and $M\varrho = 0$. Due to Lemma 3.1 and (3.2) this implies

$$\Big|\partial_t\|\psi(t)\|^2_{Y(t)}\Big| \le C\left(\|\psi\{Lu\}(t)\|^2_{Y(t)} + \|\psi(t)\|^2_{Y(t)}\right) \le C\|u\|^2_W \tag{3.8}$$

for a.e. $t \in (0, t_0)$. Hence

$$\|\psi(t_2)\|^2_{Y(t_2)} \le \|\psi(t_1)\|^2_{Y(t_1)} + C\|u\|^2_W$$

after an integration on (t_1, t_2). If $t_1 \in (0, t_0)$ is chosen according to

$$\|\psi(t_1)\|^2_{Y(t_1)} = t_0^{-1}\int_0^{t_0} \|\psi(t)\|^2_{Y(t)}dt = t_0^{-1}\|\psi\|^2_Y$$

this proves (3.7). □

Obviously Lemma 3.2, for $u, v \in W$, implies absolute continuity of the map $t \mapsto (\psi\{u\}, \psi\{v\})_{Y(t)}$ and the estimate $(\psi\{u\}, \psi\{v\})_{Y(t)} \le C\|u\|_W\|v\|_W$ for the scalar product.

For the solution of the initial value problem for the Jacobi equation (2.32) we require smoothness of the reference flow c and the data to satisfy $\beta > 0$ and

$$\varrho \ge \varrho_0 > 0, \ \varrho W''(\varrho) \ge w_0 > 0 \ \text{ in } \bar{Q}. \tag{3.9}$$

The latter condition is met, for example, if $W = W_0\varrho^{\gamma-1}$ with $\gamma > 2$. As a matter of fact it suffices to assume, apart from (3.9) and the smoothness of data, the velocity potential ϕ of the reference flow to solve the Neumann problem (2.5) only.

If we introduce the abbreviation

$$a(u,u) = \int_Q \varrho W''(\varrho)\sigma^2 + \int_Z \left(\beta|\nabla\eta|^2 + b\eta^2\right)$$

for the spatial part of the Jacobi action (2.29), the accessory variational integral can be written

$$A''(u,u) = \|\psi\{Lu\}\|_Y^2 - a(u,u).$$

We understand solvability of the initial value problem

$$u|_{t=0} = 0, \quad \psi\{Lu\}|_{t=0} = \psi_1 \in Y(0)$$

for the inhomogeneous Jacobi equation in the weak sense of

Initial value problem. *Find $u \in W$, $u(0) = 0$ such that*

$$-A''(u,v) = \langle f, v \rangle + (\psi_1, \psi\{v\}|_{t=0})_{Y(0)} \tag{3.10}$$

for all $v \in W$ with $v(t_0) = 0$. The source term f is assumed to act on u according to

$$\langle f, u \rangle = \int_Q f_1\sigma + \int_Z f_2\varrho\eta$$

with f_1, $Df_1 \in L^2(Q)$ and f_2, $Df_2 \in \int^{\oplus} H^{-1}(\Gamma_t)dt$.

The boundary term $(\psi_1 | \psi\{v\})_{Y(0)}$ is to be interpreted in the sense of Lemma 3.2. The same remark applies to the initial values $u|_{t=0}$ etc. It should be noted, however, that the initial value of $\psi\{Lu\}$ makes sense only relative to the differential equation since a function $u \in W$ generally does not have a trace $Lu(t,\cdot)$ at any instant of time.

In order realize a smooth solution $u = (\sigma, \eta)$ of (3.10) to satisfy the equations of Proposition 2.6 we have to repeat the integration by parts argument leading to the Euler-Lagrange equations, but taking now initial values and the right-hand side into account: For smooth $u = (\sigma, \eta)$ and $v = (\bar\sigma, \bar\eta)$, $v(t_0, \cdot) = 0$ we have

$$\begin{aligned} -(\psi\{Lu\}, \psi\{Lv\})_Y &= \int_Q \bar\sigma D\psi\{Lu\} + \int_Z \varrho\bar\eta D\psi\{Lu\} \\ &\quad + \int_{\Omega_0} \bar\sigma\psi\{Lu\} + \int_{\Gamma_0} \varrho\bar\eta\psi\{Lu\} \end{aligned}$$

and

$$a(u,v) = \int_Q \varrho W''(\varrho)\sigma\bar\sigma + \int_Z (b\eta - \beta\Delta_\Gamma)\,\bar\eta$$

at the one hand and

$$\langle f, v\rangle + (\psi_1, \psi\{u\})_{Y(0)} = \int_Q \bar\sigma f_1 + \int_Z \varrho\bar\eta f_2 + \int_{\Omega_0} \bar\sigma\psi_1 + \int_{\Gamma_0} \varrho\bar\eta\psi_1$$

on the other. Neglecting the Lagrange multiplier we get

$$D\psi\{Lu\} + \varrho W''(\varrho)\sigma = f_1 \text{ in } Q, \quad \varrho D\psi\{Lu\} + b\eta - \beta\Delta_\Gamma\eta = \varrho f_2 \text{ on } Z$$

and $u|_{t=0} = 0$, $\psi\{Lu\}|_{t=0} = \psi_1$ in Ω_0 for every smooth solution of (3.10).

Proposition 3.3. *Under assumption* (3.9) *there exists a unique solution* $u \in W$ *of* (3.10)*. The solution satisfies the pointwise estimate*

$$\|u(t)\|_{X(t)} \leq C\left(\|\psi_1\|_{Y(0)} + \|f\|_{Q_t} + \|Df\|_{Q_t}\right) \tag{3.11}$$

a.e. in $(0, t_0)$ *with a constant independent of* t*, where we have set*

$$\|f\|_{Q_t}^2 = \int_0^t \left(\|f_1\|_{L^2(\Omega_\tau)}^2 + \|f_2\|_{H^{-1}(\Gamma_\tau)}^2\right) d\tau.$$

The proof refers to the existence principle, cf. [18]:

(E). *Let* Φ *be an inner product space continuously embedded into a real Hilbert space* F*, and let* E *be a bilinear form defined on* $F \times \Phi$*. Then, continuity on* F *of the mapping* $u \mapsto E(u,v)$ *for every* $v \in \Phi$ *and positivity:* $E(v,v) \geq \lambda\|v\|_\Phi^2$*,* $\lambda > 0$ *for every* $v \in \Phi$ *assure, for any linear form* $v \mapsto \theta(v)$ *which is continuous on* Φ*, the existence of* $u \in F$ *satisfying*

$$E(u,v) = \theta(v) \quad \textit{for all } v \in \Phi. \tag{3.12}$$

For later reference we sketch the proof of (E): Relative to the completion $\hat{\Phi}$ of Φ let $\hat{E}$ denote the continuation by continuity of E to $\hat{\Phi} \times \Phi$. Similarly, the embedding $J : \Phi \to F$ extends continuously to a map $\hat{J} : \hat{\Phi} \to F$, (which in general looses injectivity). Translating (3.12) to the Hilbert space $\hat{\Phi}$ leads, in view of Riesz' representation theorem, to

$$(v, Su)_{\hat{\Phi}} = (v, w_\theta)_{\hat{\Phi}}, \quad w_\theta \in \hat{\Phi}, \tag{3.13}$$

where S maps Φ into $\hat{\Phi}$. Since positivity of E implies the existence of a continuous inverse S^{-1}, the Hilbert space adjoint S^* of S is surjective. This proves the existence of a solution $u \in \hat{\Phi}$ to (3.13). Because of $\hat{E}(u,v) = E(\hat{J}u, v)$ the image $\hat{J}u \in F$ of any solution of (3.13) is a solution of the original equation (3.12). In particular, this shows that, whether Φ is dense in F or not, every solution $u \in F$ of (3.12) can be approximated by a sequence $u_n \in \Phi$ which is bounded in Φ. Additionally, it should be remarked, that there exists always a solution $u \in \hat{\Phi}$ of (3.13) which satisfies the estimate

$$\|u\|_{\hat{\Phi}} \leq \lambda^{-1}\|\theta\|, \tag{3.14}$$

where $\|\theta\| = \sup\{\theta\{v\} : \theta \in \Phi, \|\theta\| \leq 1\}$ denotes the norm of θ.

Lemma 3.4. *For sufficiently large $\omega > 0$ and data*

$$u \in \Phi = \big\{u \in W \ : \ Lu \in W,\ u(0) = \psi\{Lu\}(t_0) = 0\big\}$$

there holds

$$-A''(u, e^{-\omega t}Lu) \geq c\big(\|u\|_W^2 + \|u(t_0)\|_{X(t_0)}^2\big) + \tfrac{1}{2}\|\psi\{Lu\}(0)\|_{Y(0)}^2,$$

where $c > 0$ is independent of u. Similarly, for u in the testspace Ψ,

$$u \in \ \Psi = \big\{u \in W \ : \ Lu \in W,\ u(t_0) = \psi\{Lu\}(0) = 0\big\}$$

we have the parallel estimate

$$A''(u, e^{\omega t}Lu) \geq c\big(\|u\|_W^2 + \|u(0)\|_{X(t_0)}^2\big) + \tfrac{1}{2}\|\psi\{Lu\}(t_0)\|_{Y(0)}^2.$$

Postponing the proof for a moment we establish an interpolation-type estimate at first:

Lemma 3.5. *For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $u(t) = (\sigma(t), \eta(t)) \in X(t)$ there holds*

$$\|\eta(t)\|_{L^2(\Gamma_t)} \leq \varepsilon\big(\|\nabla_{\Gamma_t}\eta(t)\|_{L^2(\Gamma_t)} + \|\sigma(t)\|_{L^2(\Omega_t)}\big) + C_\varepsilon\|\psi\{u(t)\}\|_{Y(t)}, \tag{3.15}$$

where the constant can be chosen independently of $t \in [0, t_0]$.

Proof. To prove (3.15) for a fixed $t \in [0, t_0]$, suppose by contradiction that for some $\varepsilon > 0$ there exists a sequence $\{u_n\} \subset X(t)$ with

$$1 = \|\eta_n\|_{L^2(\Gamma_t)} \geq \varepsilon\big(\|\nabla_{\Gamma_t}\eta_n\|_{L^2(\Gamma_t)} + \|\sigma_n\|_{L^2(\Omega_t)}\big) + n\|\psi\{u_n\}\|_{Y(t)}.$$

The latter inequality shows boundedness of the sequence $u_n(t)$ in $X(t)$ as well as $\psi\{u_n(t)\} \to 0$ in $Y(t)$. Extracting a weakly convergent subsequence $u_n(t) \rightharpoonup u(t) = (\sigma(t), \eta(t))$ in $X(t)$ leads to $\psi\{u_n(t)\} \rightharpoonup \psi\{u(t)\} = 0$ in $Y(t)$ on the one hand and, in view of $\eta_n(t) \to \eta(t)$ in $L^2(\Gamma_t)$ strongly, to $\|\eta(t)\|_{L^2(\Gamma_t)} = 1$ on the other, which contradicts $\psi\{u(t)\} = 0$.

As a perturbation-of-domain argument shows, C_ε can be chosen locally independent of t: At a fixed instant $\tau \in [0, t_0]$ let $t \mapsto s(t, x)$, $s(\tau, x) = x$ denote the characteristic trajectory of the derivative D emanating at $x \in \Omega_\tau$. Then $x \mapsto s(t, x)$ defines a smooth diffeomorphism from $\bar{\Omega}_\tau$ onto $\bar{\Omega}_t$. If we set, for $u \in X(t)$,

$$\ell_u(\varphi) = \int_{\Omega_t} \sigma\varphi + \int_{\Gamma_t} \varrho\eta\varphi \quad \text{for } \varphi \in Y(t)$$

we have $\|\psi\{u\}\|_{Y(t)} = \|\ell_u\|_{Y(t)'} = \sup\{\ell_u(\varphi) : \|\varphi\|_{Y(t)} \leq 1\}$. Further

$$|\ell_{u\circ s}(\varphi \circ s) - \ell_u(\varphi)| \leq C|t - \tau|\big(\|\sigma\|_{L^2(\Omega_t)} + \|\eta\|_{L^2(\Gamma_t)}\big)\|\varphi\|_{Y(t)},$$

hence

$$|\ell_{u\circ s}(\varphi)| \le |\ell_u(\varphi\circ s^{-1})| + C|t-\tau| \left(\|\sigma\|_{L^2(\Omega_t)} + \|\eta\|_{L^2(\Gamma_t)}\right) \|\varphi\circ s^{-1}\|_{Y(t)}$$

if the testfunction φ belongs to $Y(\tau)$. This estimate implies

$$\|\psi\{u\circ s\}\|_{Y(\tau)} \le C\big(\|\psi\{u\}\|_{Y(t)} + |t-\tau| \left(\|\sigma\|_{L^2(\Omega_t)} + \|\eta\|_{L^2(\Gamma_t)}\right)\big). \tag{3.16}$$

By (3.15) we have at $t=\tau$

$$\begin{aligned}\|\eta\|_{L^2(\Gamma_t)} &\le C\|\eta\circ s\|_{L^2(\Gamma_\tau)}\\ &\le \varepsilon\big(\|\nabla_{\Gamma_\tau}(\eta\circ s)\|_{L^2(\Gamma_\tau)} + \|\sigma\circ s\|_{L^2(\Omega_\tau)}\big) + C_\varepsilon\|\psi\{u\circ s\}\|_{Y(\tau)}.\end{aligned}$$

Inserting (3.16) into the latter inequality and considering the additional estimates

$$\begin{aligned}\|\sigma\circ s\|_{L^2(\Omega_\tau)} &\le (1+C|t-\tau|)\,\|\sigma\|_{L^2(\Omega_t)},\\ \|\nabla_{\Gamma_\tau}(\eta\circ s)\|_{L^2(\Gamma_\tau)} &\le \|\nabla_{\Gamma_t}\eta\|_{L^2(\Gamma_t)} + C|t-\tau|\|\eta\|_{H^1(\Gamma_t)}\end{aligned}$$

confirms the stated local constancy of C_ε. □

Proof of Lemma 3.4. For simplicity we introduce the notation

$$|||\sigma|||_Q^2 = \|\sigma\|^2_{L^2(\Omega_{t_0})} + \omega\int_0^{t_0} e^{-\omega t}\|\sigma\|^2_{L^2(\Omega_t)}\,dt$$

and

$$|||\eta|||_Z^2 = \|\eta\|^2_{L^2(\Gamma_{t_0})} + \omega\int_0^{t_0} e^{-\omega t}\|\eta\|^2_{L^2(\Gamma_t)}\,dt,$$

tacitly assuming the involved data to be sufficiently regular. To check the asserted estimate we set

$$\psi = \psi\{Lu\}, \quad \psi_\omega = \psi\{L(e^{-\omega t}Lu)\} = e^{-\omega t}\big(\psi\{L^2u\} - \omega\psi\{Lu\}\big).$$

Then, inspecting the various terms of A, we have for $u\in\Phi$

$$-\int_Q \varrho\nabla\psi\nabla\psi_\omega = \int_Q e^{-\omega t}\varrho\big(\omega|\nabla\psi|^2 - D\nabla\psi\nabla\psi + V_{ij}\partial_i\psi\partial_j\psi\big)$$

in view of Lemma 3.1. Considering $M\varrho=0$ and $\psi\{Lu\}|_{t=0}=0$, an integration by parts leads to

$$\int_Q e^{-\omega t}\varrho D\nabla\psi\nabla\psi = \frac{\omega}{2}\int_Q e^{-\omega t}\varrho|\nabla\psi|^2 - \frac{1}{2}\|\psi\|^2_{Y(0)},$$

hence

$$-\int_Q \varrho\nabla\psi\nabla\psi_\omega \ge \frac{\varrho_0}{4}|||\nabla\psi\{Lu\}|||_Q^2 + \frac{1}{2}\|\psi\{Lu\}\|^2_{Y'(0)}, \tag{3.17}$$

$c_1 > 0$, if ω is large enough.

A similar integration by parts argument applied to the inner energy term gives

$$2\int_Q e^{-\omega t}\varrho W''\sigma M\sigma = \int_Q e^{-\omega t}\varrho W''\big(M(\sigma^2)+\sigma^2\Delta\phi\big)$$
$$= \int_Q e^{-\omega t}\big(\varrho W''(\omega+\Delta\phi) - D(\varrho W'')\big)\sigma^2 + e^{-\omega t_0}\int_{\Omega_{t_0}} \varrho W''\sigma^2,$$

if (2.27) is remembered. This implies

$$\int_Q e^{-\omega t}\varrho W''\sigma M\sigma \geq \frac{\alpha}{4}|||\sigma|||_Q^2 \tag{3.18}$$

for sufficiently large $\omega > 0$.

As to the boundary terms we have

$$2\int_Z e^{-\omega t} b\varrho^{-1}\eta N(\varrho\eta) = \int_Z e^{-\omega t}\big((\omega - 2\partial_\nu^2\phi)b - Nb\big)\eta^2 + e^{-\omega t_0}\int_{\Gamma_{t_0}} b\eta^2$$

in view of $2\varrho^{-1}\eta N(\varrho\eta) = D(\eta^2) - 2\partial_\nu^2\phi\eta^2$. Consequently

$$\int_Z e^{-\omega t} b\eta\varrho^{-1}N(\varrho\eta) \geq -b_0|||\eta|||_Z^2, \tag{3.19}$$

where we have set $b_0 = 1 + \sup_Z |b|$.

Similarly, for large enough $\omega > 0$, the capillary term may be estimated in terms of

$$\int_Z e^{-\omega t}\nabla_\Gamma\eta\nabla_\Gamma\big(\varrho^{-1}N(\varrho\eta)\big) \geq \frac{1}{4}\big(|||\nabla_\Gamma\eta|||_Z^2 - |||\eta|||_Z^2\big) \tag{3.20}$$

via integration by parts, where

$$2\nabla_\Gamma\eta\nabla_\Gamma\big(\varrho^{-1}N(\varrho\eta)\big) = D\big(|\nabla_\Gamma\eta|^2\big) - 2\big([D,\nabla_\Gamma]\eta + \nabla_\Gamma(\partial_\nu^2\phi\eta)\big)\nabla_\Gamma\eta,$$

and

$$\big|\nabla_\Gamma(\partial_\nu^2\phi\eta) + [D,\nabla_\Gamma]\eta\big| \leq C\big(|\eta| + |\nabla_\Gamma\eta|\big) \text{ on } \bar{Z},$$

is to be considered. $[D,\nabla_\Gamma] = D\nabla_\Gamma - \nabla_\Gamma D$ denotes the commutator.

For large $\omega > 0$ the estimates (3.17)-(3.20) imply

$$-A''(u, e^{-\omega t}Lu) \geq c_1\left(|||\nabla\psi\{Lu\}|||_Q^2 + |||\sigma|||_Q^2 + |||\nabla_\Gamma\eta|||_Z^2\right) - c_2|||\eta|||_Z^2$$

with constants $c_1, c_2 > 0$ independent of ω. Due to Lemma 3.5 the subtrahend $|||\eta|||_Z^2$ is bounded by

$$|||\eta|||_Z^2 \leq \varepsilon\big(|||\sigma|||_Q^2 + |||\nabla_\Gamma\eta|||_Z^2\big) + C_\varepsilon|||\nabla\psi\{u\}|||_Q^2.$$

To eliminate the norm $|||\nabla\psi\{u\}|||_Q^2$ in the former estimate we observe: Due to (3.8) we have for large ω:

$$\partial_t\big(e^{-\omega t}\|\psi\{u\}\|_{Y(t)}^2\big) + \frac{\omega}{2}e^{-\omega t}\|\psi\{u\}\|_{Y(t)}^2 \le Ce^{-\omega t}\|\psi\{Lu\}\|_{Y(t)}^2,$$

which, after integration, implies

$$|||\psi\{u\}|||_Q \le C\omega^{-1}|||\nabla\psi\{Lu\}|||_Q,$$

if $\psi\{u\}|_{t=0}$ and $\psi\{Lu\}|_{t=0}$ is remembered. Extending the above estimate that way we get

$$|||\eta|||_Q^2 \le \varepsilon\big(|||\sigma|||_Q^2 + |||\nabla_\Gamma\eta|||_Z^2\big) + C_\varepsilon\omega^{-1}|||\nabla\psi\{Lu\}|||_Q^2.$$

Hence, choosing firstly $\varepsilon > 0$ sufficiently small and, afterwards, ω large enough shows the desired estimate. $\square$

Proof of Proposition 3.3. 1^{st} *Step : Existence.* We apply the existence principle (E) to the spaces $F = \{u \in W : u(0) = 0\}$ and Φ, the latter equipped with the norm

$$\|v\|_\Phi^2 = \|v\|_W^2 + \|\psi\{Lv\}\|_{Y(0)}^2 + e^{-2\omega t_0}\|v\|_{X(t_0)}^2, \tag{3.21}$$

and to the pair

$$E(u,v) = -A''(u, e^{-\omega t}Lv), \quad \theta(v) = \big\langle f, e^{-\omega t}Lv \big\rangle + (\psi_1, \psi\{Lv\})_{Y(0)}.$$

Integration by parts

$$\big\langle f, e^{-\omega t}Lv \big\rangle = \omega\big\langle f, e^{-\omega t}v \big\rangle + \big\langle e^{-\omega t}Df, v \big\rangle + e^{-\omega t_0}\Big(\int_{Q_{t_0}} f_1\bar\sigma + \int_{\Gamma_{t_0}} f_2\varrho\bar\eta\Big)$$

shows the linear form $v \mapsto \langle f, e^{-\omega t}Lv \rangle$ to be continuous on Φ with the estimate

$$|\big\langle f, e^{-\omega t}Lv \big\rangle| \le C\left(\|v\|_X + e^{-\omega t_0}\|v\|_{X(t_0)}\right)\left(\|f\|_{Q_{t_0}} + \|Df\|_{Q_{t_0}}\right)$$

for $v = (\bar\sigma, \bar\eta) \in \Phi$. Hence Lemma 3.4 assures the existence of $u \in F$ satisfying the variational equations

$$-A''(u, e^{-\omega t}Lv) = \big\langle f, e^{-\omega t}Lv \big\rangle + (\psi_1,\, \psi\{Lv\})_{Y(0)} \;\text{ for all }\; v \in \Phi$$

if $\omega \ge \omega_0$. Since $v \mapsto e^{-\omega t}Lv$ maps Φ surjectively to the testspace of (3.10) u is a solution of (3.10). In fact, in view of $w(t_0) = 0$ for every testfunction $w \in W$, the solution v of the initial value problem for the first-order PDE

$$e^{-\omega t}Lv = w \;\text{ in }\; Q, \quad v(0) = 0$$

belongs to Φ.

2^{nd} *Step: Uniqueness.* Let $u \in W$ with $u(0) = 0$ be a solution of the homogeneous equation corresponding to (3.10), i.e. $A''(u, v) = 0$ for all testfunctions $v \in W$ with $v(t_0) = 0$. Since the solution w of the first-order PDE $e^{\omega t} Lw = u$ in Q with initial value $w(t_0) = 0$ belongs to Φ, we have

$$0 = A''(e^{\omega t} Lw, w) \geq c \|w\|_W^2,$$

hence $w = 0$ and $u = 0$.

3^{rd} *Step: Stability estimate.* Due to Lemma 3.2 there exists a representative $t \mapsto u(t)$ of the solution which is continuous in the sense that $t \mapsto (u(t), \varphi)_{Y(t)}$ defines a continuous function for every $\varphi \in C^\infty(\mathbb{R}^{n+1})$. According to the existence principle (E) u can be approximated by a sequence $u_n \in \Phi$ with

$$u_n \to u \text{ in } W, \quad \|u_n\|_\Phi \leq C\|\theta\|,$$

cf. (3.14). In particular, we have

$$\|u_n(t_0)\|_{X(t_0)} \leq C\|\theta\| \tag{3.22}$$

in view of (3.21). We may assume $u_n(t_0) \rightharpoonup v$ weakly in $X(t_0)$, hence $\psi\{u_n(t_0)\} \rightharpoonup \psi\{v\}$ weakly in $Y(t_0)$. On the other hand, convergence in W implies via Lemma 3.2:

$$(\psi\{u_n(t)\}, \varphi)_{Y(t)} \to (\psi\{u(t)\}, \varphi)_{Y(t)}$$

for every $\varphi \in C^\infty(\mathbb{R}^{n+1})$, uniformly in $[0, t_0]$ - possibly after correcting the sequence on a subset of measure zero. Comparing the limits establishes the estimate (3.11) at $t = t_0$ since v satisfies (3.22).

Restating Lemma 3.4 relative to a time intervall $[0, t_1]$ with $t_1 < t_0$ and reiterating the above arguments proves the estimate (3.11) at $t = t_1$ in view of the uniqueness of the solution. □

We conclude this Section with the simple example of a ball $\Omega : |x| < R$ at rest which, for any constant density $\varrho_0 > 0$, constitutes a stationary state of the nonlinear equations of motion. In this situation we have $\phi = 0$, and the accessory variational integral - if suitably rescaled - reads

$$A'' = \int_0^{t_0} \int_\Omega \left(|\nabla \varphi'|^2 - \alpha \sigma^2 \right) - \int_0^{t_0} \int_\Gamma \left(\beta \, |\nabla_\Gamma \eta|^2 + \gamma \eta^2 \right),$$

where φ' is subject to Neumann conditions:

$$-\Delta \varphi' = \dot{\sigma} \text{ in } \Omega, \ \partial_\nu \varphi' = \dot{\eta} \text{ on } \Gamma$$

with $\alpha, \beta, \gamma \in \mathbb{R}, \ \alpha, \beta > 0$. According to Proposition 2.6 we have

$$\ddot{\sigma} - \alpha \Delta \sigma = 0 \text{ in } \Omega \text{ and } \ddot{\sigma} + (\gamma - \beta \Delta_\Gamma) \partial_\nu \sigma = 0 \text{ on } \Gamma. \tag{3.23}$$

Separation of variables $\sigma(t,x) = Z(r)Y(\theta)e^{i\omega t}$ then leads to the eigenvalue problem

$$\Delta_{S^{n-1}} Y + \lambda Y = 0 \text{ on } S^{n-1} \tag{3.24}$$

and

$$r^2 Z'' + rZ' + (\alpha^{-1}\omega^2 + \lambda)Z = 0 \text{ in } (0,R) \tag{3.25}$$

with Fréchet boundary conditions

$$Z'(R) = \omega^2(\gamma + \beta\lambda)^{-1} Z(R) \tag{3.26}$$

respectively. Since eigenvalues and eigenfunctions of (3.24) - (3.26) are explicitely known, cf. e.g. [8], the initial value problem to (3.23) may be completely solved in terms of Fourier series. In the physical relevant case $\gamma > 0$ all solutions remain bounded in time.

4. Stellar pulsation

In this Section Propostion 3.3 is supplemented by the initial value problem for linearized adiabatic oscillations of a gaseous star driven by self-gravitation. As long as the outer pressure p_0 remains positive, replacing of the capillary term in (2.1) by the gravitational energy requires only minor changes in the existence and uniqueness theory. In contrast, due to (4.2) below, vanishing of the outer pressure implies $\varrho = 0$ at the boundary Γ_t of any solution to the nonlinear equations, this circumstance causing the linearization (2.20) to degenerate along the boundary. Here we keep the conditions (3.9) for the reference flow, avoiding that way the latter degeneration but, at the same time, accepting a physically relevant situation to be excluded from the analysis.

The gravitational energy J of a star reads

$$J(\varrho) = -\frac{G}{2}\int_\Omega \varrho(x) U\{\varrho\}(x)\, dx,$$

where

$$U\{\varrho\}(x) = \int_\Omega |x-y|^{2-n}\varrho(y)\, dy \tag{4.1}$$

denotes its Newton potential and G is the gravitational constant.

We take up former considerations. Differentiation of (4.1) along the deformation path of Lemma 2.1 gives

$$U\{\varrho\}' = U\{\varrho'\} + V\{\varrho\Gamma'\}$$

with V being the potential of a simple layer,

$$V\{\varrho\Gamma'\}(x) = \int_\Gamma |x-y|^{2-n}\varrho(y)\Gamma'(y)\, d\Gamma_y.$$

This implies

$$J' = -G\int_\Omega \varrho' U\{\varrho\} - G\int_\Gamma \varrho\Gamma' U\{\varrho\}$$

for the derivative $J' = \partial_\varepsilon(U\circ c)$ along the path. Repeating the arguments of Section 2, therefore, leads to

Proposition 4.1. *For a self-gravitating star, the first variation of the action reads*

$$\begin{aligned} A' = &-\int_Q \big(\dot\phi + \tfrac{1}{2}|\nabla\phi|^2 + \partial_\varrho(\varrho W) - GU\{\varrho\}\big)\varrho' \\ &-\int_Z \big(\dot\phi + \tfrac{1}{2}|\nabla\phi|^2 + W + \varrho^{-1}p_0 - GU\{\varrho\}\big)\varrho\Gamma', \end{aligned} \tag{4.2}$$

for fields consistent with mass end endpoint constraints.

(4.2) parallels Proposition 2.2. Similary, considering the differentiation rule

$$D_{\Gamma'}U\{\varrho\}|_\Gamma = U\{\varrho\}' + \Gamma'\partial_\nu U\,|_\Gamma = U\{\varrho'\} + V\{\varrho\Gamma'\} + \Gamma'\partial_\nu U\,|_\Gamma,$$

we obtain, as a parallel of Proposition 2.5,

Proposition 4.2. *At a critical path of* (4.2) *we have*

$$\begin{aligned} A'' = &\int_Q \big(\varrho|\nabla\psi\{Lu\}|^2 - \varrho W''(\varrho)\sigma^2 + G\sigma\,(U\{\sigma\} + V\{\varrho\eta\})\big) \\ &+\int_Z \big(\eta^2\partial_\nu p + G\varrho\eta\,(U\{\sigma\} + V\{\varrho\eta\})\big) \end{aligned} \tag{4.3}$$

for every variation $u = (\sigma,\eta)$ consistent with the linearized mass and endpoint constraints, where

$$p = -\varrho\big(\dot\phi + \tfrac{1}{2}|\nabla\phi|^2 + W - GU\big)$$

denotes the reference pressure.

The gravitational part of (4.3):

$$q(u,u) = \int_Q \sigma\varphi\{u\} + \int_Z \varrho\eta\varphi\{u\} = \int_Q \varrho\nabla\varphi\{u\}\nabla\psi\{u\}, \tag{4.4}$$

where we have set $\varphi\{u\} = U\{\sigma\} + V\{\varrho\eta\}$, will be shown a minor residue to the principal part $A'' - q$ of the Jacobi action (4.3). In fact, after adapting the configuration space to the new situation, which amounts to redefining X by

$$X = \Big\{u = (\sigma,\eta) \in L^2(Q)\times L^2(Z) : \int_{\Omega_t}\sigma + \int_{\Gamma_t}\varrho\eta = 0 \text{ a.e. in } (0,t_0)\Big\},$$

and by replacing W correspondingly we get control of q by

Lemma 4.3. *For $u \in W$ and every $\varepsilon > 0$ we have*

$$|q(u, Lu)| \leq \varepsilon \|u\|_W^2 + C_\varepsilon \|\psi\{u\}\|_Y^2 \tag{4.5}$$

with a constant C_ε independent of u.

Proof. If $\|\eta\|$ denotes the norm in $\int^{\oplus} H^{-1/2}(\Gamma_t)dt$ we have

$$\Big| \int_Q \varrho \nabla \varphi\{u\} \nabla \psi\{Lu\} \Big| \leq C \|\psi\{Lu\}\|_Y \left(\|\nabla U\{\sigma\}\|_{L^2(Q)} + \|\eta\| \right),$$

since the simple layer potential acts from $H^{-1/2}(\Gamma_t)$ into $H^1(\Omega_t)$. Inserting the counterparts

$$\|\nabla U\{\sigma\}\|_{L^2(\Omega_t)}, \ \|\eta\|_{H^{-1/2}(\Gamma_t)} \leq \varepsilon \left(\|\sigma\|_{L^2(\Omega_t)} + |\eta\|_{L^2(\Gamma_t)} \right) + C_\varepsilon \|\psi\{u\}\|_{Y(t)}$$

of (3.15) into the latter estimate implies (4.5) since the integral $\int_{\Omega_t} \varrho \nabla \varphi\{u\} \nabla \psi\{v\}$ is symmetric in u, v. □

Now the arguments of Section 3 can be carried over to get

Proposition 4.4. *Under* (3.9) *and the additional stability condition*

$$\partial_\nu p \leq p_1 < 0 \quad on \ \ Z$$

there exists a unique solution $u \in W$ to the inhomogeneous Jacobi equation of (4.3) *in the sense of* (3.10), *if the inhomogenuities f_1, Df_1 and f_2, Df_2 belong to $L^2(Q)$ and to $L^2(Z)$ respectively. The solution satisfies the estimate* (3.11), *where the norm $\|f\|_Q$ is to be modified correspondingly.*

Due to Proposition 4.1 a gaseous star in hydrodynamic equilibrium satisfies the equation

$$(\varrho W)' - GU\{\varrho\} = \lambda$$

in some time independent domain Ω. Applying the Laplacian gives

$$(\varrho W)'' \Delta \varrho + (\varrho W)''' |\nabla \varrho|^2 + \omega_n G \varrho = 0 \quad \text{in } \Omega, \tag{4.6}$$

where ω_n denotes the measure of the unit sphere S^{n-1} in $\mathbb{R}^n$. For zero outer pressure (4.6) is to be complemented by the boundary condition $\varrho|_\Gamma = 0$. Applied to a polytropic gas and a radially symmetric density distribution the equation is known as Emden's differential equation, cf. [7], [11]. As already commented on, we abstain however from pursuing the analysis of the latter singular boundary value problem in the present context.

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