Precise asymptotics for periodic orbits of the geodesic flow in nonpositive curvature

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Abstract. We establish a precise asymptotic formula for the number of homotopy classes of periodic orbits for the geodesic flow on rank one manifolds of nonpositive curvature. This extends a celebrated result of G.A. Margulis to the nonuniformly hyperbolic case and strengthens previous results by G. Knieper.

We also establish some useful properties of the measure of maximal entropy.

Key words. Geodesic flows – nonpositive curvature – rank one manifolds – closed geodesics – asymptotic counting of periodic orbits – nonuniform hyperbolicity – measure of maximal entropy.

Contents

1. Introduction

1.1. Manifolds of rank one

Let M be a compact Riemannian manifold with all sectional curvatures nonpositive. For a vector $v \in TM$, the **rank** of v is the dimension of the vector space of parallel Jacobi fields along the geodesic tangent to v. The rank of M is the minimal rank of all tangent vectors. Obvious consequences of this definition are that

$$1 \leq \operatorname{rank}(M) \leq \dim(M),$$

that the rank of $\mathbb{R}^k$ with the flat metric is k and that

$$\operatorname{rank}(M \times N) = \operatorname{rank}(M) + \operatorname{rank}(N).$$

Every manifold whose sectional curvature is never zero is automatically of rank one. Products with Euclidean n-space clearly have rank at least $n+1$. However, it is possible for a manifold to be everywhere locally a product with Euclidean space and still have rank one. It turns out that the rank of a manifold of nonpositive curvature is the algebraic rank of its fundamental group [BaEb].

Apart from manifolds of negative curvature, examples are nonpositively curved surfaces containing flat cylinders or an infinitesimal analogue of a flat cylinder, as illustrated in the following diagram.

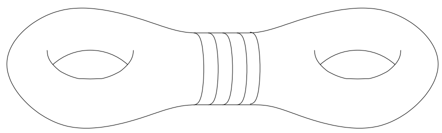

Fig. 1.1. A surface of rank one with a flat strip and a parallel family of geodesics.

In higher dimensions, examples include Gromov's (3-dimensional) graph manifolds [Gro]. There is an interesting rigidity phenomenon: Every compact 3-manifold of nonpositive curvature whose fundamental group is isomorphic to that of a graph manifold is actually diffeomorphic to that graph manifold [Sch].

We will study properties of manifolds of rank one in this article.

1.2. Reasons to study these spaces

1.2.1. Rank rigidity W. Ballmann [Bal1] and independently K. Burns and R. Spatzier [BuSp] showed that the universal cover of a nonpositively curved manifold can be written uniquely as a product of Euclidean, symmetric and rank one spaces. The first two types are understood, due to P. Eberlein and others. A general introduction to higher rank symmetric spaces is e.g. [Ebe5]; see also [BGS]. For a complete treatment of rank rigidity, see [Bal2].

Thus, in order to understand nonpositively curved manifolds, the most relevant objects to examine are manifolds of dimension at least two with rank one. This becomes even more obvious if one considers the fact that rank one is generic in nonpositive curvature [BBE]. Thus, in a certain sense, "almost all" nonpositively curved manifolds have rank one.

1.2.2. Limits of hyperbolic systems Another reason to study nonpositively curved manifolds is the following. On one hand, strongly hyperbolic systems, particularly geodesic flows on compact manifolds of negative curvature, are well understood since D.V. Anosov; see e.g. [Ano]. Later, P. Eberlein established a condition weaker than negative curvature which still ensures the Anosov property of the geodesic flow ([Ebe3], [Ebe4]). On the other hand, much less is known about the dynamics of systems lacking strong hyperbolicity. The open set of geodesic flows on manifolds with negative curvature is "essentially" understood (hyperbolicity is an open property), and hence the edge of our knowledge is mainly marked by the boundary of this set, which is a set of geodesic flows on manifolds of nonpositive curvature. Therefore it is important to study the dynamics of these.

However, the set of nonpositively curved manifolds is larger than just the closure of the set of negatively curved manifolds. This can be seen e.g. as follows: Some nonpositively curved manifolds, such as Gromov's graph manifolds, contain an embedded 2-torus. Thus their fundamental group contains a copy of $\mathbb{Z}^2$. Hence, by Preissmann's theorem, they do not admit *any* metric of negative curvature. Therefore, the investigation in this article actually deals with even more than the limits of our current knowledge of hyperbolic systems.

1.3. Statement of the result

We count homotopy classes of closed geodesics ordered by length in the following sense: The number P_t of homotopy classes of periodic orbits of length at most t is finite for all t. (For a periodic geodesic there may be uncountably many periodic geodesics homotopic to it, but in nonpositive curvature they all have the same length.) Trying to find a concrete and explicit formula for P_t which is accurate for all values of t is completely hopeless, even on very simple manifolds. Nonetheless, in this article we manage to derive an asymptotic formula for P_t, i.e. a formula which tells us the behavior of P_t when t is large. We will show (Theorem 5.28):

$$P_t \sim \frac{1}{ht} e^{ht}$$

where the notation $f(t) \sim g(t)$ means $\frac{f(t)}{g(t)} \to 1$ as $t \to \infty$. This extends a celebrated result of G.A. Margulis to the case of nonpositive curvature. It also strengthens results by G. Knieper, which were the sharpest estimates known to this date in the setup of nonpositive curvature. This is explained in more detail in the following section.

2. History

2.1. Margulis' asymptotics

The study of the functions $P_.$ and $b_.$, where $b_t(x)$ is the volume of the geodesic ball of radius t centered at x, was originated by G.A. Margulis in his dissertation [Mar1]. He covers the case where the curvature is strictly negative. His influential results were published in [Mar2]. He established that, on a compact manifold of negative curvature,

$$b_t(x) \sim c(x)e^{ht} \tag{2.1}$$

for some continuous function c on M. He also showed that

$$P_t \sim c' \frac{e^{ht}}{t} \tag{2.2}$$

for some constant c'. The exponent h is the topological entropy of the geodesic flow. See [KaHa] for a modern reference on the topic of entropy.

Margulis pointed out that if the curvature is constant then the exponential growth rate equals $(n-1)\sqrt{-K}$ and that in this case the function c is constant. Today we know that $c \equiv 1/h$ in that case. Moreover, $c' = 1/h$ for variable negative curvature.

2.2. Beyond negative curvature; Katok's entropy conjecture

The vast majority of the studies that have since been done are restricted to negative curvature; see e.g. [PaPo], [BaLe], [PoSh]. The reason is that in that case techniques from uniformly hyperbolic dynamics can be applied. From the point of view of analysis, this case is much easier to treat. However, from a geometrical viewpoint, manifolds of nonpositive curvature are a natural object to study. Already in the seventies the investigation of manifolds of nonpositive curvature became the focus of interest of geometers. (Also more general classes have been studied since, such as manifolds without *focal points*, i.e. where every parallel Jacobi field with one zero has the property that its length increases monotonically when going away from the zero, or manifolds without *conjugate points*, i.e. such that any Jacobi field with two zeroes is trivial.) In 1984 at a MSRI problem session a major list of problems which were open at the time was compiled [BuKa], including A. Katok's *entropy conjecture:* The measure of maximal entropy is unique.

One of the first result in the direction of asymptotics of closed geodesics in nonpositive curvature is G. Knieper's result [Kni3] that the growth rate of closed geodesics is h, even if the curvature is just nonpositive (instead of strictly negative). The same can be deduced from A. Manning's result [Man] that the growth rate of volume equals h in nonpositive curvature. This shows in particular that the exponent in Margulis' asymptotics must equal h (and justifies that we have already written Margulis' equations that way). A method for showing that (in the case of negative curvature) the constant c' in equation (2.2) equals $1/h$ is outlined in C. Toll's dissertation [Tol]; see also [KaHa]. The behavior of the function c in the asymptotic formula (2.1) was investigated by C. Yue in [Yue1] and [Yue2].

It took almost two decades after Knieper's and Manning's results, which in turn were published about one decade after Margulis' results, until the next step in the analysis of asymptotics of periodic orbits was completed, again by Knieper.

2.3. Knieper's multiplicative bounds

In 1996 G. Knieper succeeded in establishing asymptotic multiplicative bounds for volume and periodic orbits [Kni2] which, in the case of nonpositive curvature and rank one, were the sharpest results known until now: There exists a constant C such that for sufficiently large t,

$$\frac{1}{C} \leq \frac{b_t(x)}{e^{ht}} \leq C$$

and

$$\frac{1}{Ct} \leq \frac{P_t}{e^{ht}} \leq C.$$

These results are a groundbreaking success. The main step in the proof of these asymptotics is the proof of Katok's entropy conjecture. Knieper also demonstrated in [Kni1] that the measure of maximal entropy can be obtained via the Patterson-Sullivan construction ([Pat], [Sul]; see also [Kai1], [Kai2]). Moreover, for the case of higher rank Knieper obtained asymptotic information using rigidity. Namely,

$$\frac{1}{C} \leq \frac{b_t}{t^{(\operatorname{rank}(M)-1)/2} e^{ht}} \leq C.$$

He also estimates the number of closed geodesics for higher rank.

Knieper subsequently sharpened his results. With the same method he is able to prove that in the rank one case actually

$$\frac{1}{C} \leq \frac{P_t}{e^{ht}/t} \leq C$$

holds. (See also [Kni4].) Still, the quotient of the upper and lower bounds is a constant which cannot be made close to 1.

The question whether in this setup of nonpositive curvature and rank one one can prove more precise multiplicative asymptotics—namely without such multiplicative constants—has remained open so far. In this article we establish this result.

Remark 2.1. For non-geodesic dynamical systems no statements providing asymptotics similar the ones mentioned here are known. One of the best known results is that for some prevalent set of diffeomorphisms the number of periodic orbits of period n is bounded by $\exp(C \cdot n^{1+\delta})$ for some $\delta > 0$ [KaHu].

But even for geodesic flows in the absence of nonpositive curvature it is difficult to count—or even find—closed geodesics. The fact that every compact manifold has even *one* closed geodesic was established only in 1951 by Lyusternik and Fet [LuFe]. In the setup of positively curved manifolds and their kin, one of the strongest known results is H.-B. Rademacher's Theorem from 1990 ([Rad1], [Rad2]) stating that every connected and simply connected compact manifold has *infinitely many* (geometrically distinct) closed geodesics for a C^r-generic metric for all $r \in [2, \infty]$. See also [Rad3] for this.

For Riemannian metrics on the 2-sphere, existence of many closed geodesics took considerable effort to prove. The famous Lyusternik-Shnirelman Theorem asserts the existence of three (geometrically distinct) closed geodesics. The original proof in [LuSch] is considered to have gaps. Complete proofs were given by W. Ballmann [Bal3], W. Klingenberg (with W. Ballmann's help) [Kli] and also J. Jost ([Jos1], [Jos2]). See also [BTZ1], [BTZ2].

J. Franks [Fra] established that every metric of positive curvature on S^2 has infinitely many (geometrically distinct) geodesics. This is a consequence of his results about area-preserving annulus homeomorphisms. V. Bangert managed to show existence of infinitely many (geometrically distinct) geodesics on S^2 without requiring positive curvature by means of variational methods [Ban].

For the case of Finsler manifolds, there actually exist examples of simply connected manifolds that possess only finitely many geometrically distinct closed geodesics. On S^2 such examples were constructed by A.B. Katok in [Kat1] as a by-product of a more general construction; see also [Zil]. Explaining this particular aspect of Katok's construction is also the topic of [Mat].

In this article we derive asymptotics like the ones Margulis obtained. We prove them for nonpositive curvature and rank one using non-uniform hyperbolicity. Hence the same strong statement is true in considerably greater generality.

3. Geometry and dynamics in nonpositive curvature

Let M be a compact rank one Riemannian manifold of nonpositive curvature. As is usual, we assume it to be connected and geodesically complete. Let $S\tilde{M}$ be the unit sphere bundle of the universal covering of M. For $v \in S\tilde{M}$ let c_v be the geodesic satisfying $c'(0) = v$ (which is hence automatically parameterized by arclength). Here c' of course denotes the covariant derivative of c. Let $g = (g^t)_{t\in\mathbb{R}}$ be the **geodesic flow** on $S\tilde{M}$, which is defined by $g^t(v) := c_v'(t) =: v_t$.

3.1. Review of asymptotic geometry

Definition 3.1. *Let $\pi : TM \to M$ be the canonical projection. We say that $v, w \in S\tilde{M}$ are **positively asymptotic** (written $v \sim w$) if there exists a constant C such that $d(\pi g^t v, \pi g^t w) < C$ for all $t > 0$. This is evidently an equivalence relation. Similarly, we say that $v, w \in S\tilde{M}$ are **negatively asymptotic** (written $-v \sim -w$) if there exists a constant C such that $d(\pi g^t v, \pi g^t w) < C$ for all $t < 0$. Of course, v and w are positively asymptotic if and only if $-v$ and $-w$ are negatively asymptotic.*

Recall that $\mathrm{rank}(v) := \dim\{\text{parallel Jacobi fields along } c_v\}$. Clearly the rank is constant along geodesics, i.e. $\mathrm{rank}(c_v'(t)) = \mathrm{rank}(c_v'(0))$ for all $t \in \mathbb{R}$.

Definition 3.2. *We call a vector $v \in S\tilde{M}$, as well as the geodesic c_v, **regular** if $rank(v) = 1$ and **singular** if $rank(v) > 1$. Let* **Reg** *and* **Sing** *be the sets of regular and singular vectors, respectively.*

Remark 3.3. Since rank is semicontinuous in the sense that

$$\mathrm{rank}(\lim_n v_n) \geq \lim_n \mathrm{rank}(v_n),$$

the set **Reg** is open.

Remark 3.4. For every $v \in S\tilde{M}$ and $p \in \tilde{M}$ there exists some $w_+ \in S_p\tilde{M}$ which is positively asymptotic to v and some $w_- \in S_p\tilde{M}$ which is negatively asymptotic to v. In contrast, the existence of $w_{+-} \in T_p\tilde{M}$ which is simultaneously positively *and* negatively asymptotic to v is rare. Moreover, if $v \sim w$ and $-v \sim -w$ then v, w bound a flat strip, i.e. a totally geodesic embedded copy of $[-a, a] \times \mathbb{R}$ with Euclidean metric. Here the number a is positive if v, w do not lie on the same geodesic trajectory. In particular, if $\mathrm{rank}(v) = 1$ (hence c_v is a regular geodesic) then there does not exist such w with $w \sim v$ and $-v \sim -w$ through *any* base point in the manifold outside c_v. In other words, if $w \sim v$ and $-w \sim -v$ on a rank 1 manifold then $w = g^t v$ for some t. On the other hand, if $\mathrm{rank}(v) > 1$ (and thus c_v is a singular geodesic) then v and hence c_v may lie in a flat strip of positive width, and in that case there are vectors w with $w \sim v$ and $-w \sim -v$ at base points outside c_v, namely at all base points in that flat strip.

Since $\tilde{M}$ is of nonpositive curvature, it is diffeomorphic to $\mathbb{R}^n$ by the Hadamard-Cartan theorem, hence to an open Euclidean n-ball. It admits the **compactification** $\overline{M} = \tilde{M} \cup \tilde{M}(\infty)$ where $\tilde{M}(\infty)$, the **boundary at infinity** of $\tilde{M}$, is the set of equivalence classes of positively asymptotic vectors, i.e., $\tilde{M}(\infty) = S\tilde{M}/\sim$.

A more accurate but cumbersome way to write $\overline{M}$ would be $\overline{\tilde{M}}$; however, since M is already compact, it is clear that the compactification is that of $\tilde{M}$.

A detailed and very readable description of spaces of nonpositive curvature, even of those not equipped with a manifold structure, can be found in [Bal2].

3.2. Stable and unstable spaces

Definition 3.5. *Let $\mathcal{K} : TS\tilde{M} \to S\tilde{M}$ be the* ***connection map****, i.e. $\mathcal{K}\xi := \nabla_{d\pi\xi} Z$ where ∇ is the Riemannian connection and $Z(0) = d\pi\xi$, $\frac{d}{dt}Z(t)\big|_{t=0} = \xi$. We obtain a Riemannian metric on SM, the* ***Sasaki metric****, by setting $\langle \xi, \eta \rangle := \langle d\pi\xi, d\pi\eta \rangle + \langle \mathcal{K}\xi, \mathcal{K}\eta \rangle$ for $\xi, \eta \in T_vSM$ where $v \in SM$. Hence we can talk about length of vectors in $TS\tilde{M}$.*

There is a canonical isomorphism $(d\pi, \mathcal{K})$ between T_vSM and the set of Jacobi fields along c_v. It is given by $\xi \mapsto J_\xi$ with $J_\xi(0) = d\pi \cdot \xi$, $J'_\xi(0) = \mathcal{K}\xi$. This uses the well-known fact that a Jacobi field is determined by its value and derivative at one point.

The space $TS\tilde{M}$, i.e. the tangent bundle of the unit sphere bundle, admits a natural splitting

$$TS\tilde{M} = E^s \oplus E^u \oplus E^0,$$

i.e. $T_vS\tilde{M} = E^s_v \oplus E^u_v \oplus E^0_v$ for all $v \in S\tilde{M}$, where

$$E^0_v := \mathbb{R} \cdot \frac{d}{dt} g^t v \Big|_{t=0},$$
$$E^s_v := \{\xi \in T_vS\tilde{M} : \xi \perp E^0,\ J_\xi \text{ is the stable Jacobi field along } d\pi\xi\},$$
$$E^u_v := \{\xi \in T_vS\tilde{M} : \xi \perp E^0,\ J_\xi \text{ is the unstable Jac. field along } d\pi\xi\}.$$

Definition 3.6. *For $v \in S\tilde{M}$, define $W^s(v)$, the* ***stable horosphere*** *based at v, to be the integral manifold of the distribution E^s passing through v. Similarly, define $W^u(v)$, the* ***unstable horosphere*** *based at v, via integrating E^u. The projection of W^s (resp. W^u) to $\tilde{M}$ is again called the stable horosphere (resp. the unstable horosphere). The flow direction of course integrates to a geodesic trajectory, which one might call $W^0(v)$. The 0- and u-directions are jointly integrable, giving* *rise to an integral manifold W^{0u}, and similarly the* 0*- and s-directions give rise to an integral manifold W^{0s}. We write B^i_δ (resp. $\overline{B^i_\delta}$) for the open (resp. closed) δ-neighborhood in W^i ($i = u, s, 0u, 0s, 0$).*

On the other hand, the u- and s-directions are usually not jointly integrable. Continuity of these foliations has been proven in this form by P. Eberlein [Ebe2] and J.-H. Eschenburg [Esch]:

Theorem 3.7. *Let M be a compact manifold of nonpositive curvature. Then the foliation $\{W^s(v) : v \in S\tilde{M}\}$ of $S\tilde{M}$ by stable horospheres is continuous. The same holds for the foliation $\{W^u(v) : v \in S\tilde{M}\}$ of $S\tilde{M}$ by unstable horospheres.*

Note that due to compactness of M (hence of SM), the continuity is automatically uniform.

During the same years, Eberlein considered similar questions on *Visibility manifolds* [Ebe2]. The continuity result was improved by M. Brin [BaPe, Appendix A] to Hölderness on the Pesin sets. (See [BaPe] for the definition of these sets.) For our discussion, uniform continuity is sufficient.

The following result is easier to show in the hyperbolic case (i.e. strictly negative curvature) than for nonpositive curvature, where it is a major theorem, proven by Eberlein ([Ebe1]):

Theorem 3.8. *Let M be a compact rank one manifold of nonpositive curvature. Then stable manifolds are dense. Similarly, unstable manifolds are dense.*

3.3. Important measures

The Riemannian structure gives rise to a natural measure λ on SM, called the **Liouville measure**. It is finite since M is compact. It is the prototypical smooth measure, i.e., for any smooth chart $\varphi : U \to \mathbb{R}^{2n-1}$, $U \subset SM$ open, the measure $\varphi_*\lambda$ on a subset of $\mathbb{R}^{2n-1}$ is smoothly equivalent to Lebesgue measure.

The well-known variational principle (see e.g. [KaHa]) asserts that the supremum of the entropies of invariant probability measures on SM is the topological entropy h. The variational principle by itself of course guarantees neither existence nor uniqueness of a **measure of maximal entropy**, i.e. one whose entropy actually equals h. These two facts were established in the setup of nonpositive curvature by G. Knieper [Kni1]:

Theorem 3.9. *There is a measure of maximal entropy for the geodesic flow on a compact rank one nonpositively curved manifold. Moreover, it is unique.*

The proof uses the Patterson-Sullivan construction ([Pat], [Sul]; see also [Kai1], [Kai2]). Knieper's construction builds the measure as limit of measures supported on periodic orbits.

For the case of strictly negative curvature, the measure of maximal entropy was previously constructed (in a different way) by G.A. Margulis [Mar3]. He used it to obtain his asymptotic results. His construction builds the measure as the product of limits of measures supported on pieces of stable and unstable leaves. The measure thus obtained is often called the **Margulis measure**. It agrees with the **Bowen measure** which is obtained as limit of measures concentrated on periodic orbits. U. Hamenstädt [Ham] gave a geometric description of the Margulis measure by projecting distances on horospheres to the boundary at infinity, and this description was immediately generalized to Anosov flows by B. Hasselblatt [Has].

The measure of maximal entropy is adapted to the dynamical properties of the flow. In particular, we will see that the conditionals of this measure show uniform expansion/contraction with time. In negative curvature, this can be seen by considering the Margulis measure, where this property is a natural by-product of the construction. In nonpositive curvature, however, this property is not immediate.

The measure of maximal entropy is sometimes simply called **maximal measure**. In the setup of nonpositive curvature, we think that the name **Knieper measure** is appropriate.

Remark 3.10. It is part of Katok's entropy conjecture and shown in [Kni1] that $m(\mathbf{Sing}) = 0$ (and in fact even that $h(g|_{\mathbf{Sing}}) < h(g)$). In contrast, whether $\lambda(\mathbf{Sing}) = 0$ or not is a major open question; it is equivalent to the famous problem of ergodicity of the geodesic flow in nonpositive curvature with respect to λ. On the other hand, ergodicity of the geodesic flow in nonpositive curvature with respect to m has been proven by Knieper.

A very useful dynamical property—stronger than ergodicity—is mixing. For nonpositive curvature it has recently been proven by M. Babillot [Bab]:

Theorem 3.11. *The measure of maximal entropy for the geodesic flow on a compact rank one nonpositively curved manifold is mixing.*

We use this property in our proof of the asymptotic formula.

3.4. Parallel Jacobi fields

Lemma 3.12. *The vector $v \in SM$ is regular if and only if $W^u(v)$, $W^s(v)$ and $W^0(v)$ intersect transversally at v.*

Here transversality of the three manifolds means that

$$T_vSM = T_vW^u \oplus T_vW^0 \oplus T_vW^s.$$

Proof. $W^u(v)$ and $W^s(v)$ intersect with zero angle at v if and only if there exist

$$\xi \in TW^u(v) \cap TW^s(v) \subset T_vSM.$$

But $\xi \in TW^s(v)$ is true if and only if J_ξ is the stable Jacobi field along c_v, and $\xi \in TW^u(v)$ is true if and only if J_ξ is the unstable Jacobi field along c_v. A Jacobi field J is both the stable and the unstable Jacobi field along c_v if and only if J is parallel. The nonexistence of such J perpendicular to c_v is just the definition of rank one. □

3.5. Coordinate boxes

Definition 3.13. *We call an open set $U \subset SM$ of diameter at most δ* ***regularly coordinated*** *if for all $v, w \in U$ there are unique $x, y \in U$ such that*

$$x \in B^u_\delta(v),\ y \in B^0_\delta(x),\ w \in B^s_\delta(y).$$

In other words, v can be joined to w by means of a unique short three-segment path whose first segment is contained in $W^u(v)$, whose second segment is a piece of a flow line and whose third segment is contained in $W^s(w)$.

Proposition 3.14. *If v is regular then it has a regularly coordinated neighborhood.*

Proof. Some 4δ-neighborhood V of v is of rank one. Let

$$U = B^s_\delta(g^{(-\delta,\delta)}B^u_\delta(v)).$$

This is contained in V and hence of rank one. It is open since W^0, W^u and W^s are transversal by Lemma 3.12.

By construction, for any $w \in V$, there exists a pair (x, y) such that

$$B^u_\delta(v) \ni x \in B^0_\delta(y),\ y \in B^s_\delta(w).$$

Assume there is another pair (x', y') with this property. From

$$B^u_\delta(x) \ni v \in B^u_\delta(x')$$

we deduce $x \in B^u_{2\delta}(x')$, and from

$$B^0_\delta(x) \ni y \in B^s_\delta(w),\ w \in B^s_\delta(y'),\ y' \in B^0_\delta(x')$$

we deduce $x \in B^{0s}_{4\delta}(x')$. Hence x and x' are simultaneously positively and negatively asymptotic; therefore, they bound a flat strip. Since

V is of rank one, there is no such strip of nonzero width in $U \subset V$. Hence x and x' lie on the same geodesic. Since $x \in W^u(x')$, these two points are identical.

The same argument with u and s exchanged shows that $y = y'$. Hence the pair (x, y) is unique. $\square$

3.6. The Busemann function and conformal densities

Definition 3.15. *Let $b(.,q,\xi)$ be the* **Busemann function** *centered at $\xi \in \tilde{M}(\infty)$ and based at $q \in \tilde{M}$. It is given by*

$$b(p,q,\xi) := \lim_{p_n \to \xi} (d(q,p_n) - d(p,p_n))$$

for $p, q \in \tilde{M}$ and is independent of the sequence $p_n \to \xi$.

Remark 3.16. The function b satisfies

$$b(p,q,\xi) = \lim_{t\to\infty} (d(c_{p,\xi}(t), q) - t)$$

where $c_{p,\xi}$ is the geodesic parameterized by arclength with $c_{p,\xi}(0) = p$ and $c_{p,\xi}(t) \to \xi$ as $t \to \infty$.

For ξ and p fixed, we have

$$b(p, p_n, \xi) \to -\infty \quad \text{for} \quad p_n \to \xi$$

and

$$b(p, p_n, \xi) \to \infty \quad \text{for} \quad \lim_n p_n \in \tilde{M}(\infty) \setminus \{\xi\}.$$

Moreover it is clear that $b(p,q,\xi) = -b(q,p,\xi)$. We use the sign convention where $b(p,q,\xi)$ is negative whenever p, q, ξ lie on a geodesic in this particular order.

Definition 3.17. $(\mu_p)_{p\in\tilde{M}}$ *is a h-dimensional* **Busemann density** *(also called conformal density) if the following are true:*

- *For all $p \in \tilde{M}$, μ_p is a finite nonzero Borel measure on $\tilde{M}(\infty)$.*
- *μ_p is equivariant under deck transformations, i.e., for all $\gamma \in \pi_1(M)$ and all measurable $S \subset \tilde{M}(\infty)$ we have*

$$\mu_{\gamma p}(\gamma S) = \mu_p(S).$$

- *When changing the base point of μ_p, the density transforms as follows:*

$$\frac{d\mu_p}{d\mu_q}(\xi) = e^{-hb(q,p,\xi)}.$$

In the case of nonpositive curvature, Knieper has shown in [Kni1] that μ_p is unique up to a multiplicative factor and that it can be obtained by the Patterson-Sullivan construction.

4. The measure of maximal entropy

In section 5 we will use the fact that if m is the measure of maximal entropy then it gives rise to conditional measures m^u, m^{0u}, m^s and m^{0s} on unstable, weakly unstable, stable and weakly stable leaves which have the property that the measures m^{0u} and m^u expand uniformly with t and that m^s and m^{0s} contract uniformly with t.

Remark 4.1. In [Gun] we present an alternative and more general construction of the measure of maximal entropy in nonpositive curvature and rank one which follows the principle of Margulis' construction. Using that construction, the uniform expansion/contraction properties shown here are already a straightforward consequence of the construction. Also, that construction is more general; it is appropriate for non-geodesic flows satisfying suitable cone conditions (see [Kat2] for these). On the other hand, Knieper's approach, which substantially uses features of rank one nonpositively curved manifolds, is shorter and therefore is the one we use in this article.

First we give Knieper's definition of the measure of maximal entropy [Kni1]:

Definition 4.2. *Let $(\mu_p)_{p\in\tilde{M}}$ be a Busemann density. Let*

$$\Pi : S\tilde{M} \to \tilde{M}(\infty) \times \tilde{M}(\infty), \quad \Pi(v) := (v_\infty, v_{-\infty})$$

be the projection of a vector to both endpoints $v_{\pm\infty} = \lim_{t\to\pm\infty} \pi g^t v$ of the corresponding geodesic. Then the measure of maximal entropy of a set $A \subset S\tilde{M}$ (we can without loss of generality assume A to be regular) is given by

$$m(A) := \int\limits_{\xi,\eta\in\tilde{M}(\infty),\ \xi\neq\eta} \operatorname{len}(A \cap \Pi^{-1}(\xi,\eta)) e^{-h(b(p,q,\xi)+b(p,q,\eta))} d\mu_p(\xi) d\mu_p(\eta),$$

where $q \in \pi\Pi^{-1}(\xi,\eta)$ and $p \in \tilde{M}$ is arbitrary.

Saying that $\Pi^{-1}(\xi,\eta)$ is a geodesic already is a slight simplification, but a fully justified one since we need to deal only with the regular set.

4.1. Discussion of the conditionals

Given a vector $v \in S\tilde{M}$ with base point p, we want to put a conditional measure m^u on the stable horosphere $b(p,.,\xi)^{-1}(0)$ given by v and centered at $\xi := v_\infty$ (or on $W^s(v)$, which is the unit normal bundle of $b(p,.,\xi)^{-1}(0)$). This conditional is determined by a multiplier with respect to some given measure on this horosphere. Note that the set of points q on the horosphere is parameterized by the set $\tilde{M}(\infty)\setminus\{\xi\}$ via projection from ξ into the boundary at infinity, hence the multiplier depends on $\eta := \mathrm{proj}_\xi(q)$, i.e. is proportional to $d\mu_x(\eta)$ for some x. The canonical choice for x is p. Clearly the whole horosphere has infinite m^u-measure, but μ_x is finite for any x. Thus the multiplier of $d\mu_p$ has to have a singularity, and this has to happen at $\eta = \xi$ since any neighborhood of ξ is the projection of the complement of a compact piece of the horosphere. The term $e^{-hb(p,q,\eta)}$ has the right singularity (note that $\eta \to \xi$ means $q \to \xi$), and by the basic properties of the Busemann function the term $e^{-hb(p,q,\eta)}$ then converges to infinity. Therefore we investigate $m_p(q) := e^{-hb(p,q,\eta)} d\mu_p(\eta)$. First we prove that this is indeed the stable conditional measure for dm^s. We will parameterize dm by vectors instead of their base points.

Definition 4.3. *For $v, w \in S\tilde{M}$, let*

$$\begin{aligned} dm^u_v(w) &:= e^{-hb(\pi v, \pi w, w_\infty)} \cdot d\mu_{\pi v}(w_\infty), \\ dm^s_v(w) &:= e^{-hb(\pi v, \pi w, w_{-\infty})} \cdot d\mu_{\pi v}(w_{-\infty}). \end{aligned}$$

Proposition 4.4. *dm^s_v, dm^u_v and dt are the stable, unstable and center conditionals of the measure of maximal entropy.*

Proof. Observe that

$$\begin{aligned} dt\, dm^u_v(w)\, dm^s_v(w) &= dt\, e^{-h(b(\pi v, \pi w, w_\infty) + b(\pi v, \pi w, w_{-\infty}))} \\ &\quad \cdot d\mu_{\pi v}(w_\infty) d\mu_{\pi v}(w_{-\infty}) \\ &= dt\, e^{-h(b(p,q,\xi) + b(p,q,\eta))} d\mu_p(\xi) d\mu_p(\eta) \quad =: E \end{aligned}$$

with $p := \pi v,\ q := \pi w,\ \xi := w_\infty,\ \eta := w_{-\infty}$. This formula already agrees with the formula in Definition 4.2, although the meaning of the parameters does not necessarily do so: In Definition 4.2, p and to some extend q are arbitrary in $\tilde{M}$, while in the formula for E they are fixed. Thus we need to show that if we change them within the range allowed in Definition 4.2, the value of E does not change.

Lemma 4.5. *The term E does not change if q is replaced by any point in $\tilde{M}$ on the geodesic $c_{\eta\xi}$ from η to ξ and p by an arbitrary point in $\tilde{M}$.*

Proof. First we show that q can be allowed to be anywhere on $c_{\eta\xi}$: Parameterize $c_{\eta\xi}$ by arclength with arbitrary parameter shift in the direction from η to ξ. Replacing $q = c_{\eta\xi}(s)$ by $q' = c_{\eta\xi}(s')$ changes $b(p,q,\xi)$ to $b(p,q',\xi) = b(p,q,\xi) - (s' - s)$ since we move the distance $s' - s$ closer to ξ. It also changes $b(p,q,\eta)$ to $b(p,q',\eta) = b(p,q,\eta) + (s' - s)$ since we move the distance $s' - s$ away from to η. Thus E does not change under such a translation of q.

Now fix q anywhere on $c_{\eta\xi}$ and replace p by some arbitrary $p' \in \tilde{M}$. Note that

$$\begin{aligned} d\mu_{p'}(\xi) &= e^{hb(p',p,\xi)} d\mu_p(\xi), \\ d\mu_{p'}(\eta) &= e^{hb(p',p,\eta)} d\mu_p(\eta), \\ b(p',q,\xi) &= b(p,q,\xi) + b(p',p,\xi), \\ b(p',q,\eta) &= b(p,q,\eta) + b(p',p,\eta). \end{aligned}$$

Thus

$$e^{-h(b(p',q,\xi)+b(p',q,\eta))} d\mu_{p'}(\xi) d\mu_{p'}(\eta) = e^{-h(b(p,q,\xi)+b(p,q,\eta))} d\mu_p(\xi) d\mu_p(\eta).$$

Hence E also does not change if p is changed to any arbitrary point. □

This also concludes the proof of Proposition 4.4. □

4.2. Proof of uniform expansion/contraction of the conditionals

Let w_t denote $g^t w$.

Theorem 4.6 (Uniform expansion/contraction of the conditionals). *For all $t \in \mathbb{R}$ and all $v, w \in S\tilde{M}$ we have*

$$\begin{aligned} dm_v^u(w_t) &= e^{ht} \cdot dm_v^u(w), \\ dm_v^s(w_t) &= e^{-ht} \cdot dm_v^s(w). \end{aligned}$$

The same uniform expansion holds with m^u replaced by m^{0u} and the same uniform contraction with m^s replaced by m^{0s}.

Here $dm^{0u} := dm^u dt$, $dm^{0s} := dm^s dt$.

Proof.

$$\begin{aligned} dm_v^s(w_t) &= e^{-hb(\pi v, \pi w_t, w_{-\infty})} d\mu_{\pi v}(w_{-\infty}) \\ &= e^{-h(b(\pi v, \pi w, w_{-\infty}) + b(\pi w, \pi w_t, w_{-\infty}))} d\mu_{\pi v}(w_{-\infty}) \\ &= e^{-hb(\pi v, \pi w, w_{-\infty}) - ht} d\mu_{\pi v}(w_{-\infty}) \\ &= e^{-ht} \cdot e^{-hb(\pi v, \pi w, w_{-\infty})} d\mu_{\pi v}(w_{-\infty}) \\ &= e^{-ht} \cdot dm_v^s(w). \end{aligned}$$

Similarly

$$\begin{aligned} dm_v^u(w_t) &= e^{-hb(\pi v, \pi w_t, w_{+\infty})} d\mu_{\pi v}(w_{+\infty}) \\ &= e^{-hb(\pi v, \pi w, w_{+\infty}) + ht} d\mu_{\pi v}(w_{+\infty}) \\ &= e^{ht} \cdot dm_v^u(w). \end{aligned}$$

This shows the desired uniform expansion of m^u and the uniform contraction of m^s. From this we also immediately see the uniform expansion of m^{0u} and the uniform contraction of m^{0s} since dt is evidently invariant under g^t. □

4.3. Proof of holonomy invariance of the conditionals

Another important property of the conditional measures on the leaves is holonomy invariance. We formulate holonomy invariance on infinitesimal unstable pieces here, but of course this is equivalent to holonomy invariance that deals with pieces of leaves of (small) positive size.

We consider positively asymptotic vectors w, w' and calculate the infinitesimal $0u$-measure on corresponding leaves. We let v, v' be some (arbitrary) base points used as parameters for the pieces of leaves, so that w lies in the same $0u$-leaf of v and similarly w' in that of v'. The factor dt is evidently invariant, so we do not have to mention it any further.

Theorem 4.7 (Holonomy invariance of the conditionals of the measure of maximal entropy).

$$dm^u_v(w) = dm^u_{v'}(w')$$

whenever $v' \in W^s(v)$, $w' \in W^s(w)$, $w \in W^{0u}(v)$ *and* $w' \in W^{0u}(v')$.

Proof. Note that the equation $w' \in W^s(w)$ is equivalent to the two equations

$$w'_\infty = w_\infty,$$
$$b(\pi w, \pi w', w_\infty) = 0.$$

The latter equation is equivalent to $b(p, \pi w, w_\infty) = b(p, \pi w', w_\infty)$ for all $p \in \tilde{M}$. Thus clearly

$$\begin{aligned} dm^u_{v'}(w') &= e^{-hb(\pi v', \pi w', w_\infty)} d\mu_{\pi v'}(w'_\infty) \\ &= e^{-hb(\pi v', \pi w', w_\infty)} d\mu_{\pi v'}(w_\infty). \end{aligned}$$

Now

$$\begin{aligned} b(\pi v', \pi w', w_\infty) &= b(\pi v', \pi v, w_\infty) + b(\pi v, \pi w', w_\infty) \\ &= b(\pi v', \pi v, w_\infty) + b(\pi v, \pi w, w_\infty) \end{aligned}$$

and $d\mu_{\pi v'}(w_\infty) = e^{-hb(\pi v, \pi v', w_\infty)} d\mu_{\pi v}(w_\infty)$. Thus

$$\begin{aligned} dm^u_{v'}(w') &= e^{-h(b(\pi v', \pi w', w_\infty) + b(\pi v, \pi v', w_\infty))} d\mu_{\pi v}(w_\infty) \\ &= e^{-h(b(\pi v', \pi v, w_\infty) + b(\pi v, \pi w, w_\infty) + b(\pi v, \pi v', w_\infty))} d\mu_{\pi v}(w_\infty) \\ &= e^{-hb(\pi v, \pi w, w_\infty)} d\mu_{\pi v}(w_\infty) \\ &= dm^u_v(w). \end{aligned}$$

□

Corollary 4.8.

$$dm_v^s(w) = dm_{v'}^s(w')$$

whenever $v' \in W^u(v)$, $w' \in W^u(w)$, $w \in W^{0s}(v)$ *and* $w' \in W^{0s}(v')$.

Proof. This is the same proof as before with u and s exchanged and with w_∞ and $w_{-\infty}$ exchanged. □

Note that m^{0u} is invariant under holonomy along s-fibers and m^{0s} under holonomy along u-fibers, but m^u is not invariant under holonomy along $0s$-fibers and m^s not invariant under holonomy along $0u$-fibers due to expansion (resp. contraction) in the flow direction.

5. Counting closed geodesics

In this final section we count the periodic geodesics on M. The method used here is a generalization of the method which, for the special case of negative curvature, was outlined in [Tol] and provided with more detail in [KaHa].

Definition 5.1. *Let f and g be expressions depending on t and ε. Write*

$$f \sim g$$

if $f(t)/g(t) \to 1$ as $t \to \infty$. In other words, $f/g - 1 = o(1)$ (here 1 is understood to be a (constant) function of t). Write

$$f \bowtie g$$

if $f/g - 1 = O(\varepsilon)$ as $\varepsilon \to 0$. We write

$$f \cong g$$

if there exists f' with $f \sim f' \bowtie g$, i.e. if $f/g = (1+O(\varepsilon))(1+o(t \mapsto 1))$. Thus the equivalence relation "$\cong$" is implied by both "$\sim$" and "$\bowtie$", but the equivalence relations "$\sim$" and "$\bowtie$" are independent.

5.1. The flow cube

Fix any $v_0 \in \mathbf{Reg}$. Choose sufficiently small ε and δ such that 2ε is smaller than $\mathrm{inj}(M)$ (the injectivity radius of M), such that $2\delta < \varepsilon$, such that $B_{4\varepsilon}(v_0) \subset \mathbf{Reg}$, and such that further requirements on the smallness of these which we will mention later are satisfied.

Definition 5.2. *Let the* ***flow cube*** *be* $A := \overline{B^s}(g^{[0,\varepsilon]}(\overline{B^u_\delta}(v_0))) \subset$ **Reg**. *Here $\overline{B^u_\delta}(v_0)$ is the closed unstable ball of radius δ around v_0. We choose $\overline{B_s} = \overline{B_s}(v)$ as the closure of an open set contained in the closed stable ball of radius δ around $v \in g^{[0,\varepsilon]}(\overline{B^u_\delta}(v_0))$; this set, which depends on v, can be chosen in such a way that it contains v and that A has the following local product structure: For all $w, w' \in A$ there exists a unique $\beta \in [-\varepsilon, \varepsilon]$ such that*

$$\overline{B^s}(w) \cap \overline{B^u_{2\delta}}(g^\beta w')$$

is exactly one point. This is the local product structure in **Reg** *described in Proposition 3.14. We call $\overline{B^s}(v)$ the* **stable fiber** *(or stable ball) in A containing v.*

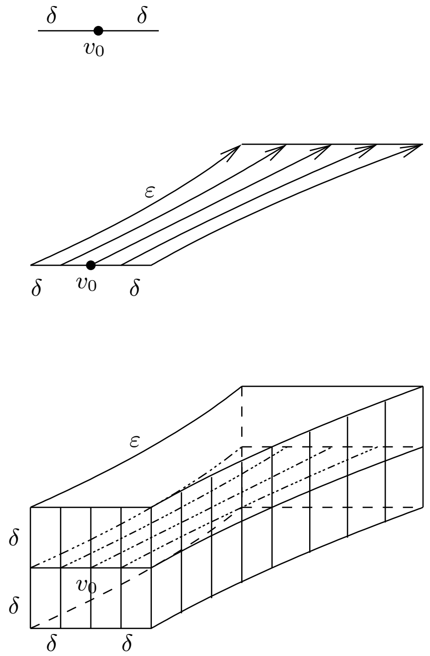

Fig. 5.1. The flow cube A: an unstable neighborhood of v_0 (top) is iterated (center) and a stable neighborhood of that is formed (bottom).

In the following arguments, the cube A will first be fixed. In particular, ε and δ are considered fixed (although subject to restrictions on their size). At the end of the article, we will consider what happens when $\varepsilon \to 0$.

Definition 5.3. *Let the* ***depth*** $\tau : A \to [0, \varepsilon]$ *be defined by*

$$v \in \overline{B^s}(g^{\tau(v)}\overline{B^u_\delta}(v_0)).$$

Lemma 5.4. *For all $v \in A$, $w \in \overline{B^u_{2\delta}}(v) \cap A$ it is true that*

$$|\tau(w) - \tau(v)| < \varepsilon^2/2.$$

Proof. The foliation W^u is uniformly continuous by Theorem 3.7 and compactness of SM, and without loss of generality δ was chosen small enough. □

Lemma 5.5 (Stable fiber contraction). *There is a function $\sigma = \sigma(t)$ such that*

$$m^s(g^t B^s(v)) \bowtie \sigma(t)$$

for all $v \in A$. In particular, for all v, w in A we have

$$m^s(g^t B^s(v)) \bowtie m^s(g^t B^s(w)).$$

Proof. First we show that $m^s(B^s(v)) \bowtie m^s(B^s(w))$. Observe that for $0 < a < \mathrm{inj}(M)$ (a independent of ε), $g^{[0,a]}B^s(v)$ is u-holonomic to a subset of $g^{[-2\varepsilon,a+2\varepsilon]}B^s(w)$. Thus

$$\frac{m^s(B^s(v))}{m^s(B^s(w))} = \frac{m^{0s}(g^{[0,a]}B^s(v))}{m^{0s}(g^{[0,a]}B^s(w))} \le \frac{\int_{-2\varepsilon}^{a+2\varepsilon} e^{-ht}dt}{\int_0^a e^{-ht}dt} \bowtie 1.$$

By exchanging v and w, the opposite inequality is also proven. Hence $m^s(B^s(v)) \bowtie m^s(B^s(w))$. Applying g^t therefore gives

$$\begin{aligned} m^s(g^t B^s(v)) &= e^{ht} m^s(B^s(v)) \\ &\bowtie e^{ht} m^s(B^s(w)) \\ &= m^s(g^t B^s(w)) \end{aligned}$$

as claimed. It immediately follows that we can define

$$\sigma(t) := m^s(g^t B^s(v))$$

for some arbitrary $v \in A$, and this definition does not depend on v (up to $\bowtie$-equivalence). □

Remark 5.6. Uniform contraction shows that $\sigma(t) \bowtie \mathrm{const} \cdot e^{-ht}$.

5.2. Expansion at the boundary

Definition 5.7. *For the cube A as above, we call*

$\partial^u A := \overline{B^s}(g^{[0,\varepsilon]}(\partial B^u_\delta(v_0)))$ *the* ***unstable end*** *of the cube,*

$\partial_0 A := \overline{B^s}(\overline{B^u_\delta}(v_0))$ *the* ***back end*** *and*

$\partial_\varepsilon A := \overline{B^s}(g^\varepsilon(\overline{B^u_\delta}(v_0)))$ *the* ***front end*** *of the cube.*

For $v \in A$ define

$$s(v) := \sup\{r : B^u_r(v) \subset A\}$$

to be the ***distance to the unstable end*** *of the flow cube.*

The stable and the unstable end are topologically the product of an interval, a k-ball and a $(k-1)$-sphere, where $k = \dim W^u(v) = \dim W^s(v) = (\dim SM - 1)/2$; hence they are connected iff $k \neq 1$.

Lemma 5.8 (Expansion of distance to unstable end). *There exists a monotonous positive function $S : \mathbb{R} \to \mathbb{R}$ satisfying $S(t) \to 0$ as $t \to \infty$ and such that if $s(v) > S(t)$ for an element $v \in A$ which satisfies $g^t v \in A$ then*

$$\overline{B^u_{2\delta}}(g^t v) \cap A \subset g^t \overline{B^u_{S(t)}}(v).$$

That means that if a point v is more than $S(t)$ away from the unstable end of the cube then the the image of a small u-disc (of size $> S(t)$) around v has the property that its unstable end is completely outside A.

Proof. By nonpositivity of the curvature, B^u_δ noncontracts, i.e., distances on it are nondecreasing in length. This is true even infinitesimally, i.e. for unstable Jacobi fields. By convexity of Jacobi fields, such distances also cannot stay bounded. Hence the radius of the largest u-ball contained in $g^t B^u_\delta$ becomes unbounded for $t \to \infty$.

Hence for all $\gamma > 0$ we can find T_γ such that

$$g^{T_\gamma} B^u_\gamma(v) \supset B^u_{2\delta}(g^{T_\gamma}(v)). \tag{5.1}$$

By compactness of A, this choice of T_γ can be made independently of $v \in A$. Without loss of generality T_γ is a strictly decreasing function of γ. Choose a function $S > 0$ so that $S(t) \leq \gamma$ for $t > T_\gamma$. E.g., choose $S(.) = T_.^{-1}$, i.e. $T_{S(t)} = t$ for $t \geq 0$. S can be chosen decreasing since $T_.$ can be. Therefore, given $v \in A$, if $t > T_{s(v)}$ then $s(v) > S(t)$, and thus equation (5.1) shows the claim. □

Remark 5.9. The convergence of S to zero in the previous Lemma is not necessarily exponential, as opposed to the case where the curvature of M is negative (i.e. the uniformly hyperbolic case). However, we do not need this property of exponential convergence.

If the smallest such S would not converge to zero, it would require the existence of a flat strip of width $\liminf_{t\to\infty} S(t) = \lim_{t\to\infty} S(t)$, which would intersect A. Since a neighborhood of A is regular, this cannot happen.

5.3. Intersection components and orbit segments

Definition 5.10. *Let A'_t be the set of $v \in A$ with $s(v) \geq S(t)$ and $\tau(v) \in [\varepsilon^2, \varepsilon - \varepsilon^2]$. Thus A'_t is the set A with a small neighborhood of the unstable end and of the front end and back end removed.*

Definition 5.11. *Let Φ_t be the set of all* ***full components of intersection*** *at time t: If I is a connected component of $A'_t \cap g^t(A'_t)$ then define*

$$\phi_t^I := g^{[-\varepsilon,\varepsilon]}(I) \cap A \cap g^t(A)$$

and

$$\Phi_t := \{\phi_t^I : I \text{ is a connected component of } A'_t \cap g^t(A'_t)\}.$$

Let

$$N(A,t) := \#\Phi_t$$

be the number of its elements.

We call the set $g^{[-\varepsilon,\varepsilon]}v \cap A$ the geometric ***orbit segment*** *of length ε in A through v. Similarly we speak about the orbit segment of length $\varepsilon - 2\varepsilon^2$ in A'_t.*

Lemma 5.12. *For every orbit segment of length $\varepsilon - 2\varepsilon^2$ in A'_t that belongs to a periodic orbit of period in $[t-\varepsilon+2\varepsilon^2, t+\varepsilon-2\varepsilon^2]$ there exists a unique $\phi_t^I \in \Phi_t$ through which the orbit segment passes.*

Proof. Existence: If $g^L o = o$ for an orbit segment o of length $\varepsilon - 2\varepsilon^2$ of A'_t that belongs to a periodic orbit of period $L \in [t-\varepsilon+2\varepsilon^2, t+\varepsilon-2\varepsilon^2]$ then o also intersects $g^t A'_t$, hence some component of $A'_t \cap g^t A'_t$.

Uniqueness: Assume that o passes through $\phi_t^I, \phi_t^J \in \Phi_t$, i.e. $p = o(a) \in \phi_t^I$, $q = o(b) \in \phi_t^J$ for $|b-a| \le \varepsilon$. Then o passes through I, J (the connected components corresponding to ϕ_t^I, ϕ_t^J) respectively. Since $g^t A'_t$ is pathwise connected, there is a path c in $g^t A'_t$ from p to q. We can assume that c consists of a segment in W^u, followed by a segment in W^0, followed by a segment in W^s. By applying g^{-t}, we get a path $g^{-t} \circ c$ in A'_t from $o(a-t) \in A'_t$ to $o(b-t) \in A'_t$. The local product structure in A'_t shows that the u-segment and the s-segment of $g^{-t} \circ c$ have length 0. Therefore $g^{-t} \circ c$ and hence c is an orbit segment. This means that c lies in A'_t and in $g^t A'_t$. Hence p and q lie in the same component, i.e. $\phi_t^I = \phi_t^J$. □

In the other direction, we have the following Lemma:

Lemma 5.13. *For every $\phi_t^I \in \Phi_t$ there exists a unique periodic orbit with period in $[t-\varepsilon, t+\varepsilon]$ and a unique segment on that orbit passing through ϕ_t^I.*

In other words, up to a small error, intersection components correspond to periodic orbits, and of all orbit segments that belong to such a periodic orbit, just one orbit segment goes through any particular full component of intersection.

Proof. Choose ϕ_t^I. Here we of course only need to consider the case $t > 0$. Since $A'_t \subset A$ has rank one, it follows that for every $v \in A'_t$ any nonzero stable Jacobi field along c_v is strictly decreasing in length, and any nonzero unstable Jacobi field is strictly increasing in length. Since the set of stable (resp. unstable) Jacobi fields is linearly isomorphic to E^s (resp. E^u) via $(d\pi, \mathcal{K})^{-1}$, it follows that for all $v \in A'_t \cap g^t A'_t$:

$$|dg^t \xi| < |\xi| \quad \forall \xi \in E^s(v) \setminus \{0\},$$

$$|dg^{-t} \xi| < |\xi| \quad \forall \xi \in E^u(v) \setminus \{0\}.$$

By compactness of A'_t and hence of ϕ_t^I there exists $c < 1$ such that for all $v \in A'_t \cap g^t A'_t$:

$$|dg^t \xi| < c|\xi| \quad \forall \xi \in E^s(v) \setminus \{0\},$$

$$|dg^{-t} \xi| < c|\xi| \quad \forall \xi \in E^u(v) \setminus \{0\}.$$

Hence g^t restricted to ϕ_t^I is (apart from the flow direction) hyperbolic. Thus the first return map on a transversal to the flow is hyperbolic. Hence it has a unique fixed point.

Therefore there exists a unique periodic orbit through ϕ_t^I. Two geometrically different (hence disjoint) orbit segments would give rise to two different fixed points. Hence the geometric segment on the periodic orbit is also unique. □

5.4. Intersection thickness

Definition 5.14. *Define the* **thickness** *(or length)* $\theta : \Phi_t \to [0, \varepsilon]$ *by*

$$\theta(\phi_t^I) := \varepsilon - \sup \left\{ \tau(v) : v \in g^t \left(\bigcup_{w \in A,\, g^t w \in I} g^{[-\varepsilon,\varepsilon]} w \cap \partial_0 A \right) \right\}$$

for such ϕ_t^I *which intersect* $\partial_\varepsilon A$ *(the front end of* A*) and*

$$\theta(\phi_t^I) := \inf \left\{ \tau(v) : v \in g^t \left(\bigcup_{w \in A,\, g^t w \in I} g^{[-\varepsilon,\varepsilon]} w \cap \partial_\varepsilon A \right) \right\}$$

for such ϕ_t^I *which intersect* $\partial_0 A$ *(the back end of* A*).*

Lemma 5.15 (The average thickness is asymptotically half that of the flow box).

$$\frac{1}{N(A,t)} \sum_{\phi_t^I \in \Phi_t} \theta(\phi_t^I) \cong \frac{\varepsilon}{2}.$$

Proof. Take any full component of intersection $\phi_t^I \in \Phi_t$. Assume that it intersects the front end of A. We cut A along flow lines in $n := \lfloor 1/\varepsilon \rfloor$ pieces

$$A_i := \left\{ v \in A : \tau(v) \in \left[\frac{i\varepsilon}{n}, \frac{(i+1)\varepsilon}{n} \right) \right\}$$

of equal measure ($i = 0, \ldots, n-1$). By the mixing property, $m(A_i \cap g^t A_0)$ is asymptotically independent of i as $t \to \infty$. Hence the number of full components of intersection of A_i with $g^t A_0$ is asymptotically independent of i. Since any intersection component of $A_i \cap g^t A_0$ has depth τ with $|\tau - i\varepsilon/n| < \varepsilon/n$, we see that the average of θ is $\varepsilon/2$ up to an error of order ε^2.

The same reasoning applies if A_0 is changed to A_{n-1}, hence for ϕ_t^I intersecting the back end of A instead of the front end. □

Note that if we compute the measure of an intersection $A_0 \cap g^t A_{n-1}$ for t large, the terms which are not in full components of intersection contribute only a fraction which by mixing is asymptotically zero because $m(A_t') \cong m(A)$, which follows from

$$m(\{v \in A : s(v) < S(t)\}) \to 0 \text{ as } t \to \infty$$

and

$$m(\{v \in A : \tau(v) \in [0, \varepsilon^2] \cup [\varepsilon - \varepsilon^2, \varepsilon]\}) = 2\varepsilon m(A).$$

5.5. Counting intersections

Proposition 5.16. *The number $N(A,t)$ satisfies*

$$N(A,t) \cong 2e^{ht} m(A).$$

Proof. Let $F := g^{[0,\varepsilon]} \overline{B_\delta^u}(v_0)$. First note that

$$m(\phi_t^I) \bowtie \frac{\theta(\phi_t^I)}{\varepsilon} m^{0u}(F)\sigma(t)$$

for $\phi_t^I \in \Phi_t$ since by Lemma 5.5 the stable measure of the pieces of stable fibers in ϕ_t^I is equal to $\sigma(t)$ up to an error term that disappears as $\varepsilon \to 0$ and since by holonomy invariance (Theorem 4.7) and by Lemma 5.8 the m^{0u}-measure of $0u$-leaves of ϕ_t^I is the same as that of F, except that the thickness of the intersection is not ε but $\theta(\phi_t^I)$. Since by Lemma 5.15 the average of the $\theta(\phi_t^I)$ is asymptotically $\varepsilon/2$, we get

$$\frac{1}{N(A,t)} \sum_{\phi_t^I \in \Phi_t} m(\phi_t^I) \cong \frac{1}{2}\sigma(t) m^{0u}(F).$$

Since the measure of $A \cap g^t A$ is asymptotically the sum of the measures of the full components of intersection (there are $N(A,t)$ of those), we obtain

$$m(A \cap g^t A) \cong \frac{1}{2} N(A,t) \sigma(t) m^{0u}(F). \tag{5.2}$$

Moreover note that by the mixing property of g,

$$m(A \cap g^t A) \cong e^{ht} m(A) \sigma(t) m^{0u}(F).$$

The claim follows from combining these two equations. □

Lemma 5.17. *The number of orbit segments passing through A that belong to periodic orbits with period in $[t-\varepsilon, t+\varepsilon]$ is $\cong N(A,t)$.*

Proof. Let $\mathbf{G}(t,\varepsilon)$ be the set of all geometric orbit segments in A of periodic orbits with period in $[t-\varepsilon, t+\varepsilon]$. As before, $\Phi_t = \Phi_t(\varepsilon)$ is the set of all full components for given t, ε. Let $G(t,\varepsilon) := \#\mathbf{G}(t,\varepsilon)$ and $N(t,\varepsilon) := \#\Phi_t(\varepsilon)$. We want to show

$$N(t,\varepsilon) \cong G(t,\varepsilon).$$

By Lemma 5.12 we have a map $\mathbf{G}(t, \varepsilon - 2\varepsilon^2) \to \Phi_t(\varepsilon)$ and by Lemma 5.13 a map $\Phi_t(\varepsilon) \to \mathbf{G}(t,\varepsilon)$. These maps are invertible between their domains and images; hence they are injective. Thus we have

$$G(t, \varepsilon - 2\varepsilon^2) \le N(t,\varepsilon) \le G(t,\varepsilon).$$

Since $N(t, \varepsilon - 2\varepsilon^2) \le G(t, \varepsilon - 2\varepsilon^2)$, it suffices to show

$$N(t,\varepsilon) \cong N(t, \varepsilon + \varepsilon^2).$$

Partition A again into $n := \lfloor 1/\varepsilon \rfloor$ pieces

$$A_i := \{v \in A : \tau(v) \in [i\varepsilon/n, (i+1)\varepsilon/n)\}$$

of equal measure ($i = 0, \dots, n-1$). Mixing implies that

$$m(A_0 \cap g^t A_{n-1}) \cong \varepsilon^2 m(A)^2.$$

Observe that in analogy to equation (5.2) we have

$$\varepsilon^2 m(A)^2 \cong m(A_0 \cap g^{t+\varepsilon^2} A_{n-1}) \cong \frac{1}{2} \varepsilon^2 N(t,\varepsilon) m^{0u}(F) \sigma(t).$$

The full components of $A_0 \cap g^{t+\varepsilon^2} A_{n-1}$ which are newly created at the back end of A by increasing t to $t + \varepsilon^2$ have average thickness $\varepsilon^2/2$ and hence average measure $\frac{1}{2} \varepsilon m^{0u}(F) \sigma(t)$. Hence this increase of t can produce at most $\cong \varepsilon N(t,\varepsilon)$ such full components. Thus

$$N(t + \varepsilon^2, \varepsilon) \overset{\sim}{\le} N(t,\varepsilon) + \varepsilon N(t,\varepsilon).$$

(The notation $f_1(t,\varepsilon) \overset{\sim}{\leq} f_2(t,\varepsilon)$ means $f_1(t,\varepsilon) \leq f_3(t,\varepsilon) \cong f_2(t,\varepsilon)$ for some f_3.) It follows that $N(t+\varepsilon^2,\varepsilon) \cong N(t,\varepsilon)$.

Since increasing ε by ε^2 leads to a gain in the number of full components by making more of them enter the back end of the flow cube exactly like increasing t by ε^2 does, plus a similar increase in number by making some of them delay their departure through the front end of the flow cube, we get

$$N(t,\varepsilon+\varepsilon^2) \overset{\sim}{\leq} N(t,\varepsilon) + 2\varepsilon N(t,\varepsilon).$$

This shows that $N(t,\varepsilon+\varepsilon^2) \cong N(t,\varepsilon)$. Hence $N(t,\varepsilon) \cong G(t,\varepsilon)$ as claimed. □

5.6. A Bowen-type property of the measure of maximal entropy

Definition 5.18. *Let P_t be the number of homotopy classes of closed geodesics of length at most t. Let $P_t(A)$ be the number of closed geodesics of length at most t that intersect A. Let P_t' be the number of* regular *closed geodesics of length at most t.*

Remark 5.19 (Terminology). When we say "closed geodesic", we mean "periodic orbit for the geodesic flow", i.e. with parameterization (although always by arclength and modulo adding a constant to the parameter). Thus a locally shortest curve is counted as two geodesics (i.e. periodic orbits for the geodesic flow), namely one for each direction.

$P_t', P_t(A)$ are finite because there is only one regular geodesic in each homotopy class. Clearly

$$P_t(A) \leq P_t' \leq P_t$$

for any t. We will show that these are in fact asymptotically equal.

Lemma 5.20.

$$P_t' \sim P_t.$$

Proof. Singular geodesics have a smaller exponential growth rate than regular ones because the entropy of the singular set is smaller than the topological entropy [Kni2] whereas the entropy of the regular set equals the topological entropy. □

In the case that M is a surface, the growth rate of **Sing** is in fact zero, since a parallel perpendicular Jacobi field gives rise to the largest Liapunov exponent being zero.

Definition 5.21. *Let μ_t be the arclength measure on all regular periodic orbits of length at most t, normalized to 1:*

$$\begin{aligned}
\mathbf{P}_t \quad &:= \{\textit{regular closed geodesics of length} \leq t\},\\
\mathbf{P}_t(A) &:= \{\textit{geodesics in } \mathbf{P}_t \textit{ which pass through } A\},\\
\mu_t \quad &:= \frac{1}{\#\mathbf{P}_t} \sum_{c \in \mathbf{P}_t} \frac{1}{len(c)} \delta_c,\\
\mu_t^A \quad &:= \frac{1}{\#\mathbf{P}_t(A)} \sum_{c \in \mathbf{P}_t(A)} \frac{1}{len(c)} \delta_c.
\end{aligned}$$

Here δ_c is the length measure on $\dot{c}$.

Theorem 5.22. *For any weak limit μ of $(\mu_t)_{t>0}$ we have*

$$m \sim \mu.$$

Moreover, for any weak limit μ^A of $(\mu_t^A)_{t>0}$ we have

$$m \sim \mu^A.$$

In other words, for any $t_k \to \infty$ such that $(\mu_{t_k}^A)_{t_k \in \mathbb{R}}$ converges weakly and for any measurable U the following holds:

$$\lim_{k \to \infty} \mu_{t_k}^A(U) = m(U).$$

Similarly with μ^A replaced by μ.

Proof. Knieper showed in [Kni1] that m can be obtained as a weak limit of the measures μ_{t_k} which are Borel probability measures supported on $\mathbf{P}_{t_k}$; see also [Pol]. The singular closed geodesics can be neglected because the singular set has entropy smaller than h. Hence any weak limit of μ_t equals m.

Since

$$P_t(A) \geq C \frac{e^{ht}}{t}$$

[Kni1, Remark after Theorem 5.8], any weak limit of the measures $\mu_{t_k}^A$ concentrated on $\mathbf{P}_{t_k}(A)$ has entropy h. Since the measure of maximal entropy is unique, any such weak limit equals m. □

Corollary 5.23. $P_t(A) \sim P_t$.

Proof. The measure on the geodesics in $\mathbf{P}_t \backslash \mathbf{P}_t(A)$ (which assigns zero measure to A) would otherwise also converge weakly to the measure of maximal entropy. □

Remark 5.24. This means that we can approximate the measure of maximal entropy m of a measurable set by its μ_{t_k}-measure for k sufficiently large. Moreover, when counting orbits, an arbitrarily small regular local product cube A will suffice to count periodic orbits in such a way that the fraction of those not counted will converge to zero as the period of these orbits becomes large. We use this fact in the proof of Theorem 5.26.

Definition 5.25. *Let $P_{t,\varepsilon}$ be the number of regular geodesics with length in $(t-\varepsilon, t+\varepsilon]$.*

Again, $P_{t,\varepsilon}$ is finite because there is only one regular geodesic in each homotopy class.

Theorem 5.26. *The number $P_{t,\varepsilon}$ of regular closed geodesics with prescribed length is given by the asymptotic formula*

$$P_{t,\varepsilon} \cong \frac{\varepsilon N(A,t)}{t \cdot m(A)}.$$

Proof. By Theorem 5.22, for a typical closed geodesic c with sufficiently large length,

$$\frac{1}{\operatorname{len}(c)} \delta_c(A) = \frac{1}{\operatorname{len}(c)} \int_{\dot{c}\cap A} d\mathrm{len} \cong m(A).$$

Here "typical" means that the number of closed geodesics of length at most t that have this property is asymptotically the same as the number of all closed geodesics of length at most t; in other words, the ratio tends to 1.

Hence such a geodesic (which consists of t/ε segments of length ε) will have asymptotically $m(A)t/\varepsilon$ segments of length ε intersecting A. Thus $P_{t,\varepsilon} \cong \frac{\varepsilon G(t,\varepsilon)}{tm(A)}$ where G is as in the proof of Lemma 5.17. The statement of Lemma 5.17 then shows the claim.

Note that it suffices to consider closed orbits which are not multiple iterates of some other closed orbit for the following reason: If $H(t,k)$ is the number of periodic orbits passing through A of length at most t which are k-fold iterates, then $H(t,k) = 0$ for $k > t/\operatorname{inj}(M)$. Thus the number of segments of A which are transversed by all multiple iterates is at most $\sum_{k=2}^{\lfloor t/\operatorname{inj}(M)\rfloor} kH(t,k)$. By Knieper's multiplicative estimate (see Section 2.3), this number is at most const $\cdot \sum_{k=2}^{\lfloor t/\operatorname{inj}(M)\rfloor} ke^{ht/k}$, thus at most const $\cdot\, t^2 e^{ht/2}$. This contributes only a zero asymptotic fraction of the segments and can thus be ignored. □

Proposition 5.16 and Theorem 5.26 combined yield:

Corollary 5.27.

$$P_{t,\varepsilon}(A) \cong \frac{2\varepsilon e^{ht}}{t}. \qquad \square$$

5.7. Proof of the main result

The desired asymptotic formula is now derived:

Theorem 5.28 (Precise asymptotics for periodic orbits). *Let M be a compact Riemannian manifold of nonpositive curvature whose rank is one. Then the number P_t of homotopy classes of periodic orbits of length at most t for the geodesic flow is asymptotically given by the formula*

$$P_t \sim \frac{e^{ht}}{ht}$$

where $\sim$ means that the quotient converges to 1 as $t \to \infty$.

Proof. We use the standard limiting process

$$\int_a^b f(x)dx \bowtie \sum_{i=\lfloor a/2\varepsilon \rfloor}^{\lfloor b/2\varepsilon \rfloor} 2\varepsilon f((2i+1)\varepsilon)$$

for suitable functions f (in particular, if f is continuous and piecewise monotonous, as is the case for $f(x) = e^{hx}/x$). Choose some fixed sufficiently large number $t_0 > 0$. Since we can ignore all closed geodesics of length at most t_0 for the asymptotics, we see that for $t > t_0$ by Corollary 5.27 we get

$$\begin{aligned}
P'_t &\cong P_t(A) \\
&\cong \sum_{i=\lfloor t_0/2\varepsilon \rfloor}^{\lfloor t/2\varepsilon \rfloor} P_{(2i+1)\varepsilon,\varepsilon} \\
&\cong \sum_{i=\lfloor t_0/2\varepsilon \rfloor}^{\lfloor t/2\varepsilon \rfloor} 2\varepsilon \frac{e^{h(2i+1)\varepsilon}}{(2i+1)\varepsilon} \\
&\cong \int_{t_0}^{t} \frac{e^{hx}}{x} dx \\
&= \left.\frac{e^{hx}}{hx}\right|_{t_0}^{t} + \int_{t_0}^{t} \frac{e^{hx}}{hx^2} dx \\
&\cong \frac{e^{ht}}{ht} - \frac{e^{ht_0}}{ht_0} \\
&\cong \frac{e^{ht}}{ht}.
\end{aligned}$$

Note that in the last term there is no dependence on ε and that $P'_t \sim P_t$. Hence

$$P_t \sim \frac{e^{ht}}{ht}.$$

This concludes the proof. □

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