# Conformal geometry of gravitational plane waves

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Abstract: It is well known that the conformal group of a non-flat vacuum *pp*-wave is at most 7-dimensional. Here we explicitly determine all solutions with a 7-dimensional group in three particular families of gravitational plane waves. All of them are exact solutions in terms of elementary functions. Furthermore, it turns out that a gravitational plane wave with a 6-dimensional homothety group (larger than the isometry group) does not have to be real analytic. It may contain an open flat part even if it is not conformally flat. Finally we classify all vacuum *pp*-waves admitting a particular standard conformal vector field arising from flat Minkowski space.

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## 1. Introduction and Results

Notations: We consider 4-dimensional Lorentzian manifolds $(M, g)$ with a metric tensor $g$ of type $(3, 1)$. A *conformal diffeomorphism* $f\colon (M, g) \to (\overline{M}, \overline{g})$ is a mapping preserving the angles (or the orthogonality) between any two directions and the type (space-like or time-like) of vectors. An equivalent formulation is that the induced metric $f^*\overline{g}$ is everywhere a positive multiple of $g$. We denote this positive factor by $f^*\overline{g} = \varphi^{-2}g$ where $\varphi\colon M \to \mathbb{R}_+$ is a function. We can then consider two conformally equivalent metrics $g$ and $\overline{g} = \varphi^{-2}g$ on the manifold $M$. The mapping $f$ is called an *isometry* if $\varphi = 1$, it is called a *homothety* or *similarity* if $\varphi$ is constant. The classical case of a similarity in flat Minkowski space is the mapping $x \mapsto cx$ with a real constant $c$. Let $\nabla$ denote the Levi-Civita connection induced by $g$. For any given smooth function $\varphi$ on $(M, g)$ let $\nabla\varphi$ denote the *gradient* of $\varphi$, $\nabla^2\varphi$ denotes the *Hessian* (0,2)-tensor, $\Delta\varphi = trace_g\nabla^2\varphi$ is the *Laplacian* of $\varphi$. A *conformal vector field* $V$ is generated by a local 1-parameter group of conformal mappings. A vector field $V$ is conformal if and only if the equation

$$\mathcal{L}_V g = 2\psi \cdot g$$

holds for a real function $\psi$ where necessarily $4\psi = \mathrm{div}(V)$. If $\psi$ is constant then $V$ is called homothetic. By the equation $\mathcal{L}_{\nabla\varphi} g = 2\nabla^2\varphi$ a gradient field $\nabla\varphi$ is conformal if and only if $\nabla^2\varphi$ is a scalar multiple of $g$. The local theory of conformal gradient fields is known, see [17].

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**Theorem A.** (Eardley et al. [7, Thm.3]) *A vacuum spacetime admitting a non-homothetic conformal vector field is locally either flat or is a pp-wave (plane–fronted wave).*

This result is already implicitly contained in a classical paper by Brinkmann [3]. Moreover, from the mathematical point of view from Brinkmann's paper one can formulate the following slight modification:

**Theorem B.** *A* 4*-dimensional Riemannian or Lorentzian Einstein space admitting a non-homothetic conformal vector field is either locally conformally flat (i.e., of constant sectional curvature) or is locally a vacuum pp-wave (plane–fronted wave).*

The first case is called the "proper case" in [3], the second case is the "improper case". This refers to the distinction whether the gradient of the conformal factor is a null vector ("improper") or not ("proper"), compare [13]. If the Ricci tensor vanishes it is trivially preserved by the corresponding conformal diffeomorphisms. Thus we have the special type of what is called a *Liouville transformation* in [16]. The conformal factor in this case must be very special, compare [10].

However, this result does not immediately lead to a classification of the metrics and the conformal vector fields. The "improper" case of a *pp*-wave does not yet seem to be completely understood. The converse to Theorem A is not true: Not every vacuum *pp*-wave admits a non-homothetic conformal vector field, see Corollary 2 below. It seems that an explicit example of a non-flat vacuum *pp*-wave carrying a non-homothetic conformal vector field is not described in the literature, compare [19]. An implicit example can be found in [7, Appendix A]. However, it is possible to write down exact solutions in terms of elementary functions, see Theorem 1 below. Our main results are the following:

**Theorem C.** *There are exact solutions of vacuum spacetimes admitting a non-homothetic conformal vector field. Moreover, all vacuum spacetimes admitting a* 7*-dimensional conformal group (together with the vector fields themselves) can be explicitly determined in terms of elementary functions and a finite number of parameters.*

**Theorem D.** *The class of all vacuum spacetimes admitting a* 6*-dimensional conformal (or homothety) group cannot be described in terms of a finite dimensional space of parameters. More precisely, there are solutions which are $C^\infty$ but not real analytic, depending on the arbitrary choice of a real $C^\infty$-function.*

**Corollary** *A Lorentzian Einstein manifold is not necessarily real analytic.*

This is in sharp contrast with the Riemannian case where Einstein spaces are always real analytic in appropriate coordinates, see [2, Sect.5F].

## 2. The conformal geometry of flat Minkowski space

The conformal mappings and the conformal vector fields on the flat Minkowski space $R^n_1$ with $n \geq 3$ are well known by the analogue of Liouville's theorem, see [12], [21]. Up to isometries, a non-isometric conformal mapping is necessarily a composition of the following three types:

1. a *similarity* (or dilatation) $x \mapsto cx$,
2. an *inversion* $x \mapsto x/\langle x, x\rangle$,
3. a special type of a transformation with a parallel null vector field as the gradient of the conformal factor.

A simple example for type 3 is the space $\mathbb{R}^4_1 = \{(u, v, x, y)\}$ with the metric $ds^2 = 2dudv + u^2(dx^2 + dy^2)$. Here the mapping $F(u, v, x, y) = (\frac{1}{u}, -v - \frac{1}{2}u(x^2 + y^2), ux, uy)$ is conformal with $F^*(ds^2) = \frac{1}{u^2}ds^2$. The gradient $\nabla u$ of the conformal factor is the null vector $\frac{\partial}{\partial v}$. Type 3 was first described by Haantjes [12]. It occurs also for *pp*-waves and for gravitational plane waves, see below. Type 1 and type 2 occur also on cones where the apex need not be part of the manifold. A special example for type 1 is the flat Lorentzian warped product metric $ds^2 = -dr^2 + r^2 ds^2_{-1}$ where $ds^2_{-1}$ denotes a 3-dimensional compact Riemannian manifold of constant negative curvature. The mapping $F(r, x) = (cr, x)$ is such a similarity. By the transformation $r = e^t$ the metric can also be written in the form $ds^2 = e^{2t}(-dt^2 + ds^2_{-1})$. This is called the *expanding hyperbolic spacetime* in [7].

A non-isometric conformal vector field $V$ on the Minkowski space is of one of the following three types (compare [21]):

1. The radial vector field $V(X) = X$,
2. The field $V(X) = 2\langle X, T\rangle X - \langle X, X\rangle T$ for a fixed vector $T$ with $\langle T, T\rangle \neq 0$ (also called *general conformal vector field*),
3. The field $V(X) = 2\langle X, T\rangle X - \langle X, X\rangle T$ for a fixed vector $T$ with $\langle T, T\rangle = 0$ (also called *special conformal vector field*). In this case the gradient of the conformal factor is a parallel null vector field.

If we write the flat metric in the form $ds^2 = -2dudv + dx^2 + dy^2$ and if we choose $T = \partial_v$ then the special conformal vector field can be written in the form

$$V = 2u^2\partial_u + (x^2 + y^2)\partial_v + 2ux\partial_x + 2uy\partial_y.$$

We will refer to it as the *standard special conformal vector field*. Obviously, this field is globally defined. It vanishes precisely along the $v$-coordinate axis. The conformal factor of $V$ is the function $2u$, its gradient $\nabla u$ is the vector $2\partial_v$. We shall come back to this vector field several times because it has a chance to survive in a Ricci flat *pp*-wave manifold since all of these manifolds carry parallel null vectors $\partial v$.

If we combine the radial vector field with the vector field $u\partial_u - v\partial_v$ then we obtain the homothetic vector field $2v\partial_v + x\partial_x + y\partial_y$ which is also of some special type. It vanishes along the $u$-coordinate axis. This vector field is important because it survives in a large class of Ricci flat *pp*-waves, see below. Consequently, these two special types gives us an idea how homothetic or conformal symmetries can look like in vacuum space-times or in *pp*-waves.

The conformal group of Minkowski space is 15-dimensional. It contains the 4-dimensional abelian group of translations, the 6-dimensional group of rotations, one additional homothety and 4 independent conformal transformations defined by the various choices of the vector $T$ above. Among them we find the standard special conformal vector field. Clearly all conformally flat manifolds have (locally) the same 15-dimensional conformal group.

## 3. Homothetic and conformal vector fields on *pp*-waves

The class of *pp*-waves in general is given by all Lorentzian metrics on $I\!R^4 = \{(u, v, x, y)\}$ which are of the form

$$ds^2 = -2H(u, x, y)du^2 - 2dudv + dx^2 + dy^2$$

with an arbitrary function $H$, the *potential*, which does not depend on $v$.

The subclass of *plane waves* is given by all $H$ of the form [11], [23]

$$H(u, x, y) = a(u)x^2 + 2b(u)xy + c(u)y^2.$$

The difference between these cases as well as the history is very well explained in a paper by Schimming [20]. A *pp*-wave is a plane wave if and only if the curvature tensor is parallel in $\partial_x$- direction and $\partial_y$-direction, compare [11].

The following lemma is well known [11].

**Lemma 1:**

1. *A pp-wave is conformally flat if and only if the Hessian of $H$ with respect to $x, y$ is a multiple of the identity matrix.*
2. *It is Ricci flat if and only the Laplacian of $H$ with respect to $x, y$ vanishes identically:* $\Delta H = H_{xx} + H_{yy} = 0$.
3. *Consequently, it is flat if and only if $H$ is linear in $x$ and $y$.*
4. *Furthermore, a plane wave is Ricci flat if and only if $a(u) + c(u) = 0$.*

In Relativity a Ricci flat *pp*-wave (or plane wave) is called a *vacuum pp-wave* (or a *gravitational plane wave*, respectively) since it satisfies the Einstein field equations $R_{ab} = 0$ for the vacuum.

The study of conformal vector fields $V$ on *pp*-wares is based on an examination of the 10 equations

$$\mathcal{L}_V g_{ik} = \nabla_i V^j g_{jk} + \nabla_k V^j g_{ji} = 2\psi g_{ik}$$

or, equivalently,

$$\partial_i V^j g_{jk} + \partial_k V^j g_{ji} + V^j \Gamma_{ij,k} + V^j \Gamma_{kj,i} = 2\psi g_{ik}$$

where $i, j, k$ range from 1 to 4. The only Christoffel symbols $\Gamma_{ij,k}$ which do not always vanish identically are

$$\Gamma_{11,1} = -H_u, \quad \Gamma_{11,3} = -\Gamma_{13,1} = H_x, \quad \Gamma_{11,4} = -\Gamma_{14,1} = H_y.$$

In more detail a given vector field

$$V = V^u \partial_u + V^v \partial_v + V^x \partial_x + V^y \partial_y$$

is conformal on a *pp*-wave with conformal factor $\psi$ if and only if the following equations are satisfied:

$$\begin{aligned}
\partial_u V^v + 2\partial_u V^u H + V^u \partial_u H + V^x \partial_x H + V^y \partial_y H &= 2\psi H \\
\partial_v V^u &= 0 \\
\partial_u V^u + \partial_v V^v &= 2\psi \\
2H \partial_x V^u + \partial_x V^v - \partial_u V^x &= 0 \\
2H \partial_y V^u + \partial_y V^v - \partial_u V^y &= 0 \\
\partial_x V^u - \partial_v V^x &= 0 \\
\partial_y V^u - \partial_v V^y &= 0 \\
\partial_x V^x &= \psi \\
\partial_y V^y &= \psi \\
\partial_x V^y + \partial_y V^x &= 0
\end{aligned}$$

**Corollary 1:** On any *pp*-wave the following hold:

1. $X_0 := \partial_v$ is always isometric and even parallel.
2. $X_1 := \partial_u$ is isometric iff $H_u = 0$.
3. The boost $X_2 := u\partial_u - v\partial_v$ is isometric iff $uH_u + 2H = 0$.
4. The rotation vector field $X_3 := y\partial_x - x\partial_y$ is isometric iff $H$ depends only on $u$ and $x^2 + y^2$.
5. The radial vector field $Y_1 := u\partial_u + v\partial_v + x\partial_x + y\partial_y$ is homothetic with $\psi = 1$ iff $uH_u + xH_x + yH_y = 0$
6. $Y_2 := Y_1 - X_2 = 2v\partial_v + x\partial_x + y\partial_y$ is homothetic with $\psi = 1$ iff $xH_x + yH_y = 2H$.

   This type was studied by D.Alekseevskii in a more general context. By [1] the class of plane waves forms a subclass of all Lorentzian manifolds admitting a self-similarity which fixes pointwise a ligtlike geodesic line which in our case is $\gamma(u) = (u, 0, 0, 0)$ since $H_u(u, 0, 0, 0) = 0$.
7. The standard vector field $Z_1 := u^2\partial_u + \frac{1}{2}(x^2 + y^2)\partial_v + ux\partial_x + uy\partial_y$ is conformal with $\psi = u$ iff $uH_u + xH_x + yH_y = -2H$.

   $Z_1$ coincides with the standard special conformal vector field in Minkowski space above (where $T$ is a null vector).

## 4. Vacuum *pp*-waves with a large conformal group

Conformal mappings between Ricci flat manifolds were studied in a pioneering paper by Brinkmann [3]. First of all, Brinkmann proved that an Einstein space of non-zero scalar curvature admitting a non-homothetic conformal mapping onto some other Einstein space is conformally flat, i.e., of constant curvature. Secondly Brinkmann proved that a Ricci flat (but non-flat) 4-manifold can admit a nontrivial conformal mapping onto another Einstein 4-manifold only if it is of a fairly special type which later was called a *pp*-wave. The transition from Brinkmann's presentation of the solution to the notion of a *pp*-wave is discussed in [11] and [20], compare [4]. On the other hand it does not seem to be known which *pp*-waves precisely occur although exact solutions to Einstein's equtions with nontrivial homothety or conformal groups were discussed in the literature quite extensively, compare [5] for the homothetic case and [7] for the conformal case. Maartens and Maharaj [19] investigated the differential equation governing conformal vector fields on *pp*-waves, and Eardley et al. [7, Appendix A] found the existence of conformal vector fields on some *pp*-waves by the method of evolution equations (somehow implicitly) but it seems they missed the explicit solutions below which, in the simplest cases, are just rational functions. This situation is improved by our following main theorem:

**Theorem 1:** *Assume that $(M^4, g)$ is a pp-wave with metric tensor of the form*

$$g = -2H(u,x,y)du^2 - 2dudv + dx^2 + dy^2$$

*which is not flat and which satisfies the Einstein field equations for the vacuum (i.e., which is Ricci flat). Assume that $M$ admits a conformal group which is larger than the isometry group.*

*Then the conformal group is at most 7-dimensional, and if it is 7-dimensional, then $H$ is a linear combination of the real and imaginary parts of one of the following three cases (up to a shift of the coordinates and up to a fixed rotation in the $(x,y)$-plane):*

Case 1: $2H(u,x,y) = c(x+iy)^2 \exp(2\kappa u i)$ *where $c$ and $\kappa$ are constants,*

Case 2: $2H(u,x,y) = c\dfrac{(x+iy)^2}{u^2} \exp(2\kappa u i)$ *where $c$ and $\kappa$ are constants,*

Case 3: $2H(u,x,y) = c \cdot \dfrac{(x+iy)^2}{(u^2+\alpha u+\beta)^2} \exp\Big(2\gamma i \displaystyle\int \frac{du}{u^2+\alpha u+\beta}\Big)$

*where $c, \alpha, \beta, \gamma$ are constants and where a non-homothetic conformal vector field can be chosen as*

$$V = Z_1 + \alpha(u\partial_u - v\partial_v) + \beta\partial_u + \gamma(y\partial_x - x\partial_y)$$

*where $Z_1$ is the standard conformal field $Z_1 = u^2\partial_u + \frac{1}{2}(x^2+y^2)\partial_v + ux\partial_x + uy\partial_y$ which we know from Minkowski space (Section 2).*

*In each of these three cases a non-isometric homothetic vector field $Y$ can be chosen as $Y_2 = 2v\partial_v + x\partial_x + y\partial_y$ (Alekseevskii's type). The radial vector field $Y_1$ is homothetic only in Case 2 for $\kappa = 0$.*

*In addition, for $\kappa = 0$ the metric in Case 2 admits a conformal diffeomorphism onto itself (a conformal inversion). The same conformal diffeomorphism interchanges the special subcases $2H = x^2 - y^2$ and $2H = (x^2-y^2)/u^4$ of the Cases 1 and 3.*

Note that the integral in case 3 can always be evaluated in terms of elementary functions, depending on the discriminant $4\beta - \alpha^2$. The metric in Case 1 (see also [6]) is clearly complete on the entire 4-space, in the Cases 2 and 3 the metric is globally defined on $\{(u, v, x, y) \mid u > 0\}$ and complete in Case 2. In particular, no singularity is involved, and the $(x, y)$-planes are always complete Euclidean planes. So these metric are honest gravitational plane-fronted waves. The isometry group is 6-dimensional in the Cases 1 and 2, it is 5-dimensional in Case 3. The conformal group is purely homothetic in the Cases 1 and 2.

PROOF. First we restate some basic facts.

1. The dimension of the conformal group does not exceed the dimension of the homothety group by more than 1. This can be seen as follows: It is well known that for Ricci flat (but non-flat) space-times the divergence of any conformal vector field has a parallel gradient. This is the *improper case* discussed in [3]. The proper case would lead to a warped product metric with an Einstein fibre which would necessarily have to be flat, compare [18]. So if $\mathcal{L}_V g = 2\psi \cdot g$ and $\mathcal{L}_W g = 2\phi \cdot g$ then the gradients of $\psi, \phi$ are both parallel vector fields. On a non-flat *pp*-wave this is possible only if both are linearly dependent and pointing into the $\partial_v$-direction. Consequently, a linear combination of $\psi, \phi$ is constant, hence a linear combination of $V, W$ is homothetic.

2. The dimension of the homothety group does not exceed the dimension of the isometry group by more than 1. This follows similarly: If $\mathcal{L}_V g = 2\psi \cdot g$ and $\mathcal{L}_W g = 2\phi \cdot g$ with two constants $\psi, \phi$ then the linear combination $\phi V - \psi W$ is isometric.

3. The possible isometry groups of vacuum *pp*-waves were classified in [8], compare [15], [19], [22]. Such a group is at most 6-dimensional. There are three cases of large groups of dimensions 5 and 6. In these cases the function $H$ has the form $H(u, z) = F(u)z^2$ in complex notation where $z = x + iy$. Its real part is $H = \mathrm{Re}(F(u))(x^2 - y^2) - \mathrm{Im}(F(u))2xy$. It follows that the conformal group is at most 8-dimensional, and if it is 8-dimensional, then the corresponding isometric part must coincide with one of the two possible 6-dimensional groups.

In any case the function $H$ is of the type $H(u, x, y) = a(u)(x^2 - y^2) - 2b(u)xy$. This is a particular form of a plane wave. By Lemma 1 and Corollary 1 the metric admits the homothetic vector field $Y_2 = 2v\partial_v + x\partial_x + y\partial_y$, compare [1] where this type of homothety is studied in the context of generalized *pp*-waves. So in the two cases of a 6-dimensional isometry group we have a 7-dimensional homothety group where either $F(u) = c\exp(2i\kappa u)$ or $F(u) = c\exp(2i\kappa u)u^{-2}$, with the real part

$$H(u, x, y) = c\cos(2\kappa u)(x^2 - y^2) - 2c\sin(2\kappa u)xy$$

in Case 1 or

$$H(u,x,y) = c\cos(2\kappa u)u^{-2}(x^2-y^2) - 2c\sin(2\kappa u)u^{-2}xy$$

in Case 2 with a constant $c$ and with another constant $\kappa$ determining a certain rotation. This rotation induces a term $\kappa(y\partial_x - x\partial_y)$ which is added to one of the generators in the space of isometric vector fields. Note that $y\partial_x - x\partial_y$ itself is never part of the isometry group since the function $H$ is not rotationally symmetric in $x, y$.

It remains to discuss Case 3 and all possible conformal vector fields on a *pp*-wave admitting a 5-dimensional isometry group. Again the function $H$ is of the form

$$H(u,x,y) = a(u)(x^2-y^2) - 2b(u)xy.$$

Up to isometric vector fields and up to a constant multiple of the homothetic vector field $Y_1$, any conformal vector field has to be of the type

$$V = Z_1 + \alpha(u\partial_u - v\partial_v) + \beta\partial_u + \gamma(y\partial_x - x\partial_y)$$

with constants $\alpha, \beta, \gamma$ where $Z_1 = u^2\partial_u + \frac{1}{2}(x^2+y^2)\partial_v + ux\partial_x + uy\partial_y$, see [19, p.511]. This vector field is conformal with (necessarily) the conformal factor $\psi = u$ if and only if the equation

$$2(u+\alpha)H + (u^2+\alpha u+\beta)H_u + u(xH_x + yH_y) + \gamma(yH_x - xH_y) = 0$$

is satisfied. If we put in the special form of $H$ we obtain the following linear system of ODEs:

$$(u^2+\alpha u+\beta)\begin{pmatrix} a \\ b \end{pmatrix}' + 2\begin{pmatrix} 2u+\alpha & \gamma \\ -\gamma & 2u+\alpha \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The eigenvalues of the matrix are $\lambda = 2u+\alpha \pm \gamma i$ with corresponding eigenvectors $\binom{1}{\pm i}$, independently of $u$. Hence the matrix is simultaneously diagonalizable for all $u$. This implies that the system of ODEs has an explicit complex solution $\Theta\binom{1}{\pm i}$ as in the case of constant coefficients where $\Theta$ ist a solution

$$\Theta' + 2\frac{2u+\alpha\pm\gamma i}{u^2+\alpha u+\beta}\Theta = 0.$$

In more detail we have

$$\Theta_\pm(u) = \frac{c}{2}\cdot\frac{1}{(u^2+\alpha u+\beta)^2}\exp\Big(\mp 2\gamma i\int\frac{du}{u^2+\alpha u+\beta}\Big)$$

with a constant $c$. We then obtain the solution $\binom{a}{b}$ as the real or the imaginary part of $\Theta_{\pm}\binom{1}{\pm i}$. The real part is

$$a(u) = \frac{c}{2} \cdot \frac{1}{(u^2+\alpha u+\beta)^2} \cos\Big(2\gamma \int \frac{du}{u^2+\alpha u+\beta}\Big),$$

$$b(u) = \frac{c}{2} \cdot \frac{1}{(u^2+\alpha u+\beta)^2} \sin\Big(2\gamma \int \frac{du}{u^2+\alpha u+\beta}\Big).$$

The simplest subcase is the case $\alpha = \beta = \gamma = 0$ with the solution $2H(u,x,y) = c \cdot \frac{x^2-y^2}{u^4}$. This leads to a 7-dimensional conformal group in case 3. It also shows that the 7-dimensional homothety groups in the cases 1 and 2 are maximal since the equation for an additional conformal field is not satisfied, see Corollary 2. Consequently, the 8-dimensional case cannot occur. This completes the proof of the theorem. In fact the conformal group of any non-flat vacuum spacetime is at most 7-dimensional [14] but this result is not used above.

**Remarks:** In Case 1 with $\kappa = 0$ and $c > 0$ the conformal group is generated by the following vector fields where only the last one is not isometric:

$$\begin{array}{l}
\partial_v \\
\partial_u \\
\sinh(\sqrt{c}u)\partial_y + \sqrt{c}\cosh(\sqrt{c}u)y\partial_v \\
\cosh(\sqrt{c}u)\partial_y + \sqrt{c}\sinh(\sqrt{c}u)y\partial_v \\
\sin(\sqrt{c}u)\partial_x + \sqrt{c}\cos(\sqrt{c}u)x\partial_v \\
\cos(\sqrt{c}u)\partial_x - \sqrt{c}\sin(\sqrt{c}u)x\partial_v \\
2v\partial_v + x\partial_x + y\partial_y
\end{array}$$

In Case 2 with $\kappa = 0$ and $c > 0$ the conformal group is generated by the following vector fields where only the last one is not isometric:

$$\begin{array}{l}
\partial_v \\
u\partial_u - v\partial_v \\
u^\alpha\partial_y + \alpha u^{\alpha-1}y\partial_v \\
\qquad\qquad\qquad \text{where} \quad \alpha^2 - \alpha - c = 0 \qquad \text{(two roots)} \\
\mathrm{u}^\beta\partial_x + \beta u^{\beta-1}x\partial_v \\
\qquad\qquad\qquad \text{where} \quad \beta^2 - \beta + c = 0 \qquad \text{(two roots)} \\
u\partial_u + v\partial_v + x\partial_x + y\partial_y
\end{array}$$

The corresponding ODE for the coefficients is Euler's equation $f'' \pm \frac{c}{u^2} f = 0$ which can be solved by (real or complex) powers of $u$. There are two distinct roots in all cases except for $c = \frac{1}{4}$. Here we have two roots $\alpha = \frac{1}{2}(1 \pm \sqrt{2})$ but only one $\beta = \frac{1}{2}$. In this case we have the solutions $u^{1/2}$ and $u^{1/2} \log u$ instead. In the special case $c = 2$ we obtain $\alpha = 2, -1$, i.e., $u^2\partial_y + 2uy\partial_v$ (a null rotation) and $u^{-1}\partial_y - u^{-2}y\partial_v$.

In addition, there is the conformal inversion

$$Inv(u, v, x, y) = \frac{1}{u}(1, -uv + \tfrac{x^2+y^2}{2}, x, y)$$

which commutes with the 1-parameter group generated by Alekseevskii's vector field $Y_2 = 2v\partial_v + x\partial_x + y\partial_y$.

In Case 3 for the special choice of $2H(u, x, y) = c \cdot \frac{x^2-y^2}{u^4}$ with $c > 0$ the conformal group is generated by the following vector fields where the first five generate the isometry group. Here the corresponding ODE is $f'' \pm \frac{c}{u^4} f = 0$. The reader might be amused when seeing the function $\sin(1/u)$ involved in a 1-parameter group of isometries of a spacetime satisfying Einstein's field equations for the vacuum, even though all entries of the metric tensor are rational functions.

$$\begin{aligned}
&\partial_v \\
&u \sinh \tfrac{\sqrt{c}}{u} \partial_y + \left( \sinh \tfrac{\sqrt{c}}{u} - \tfrac{\sqrt{c}}{u} \cosh \tfrac{\sqrt{c}}{u} \right) y\partial_v \\
&u \cosh \tfrac{\sqrt{c}}{u} \partial_y + \left( \cosh \tfrac{\sqrt{c}}{u} - \tfrac{\sqrt{c}}{u} \sinh \tfrac{\sqrt{c}}{u} \right) y\partial_v \\
&u \sin \tfrac{\sqrt{c}}{u} \partial_x + \left( \sin \tfrac{\sqrt{c}}{u} - \tfrac{\sqrt{c}}{u} \cos \tfrac{\sqrt{c}}{u} \right) x\partial_v \\
&u \cos \tfrac{\sqrt{c}}{u} \partial_x + \left( \cos \tfrac{\sqrt{c}}{u} + \tfrac{\sqrt{c}}{u} \sin \tfrac{\sqrt{c}}{u} \right) x\partial_v \\
&2v\partial_v + x\partial_x + y\partial_y \\
&u^2\partial_u + \tfrac{1}{2}(x^2 + y^2)\partial_v + ux\partial_x + uy\partial_y
\end{aligned}$$

All these generators are in accordance with [8, p.235], just by solving the corresponding differential equations.

**Corollary 2:** *The gravitational plane wave with metric*

$$g = -(x^2 - y^2)du^2 - 2dudv + dx^2 + dy^2$$

*does not admit any conformal vector field which is not isometric or homothetic.*

PROOF. If there were such a conformal vector field $V$ then a constant multiple of $V$ would have to be of the form in Case 3 above, i.e.,

$$cV = u^2\partial_u + \frac{1}{2}(x^2 + y^2)\partial_v + ux\partial_x + uy\partial_y + \alpha(u\partial_u - v\partial_v) + \beta\partial_u + \gamma(y\partial_x - x\partial_y)$$

with constants $\alpha, \beta, \gamma$. Hence the equation

$$2(u + \alpha)(x^2 - y^2) + 2u(x^2 - y^2) + 4\gamma xy = 0$$

would have to be satisfied for any $u, x, y$, see above. This is impossible.

If the group is only 6-dimensional, then a theorem analogous to Theorem 1 cannot hold. Moreover, even though the function $H$ must be analytic in $x, y$ because it is harmonic, the function $H$ itself does not have to be analytic in $u$, as is demonstrated by the following example:

**Proposition 1:** *There is a complete gravitational plane wave admitting a 6-dimensional conformal group (larger than the isometry group) which is $C^\infty$ but not real analytic. In particular, a Ricci flat and self-similar Lorentzian manifold is not necessarily analytic.*

We can choose the following function

$$H(u, x, y) = \begin{cases} \exp\left(\frac{1}{u^2-1}\right)(x^2 - y^2) & \text{if } u^2 < 1 \\ 0 & \text{if } u^2 \geq 1. \end{cases}$$

The Ricci tensor vanishes everywhere identically. The curvature tensor vanishes for $u^2 > 1$ but does not vanish for $u^2 < 1$. Therefore, the metric is not analytic, independently of the choice of coordinates. For the completeness compare [9, Remark 3.3]. The isometry group is 5-dimensional, together with the homothetic vector field we obtain a 6-dimensional conformal group. The isometric vector fields are $\partial_v$ and four vector fields determined by four solutions of the equation equation $f'' \pm A(u)f = 0$ where $H = A(u)(x^2 - y^2)$. These vector fields are not real analytic either since $A$ is not.

Another example is given by

$$H(u, x, y) = \begin{cases} \exp\left(\frac{1}{u^2-1}\right)\log(x^2 + y^2) & \text{if } u^2 < 1 \\ 0 & \text{if } u^2 \geq 1. \end{cases}$$

but this is not complete in the $(x, y)$-planes, because there is a spatial singularity. However, this admits a 3-dimensional conformal group containing the rotation vector field $y\partial_x - x\partial_y$.

## 5. The standard special conformal vector field

In the following theorem we characterize those vacuum *pp*-waves which admit the standard special vector field from Section 2 as a conformal vector field. This gives a large number of additional explicit examples.

**Theorem 2** *Assume that a vacuum pp-wave with metric*

$$g = -2H(u,x,y)du^2 - 2dudv + dx^2 + dy^2$$

*admits the standard special conformal vector field* $Z_1 = u^2\partial_u + \frac{1}{2}(x^2+y^2)\partial_v + ux\partial_x + uy\partial_y$. *Assume further that the function* $H$ *is defined in a neighborhood of* $x = y = 0$ *for any fixed* $u$. *Then* $H$ *can be written as*

$$H(u,x,y) = \sum_{n\geq 0} u^{-(n+2)} P_n(x,y)$$

*where* $P_n$ *denotes a homogeneous polynomial of degree* $n$ *in the variables* $x, y$ *which is harmonic, i.e.* $\Delta P_n = 0$.
*If the pp-wave is not Ricci flat then the same expression holds (without harmonicity of* $P_n$*) under the assumption that* $H$ *is real analytic in a neighborhood of* $x = y = 0$. *Vice versa, any function* $H$ *of that type admits the standard conformal vector field* $Z_1$.

PROOF. By assumption $H$ is analytic in $x, y$ in a neighborhood of $x = y = 0$ for any fixed $u$. Therefore there is a power series expansion

$$H(u,x,y) = \sum_{n\geq 0} \sum_{i+j=n} a_{ij}(u) x^i y^j$$

for any fixed $u$. Now we come back to the equation $uH_u + xH_x + yH_y = -2H$ from Lemma 1 and Corollary 1 which must be satisfied if $Z_1$ is conformal. This leads to the equation

$$\sum u a'_{ij}(u) x^i y^j + \sum a_{ij}(u) n x^i y^j = -2 \sum a_{ij}(u) x^i y^j.$$

Therefore for any fixed $i, j$ the coefficient $a_{ij}$ has to satisfy the ODE $ua'_{ij} = -(n+2)a_{ij}$ with solution $a_{ij}(u) = c_{ij} u^{-(i+j+2)}$ with a constant $c_{ij}$. This leads to the representation above.

Vice versa, any power series of this type leads to a *pp*-wave admitting the standard special conformal vector field $Z_1$ provided that the series converges. As a special case we recognize the solution $H = u^{-4}(x^2 - y^2)$ of Theorem 1 above. Another special case is the solution $H = u^{-6}(x^4 + ax^3y - 6x^2y^2 - axy^3 + y^4)$.

# References


[1] D. V. Alekseevskii: *Selfsimilar Lorentzian manifolds*, Ann. Glob. Anal. Geom. **3** (1985), 59–84

[2] A. Besse, *Einstein manifolds*, Springer, Berlin 1987

[3] H. W. Brinkmann: *Einstein spaces which are mapped conformally on each other*, Math. Ann. **94** (1925), 119–145

[4] D. A. Catalano: *Concircular diffeomorphisms of pseudo-Riemannian manifolds.* Doctoral thesis ETH Zürich 1999

[5] A. A. Coley: *Self-Similarity and Cosmology*, Fields Institute Communications **15** (1997), 19–42

[6] L. Defrise-Carter, *Conformal groups and conformally equivalent isometry groups,* Comm. Math. Phys. **40** (1975), 273–282

[7] D. Eardley, J. Isenberg, J. Marsden and V. Moncrief: *Homothetic and conformal symmetries of solutions to Einstein's equations*, Commun. Math. Phys. **106** (1986), 137–158

[8] J. Ehlers and W. Kundt, *Exact solutions of the gravitational field equations*, Gravitation, an introduction to current research (L.Witten, ed.), pp. 49–101, Wiley, New York 1962

[9] P. E. Ehrlich and G. G. Emch, *Gravitational waves and causality*, Reviews in Math. Phys. **4** (1992), 163–221

[10] J. Ferrand: *Sur une classe de morphismes conformes*, C. R. Acad. Sci. Paris **296** (1983), 681–684

[11] P. Günther, *Huygens' principle and hyperbolic equations*, Academic Press, Boston 1988

[12] J. Haantjes: *Conformal representations of an n-dimensional euclidean space with a non-definite fundamental form on itself*, Proc. Kon. Ned. Akad. Amsterdam **40** (1937), 700–705

[13] W. D. Halford, *Brinkmann's theorem in general relativity*, Gen. Rel. Grav. **14**, 1193–1195 (1982)

[14] G. S. Hall and J. D. Steele, *Conformal vector fields in general relativity*, J. Math. Phys. **32**, 1847–1853 (1991)

[15] D. Kramer, H. Stephani, M. MacCallum, E. Herlt, *Exact solutions of Einstein's field equations*, VEB Deutscher Verlag d. Wiss., Berlin 1980

[16] W. Kühnel and H.-B. Rademacher: *Conformal diffeomorphisms preserving the Ricci tensor*, Proc. Amer. Math. Soc. **123** (1995), 2841–2848

[17] W. Kühnel and H.-B. Rademacher: *Essential conformal fields in pseudo-Riemannian geometry*, J. Math. Pures Appl. **74** (1995), 453–481, Part II: J. Math. Sci. Univ. Tokyo **4** (1997), 649–662

[18] W. Kühnel and H.-B. Rademacher: *Conformal vector fields on pseudo-Riemannian spaces*, Diff. Geom. Appl. **7** (1997), 237–250

[19] R. Maartens and S. D. Maharaj: *Conformal symmetries of pp-waves*, Class. Quantum Grav. **8** (1991), 503–514

[20] R. Schimming, *Riemannsche Räume mit ebenfrontiger und mit ebener Symmetrie*, Math. Nachr. **59**, 129–162 (1975)

[21] M. Schottenloher: *A Mathematical Introduction to Conformal field Theory*, Springer, Berlin 1997

[22] R. Sippel and H. Goenner, *Symmetry classes of pp-waves*, Gen. Relat. Grav. **18**, 1229–1243 (1986)

[23] C. G. Torre, *Gravitational waves: just plane symmetry*, arXiv:gr-qc/9907089 (1999)

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