RIGIDITY OF HYPERBOLIC PRODUCT MANIFOLDS

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Abstract. This paper presents a scalar curvature rigidity result of real hyperbolic product manifolds in analogy to M. Min–Oo's result [10]. In order to prove this, we consider Dirac bundles obtained from the spinor bundle, and we derive Killing equations trivializing these Dirac bundles. Moreover, an integrated Bochner–Weitzenböck formula is shown which allows the usage of the non–compact Bochner technique.

1. introduction

M. Min–Oo proved in [10] that a strongly asymptotically hyperbolic spin manifold (M^n, g) with scalar curvature $\mathrm{scal} \geq -n(n-1)$ is isometric to the real hyperbolic space. The key points of Min–Oo's result are the existence of imaginary Killing spinors on the real hyperbolic space as well as the non–compact Bochner technique. This technique was introduced by E. Witten [11] to give another proof of the positive energy theorem. Moreover, R. Bartnik generalized in [2] Witten's method to show scalar curvature rigidity of asymptotically flat spin manifolds. Another rigidity result of a non–compact symmetric space was given by M. Herzlich in [6]. But for this result a holonomy assumption turned out to be necessary.

In this paper a rigidity result of the Riemannian product manifold

$$(M_0, g_0) = \mathbb{R}^{m_1} \times \mathbb{R}\mathrm{H}^{m_2}(-K_2) \times \cdots \times \mathbb{R}\mathrm{H}^{m_l}(-K_l)$$

is given, where $\mathbb{R}\mathrm{H}^{m_j}(-K_j)$ means the real hyperbolic space of dimension m_j and sectional curvature $-K_j$. In order to obtain this result, we need a *weak* holonomy assumption which is specified in the definition below. Moreover, we have to show the existence of special Killing spinors on a Dirac bundle of (M_0, g_0), and we have to derive an integrated Bochner–Weitzenböck formula on this Dirac bundle which is suitable for the usage of the non–compact Bochner technique.

Definition 1.1. Let (M^n, g) be a Riemannian manifold and $\mathcal{U}_j \subset TM$, $j = 1 \ldots l$, be subbundles of rank m_j that give an orthogonal decomposition of the tangent bundle. (M, g) is said to be *strongly asymptotic* to the Riemannian product (M_0, g_0) if there is a compact manifold C with a disjoint decomposition $M = C \cup E$ and a diffeomorphism $f : E \to M_0 - \overline{B_R(0)}$ in such a way that the positive definite gauge transformation $A \in \Gamma(\mathrm{End}(TM_{|E}))$ given by

$$g(AX, AY) = (f^*g_0)(X, Y), \qquad g(AX, Y) = g(X, AY)$$

satisfies

(1) A is uniformly bounded:

$$\frac{1}{c}\,|X|_g \leq |AX|_g \leq c\,|X|_g$$

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for some constant $c > 0$.

(2)

$$\left|(f^*\nabla^0)A\right|_g + \sum_{j=2}^{l} \left|A \circ \pi_j^0 - \pi_j\right|_g \in L^1(E; e^{\alpha r}\mathrm{vol}_g) \cap L^2(E; e^{\alpha r}\mathrm{vol}_g),$$

where $f^*\nabla^0$ is the Levi–Civita connection for f^*g_0, $\pi_j^0 \in \Gamma(\mathrm{End}(TM_{|E}))$ is the pull back by $\mathrm{d}f$ of the projection $TM_0 \to T(\mathbb{R}\mathrm{H}^{m_j})$, π_j is the projection $TM \to \mathcal{U}_j$, r is the f^*g_0–distance to a fixed point and α is given by $\sqrt{\sum K_j}$.

Theorem 1.2. *Let (M, g) be a complete and connected Riemannian spin manifold which is strongly asymptotic to (M_0, g_0) with $\dim M_0 \geq 3$ and $m_2 \geq 2$. If the scalar curvature satisfies*

$$2\sum_{j=2}^{l} \sqrt{K_j\, g(\delta\pi_j, \delta\pi_j)} + \mathrm{scal}_0 \leq \mathrm{scal}, \tag{1.1}$$

(M, g) is isometric to the symmetric space (M_0, g_0). In this case $\delta\pi_j \in \Gamma(TM)$ is the divergence of π_j with respect to g, i.e. $\delta\pi_j = \sum(\nabla_{e_i}\pi_j)(e_i)$ if $e_1, \ldots, e_n$ is a g–orthonormal base. Moreover, scal_0 is the scalar curvature of (M_0, g_0) and it equals $-\sum\limits_{j=2} m_j(m_j - 1)K_j$.

If (M_0, g_0) is isometric to the real hyperbolic space, this rigidity result reduces exactly to the one of Min–Oo (cf. [10]). An example of a manifold which is strongly asymptotic to (M_0, g_0) is the following. Suppose $h : M_0 \to M_0$ to be a smooth compact supported function, then the manifold $(M_0, e^h g_0)$ is strongly asymptotic to (M_0, g_0). Since $(M_0, e^h g_0)$ is isometric to (M_0, g_0) if and only if $h \equiv 0$, equation (1.1) holds only if $h \equiv 0$.

2. Killing spinors on (M_0, g_0)

Because of irreducibility of Riemannian manifolds carrying a real or imaginary Killing spinor, it is more difficult to get some kind of Killing equation on Riemannian product manifolds. Nevertheless, we can make the following ansatz. Suppose (M, g) is a Riemannian spin manifold. Denote by $\$M$ the complex spinor bundle of M associated to the chosen spin structure and denote by $\not{\mathfrak{D}}$ the Dirac operator on $\$M$ with respect to the spin connection. Equip

$$\boldsymbol{\$}^q M = \bigoplus_{j=1}^{q} \$M$$

with the connection $\boldsymbol{\nabla}$ obtained by a diagonal extension of the usual spin connection ∇ and let $\boldsymbol{\gamma}$ be some Clifford action on $\boldsymbol{\$}^q M$ with $\boldsymbol{\gamma}(X)\boldsymbol{\gamma}(Y) = \gamma(X)\gamma(Y)\mathrm{Id}$, where γ is the Clifford multiplication on $\$M$. $\boldsymbol{\$}^q M$ becomes a Dirac bundle (cf. [8]) if the induced metric is considered. Since $\mathbf{R}_{e_i,e_j}$ as well as $\boldsymbol{\gamma}(e_i)\boldsymbol{\gamma}(e_j)$ are diagonal, we obtain the Lichnerowicz formula on $\boldsymbol{\$}^q M$

$$\boldsymbol{\not{\mathfrak{D}}}^2 = \boldsymbol{\nabla}^*\boldsymbol{\nabla} + \frac{\mathrm{scal}}{4}. \tag{2.1}$$

Assume a parallel and orthogonal splitting of the tangent bundle $TM = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$ and compute the curvature of

$$\boldsymbol{\nabla}_X \zeta = \mu_1 \boldsymbol{\gamma}(X_1)\mathcal{P}_1\zeta + \ldots + \mu_k \boldsymbol{\gamma}(X_k)\mathcal{P}_k\zeta, \tag{2.2}$$

where X_j denotes the projection of X to $\mathcal{U}_j$ and $\mathcal{P}_j \in \Gamma(\mathrm{End}(\boldsymbol{\$}^q M))$ is supposed to be parallel. If $\zeta \in \Gamma(\boldsymbol{\$}^q M)$ is a solution of (2.2), this leads to

$$\mathbf{R}^s_{X,Y}\zeta = \sum_{j=1}^{k} \mu_j^2 \left(\boldsymbol{\gamma}(Y_j)\mathcal{P}_j\boldsymbol{\gamma}(X_j)\mathcal{P}_j - \boldsymbol{\gamma}(X_j)\mathcal{P}_j\boldsymbol{\gamma}(Y_j)\mathcal{P}_j\right)\zeta + \\ + \sum_{j\neq i} \mu_j\mu_i \left(\boldsymbol{\gamma}(Y_j)\mathcal{P}_j\boldsymbol{\gamma}(X_i)\mathcal{P}_i - \boldsymbol{\Gamma}(X_i)\mathcal{P}_i\boldsymbol{\gamma}(Y_j)\mathcal{P}_j\right)\zeta.$$

Thus, if the relations

$$\begin{aligned} \boldsymbol{\gamma}(X)\mathcal{P}_i + \mathcal{P}_i\boldsymbol{\gamma}(X) &= 0 \\ \mathcal{P}_j\mathcal{P}_i + \mathcal{P}_i\mathcal{P}_j &= 0 \end{aligned} \tag{2.3}$$

are satisfied for all X and $i \neq j$, the last line in the equation for $\mathbf{R}^s$ vanishes because of the orthogonal decomposition of TM. Moreover, if $(\mathcal{P}_j)^2 = -\mathrm{Id}$ holds, we have

$$\mathbf{R}^s_{X,Y}\zeta = 2\sum_{j=1}^{k} \mu_j^2 \boldsymbol{\gamma}(Y_j \wedge X_j)\zeta.$$

But $\boldsymbol{\gamma}(X \wedge Y)$ and $\mathbf{R}^s$ are diagonal from the assumptions which implies that every component $\zeta_l \in \Gamma(\$M)$ of ζ satisfies

$$R^s_{X,Y}\zeta_l = -2\sum_{j=1}^{k} \mu_j^2 (X_j \wedge Y_j)\cdot\zeta_l.$$

This formula gives the following important fact.

Proposition 2.1. *If (M, g) is a complete and simply connected spin manifold with a parallel and orthogonal decomposition of the tangent bundle, the bundle $\boldsymbol{\$}^q M$ is trivialized by solutions of (2.2) if and only if (M, g) is a Riemannian product of the Euclidean space, hyperbolic spaces and spheres as well as $\mathcal{U}_j$ are the induced tangent bundles of the corresponding manifolds.*

Proof. If (M, g) is a product of spheres, hyperbolic spaces and the Euclidean space, the Riemannian curvature considered as endomorphism on $\Lambda^2 M$ satisfies:

$$R(X \wedge Y) = -\sum_{j=1} K_j \cdot X_j \wedge Y_j,$$

where K_j is the sectional curvature of the corresponding part of M. Therfore, the property (cf. [3])

$$R^s_{X,Y} = \frac{1}{2} R(X \wedge Y)\cdot$$

as well as [7, Ch. II, cor. 9.2] supply the first claim with $\mu_j = \pm\frac{1}{2}\sqrt{K_j}$. Since $\Lambda^2 M$ is effective on $\$M$, i.e. $\eta \cdot \psi = 0$ for all $\psi \in \$M$ implies $\eta = 0$, the converse claim also follows from the curvature operator and the fact that (M, g) is supposed to be complete and simply connected. □

What are the conditions to have parallel endomorphism $\mathcal{P}_j$ satisfying (2.3)? Suppose there is another parallel $\mathcal{P}_{k+1}$ with $\mathcal{P}_{k+1}^2 = -\mathrm{Id}$ and which anticommutes with $\mathcal{P}_j$ for all j, then $\boldsymbol{\gamma}(X)$ will be given by $\mathbf{i}\gamma(X)\mathcal{P}_{k+1}$. Thus, the problem to find such $\mathcal{P}_j$ reduces to representation theory of the Clifford algebra of a $k+1$ dimensional vector space. Note that here the Clifford relation $v^2 = -|v|^2$ is used only for convenience to get purely imaginary μ_j in the case of the real hyperbolic space. If we use the opposite Clifford convention for $\mathcal{P}_j$, the following considerations will work too with $\boldsymbol{\gamma}(X) = \gamma(X)\mathcal{P}_{k+1}$ and with $(\mathcal{P}_j)^* = \mathcal{P}_j$ instead of $(\mathcal{P}_j)^* = -\mathcal{P}_j$. In order to get minimal dimension of $\boldsymbol{\$}^q M$ choose $q = 2^{[\frac{k+1}{2}]}$. In the case that $\dim M$ is even, it is possible to reduce q to $2^{[\frac{k-1}{2}]}$ if the natural $\mathbb{Z}_2$ grading of $\not{S}M$ is taking into account like in example 2.3.

Thus, the endomorphisms $\mathcal{P}_j$ will be given by $\chi(e_j)$ where $\chi : Cl_{\mathbb{C}}(V^{k+1}) \to \mathrm{End}(S)$ is a representation of the Clifford algebra of V and $e_1, \ldots, e_{k+1}$ is an orthonormal base of V. Consider the representation in [3, Ch. 1], then the following parallel endomorphism of $\not{S}M \oplus \not{S}M$:

$$G_0 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$$

will define the required endomorphism $\mathcal{P}_j$. Set $|j| = j \mod 2$, then in the case $k+1$ is even, $\mathcal{P}_j$ $(j = 1 \ldots k+1)$ is given by the Kronecker product

$$\underbrace{E \otimes \cdots \otimes E}_{[\frac{k+1}{2}]-[\frac{j+1}{2}]} \otimes G_{|j+1|} \otimes \underbrace{T \otimes \cdots \otimes T}_{[\frac{j-1}{2}]}.$$

If $k+1$ is odd, choose $\mathcal{P}_j$ for $j = 1 \ldots k$ like in the even case and set

$$\mathcal{P}_{k+1} = \mathbf{i}\underbrace{T \otimes \cdots \otimes T}_{[\frac{k+1}{2}]}.$$

$\mathcal{P}_j$ is independent of the choice of the connection on $\not{S}M$, that means $\mathcal{P}_j$ is parallel w.r.t. any diagonal connection $\boldsymbol{\nabla}$ on $\boldsymbol{\not{S}}^q M$ obtained from ∇ on $\not{S}M$. Moreover, if B is an endomorphism on $\not{S}M$, the diagonal extension of B to an endomorphism on $\boldsymbol{\not{S}}^q M$ always commutes with $\mathcal{P}_j$ for all j.

Example 2.2. Consider the case $k = 3$. Suppose $TM = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{U}_3$ is a parallel and orthogonal splitting of the tangent bundle of M. The bundle $\boldsymbol{\not{S}}^4 M$ will be equipped with the Clifford multiplication

$$\boldsymbol{\gamma}(X) = \mathbf{i} \begin{pmatrix} 0 & 0 & 0 & \gamma(X) \\ 0 & 0 & -\gamma(X) & 0 \\ 0 & \gamma(X) & 0 & 0 \\ -\gamma(X) & 0 & 0 & 0 \end{pmatrix},$$

where γ means the usual Clifford multiplication on $\not{S}M$. Moreover, choose

$$\mathcal{P}_1 = \begin{pmatrix} \mathbf{i} & 0 & 0 & 0 \\ 0 & -\mathbf{i} & 0 & 0 \\ 0 & 0 & \mathbf{i} & 0 \\ 0 & 0 & 0 & -\mathbf{i} \end{pmatrix} \quad \mathcal{P}_2 = \begin{pmatrix} 0 & \mathbf{i} & 0 & 0 \\ \mathbf{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{i} \\ 0 & 0 & \mathbf{i} & 0 \end{pmatrix} \quad \mathcal{P}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

then $\boldsymbol{\not{S}}^4 M$ is locally trivialized by solutions of

$$\boldsymbol{\nabla}_X \zeta = \mu_1 \boldsymbol{\gamma}(X_1)\mathcal{P}_1\zeta + \mu_2 \boldsymbol{\gamma}(X_2)\mathcal{P}_2\zeta + \mu_3 \boldsymbol{\gamma}(X_3)\mathcal{P}_3\zeta$$

if and only if (M, g) is locally isometric to

$$\mathbb{M}_0^{n_1}(4\mu_1^2) \times \mathbb{M}_0^{n_2}(4\mu_2^2) \times \mathbb{M}_0^{n_3}(4\mu_3^2).$$

In this case $(\mathbb{M}_0^n(K), g_0)$ means the n–dimensional complete simply connected manifold of constant curvature K and n_j is the rank of $\mathcal{U}_j$.

Example 2.3. Consider the Riemannian product manifold

$$(M, g) = \mathbb{R}^k \times \mathbb{R}\mathrm{H}^m(-4\lambda^2) \times \mathbb{R}\mathrm{H}^n(-4\kappa^2)$$

where $\dim M$ is supposed to be even. The spinor bundle $\not{S}M = (\not{S}M)^+ \oplus (\not{S}M)^-$ is trivialized by solutions of (cf. [9])

$$\nabla_X \zeta = \lambda\gamma(X_2)\zeta + \mathbf{i}\kappa\gamma(X_3)(\zeta^+ - \zeta^-).$$

Nevertheless, since this Killing equation does not make sense if $\dim M$ is odd, we will use for convenience the above approach to prove theorem 1.2.

3. Integrated Bochner–Weitzenböck formula

Let (M,g) be a Riemannian spin manifold with an orthogonal decomposition of its tangent bundle $TM = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$. Denote by $\pi_j : TM \to \mathcal{U}_j$ the projections as well as by $m_j \neq 0$ the rank of $\mathcal{U}_j$. Consider the connection

$$\widehat{\boldsymbol{\nabla}}_X := \boldsymbol{\nabla}_X - \sum_{j=1}^{k} \mu_j \boldsymbol{\gamma}(\pi_j X)\mathcal{P}_j$$

with $q = 2^{[\frac{k+1}{2}]}$, $\mathcal{P}_j$ and $\boldsymbol{\gamma}$ as in the previous section. Suppose $N \subset M$ is a compact manifold with boundary ∂N and outward normal vector field ν. If μ_j is purely imaginary for all j, this leads to the following integrated Bochner–Weitzenböck formula

$$\int\limits_{\partial N} \left\langle \widehat{\boldsymbol{\nabla}}_\nu \psi + \nu \cdot \widehat{\boldsymbol{D}}\psi, \varphi \right\rangle = \int\limits_{N} \left\langle \widehat{\boldsymbol{\nabla}}\psi, \widehat{\boldsymbol{\nabla}}\varphi \right\rangle - \left\langle \widehat{\boldsymbol{D}}\psi, \widehat{\boldsymbol{D}}\varphi \right\rangle + \left\langle \widehat{\mathfrak{R}}\psi, \varphi \right\rangle, \tag{3.1}$$

where integration is done with respect to vol_g and

$$\widehat{\mathfrak{R}} = \frac{\mathrm{scal}}{4} - \sum_{j=1}^{k} \mu_j^2 m_j(m_j - 1) - \sum_{j=1}^{k} \mu_j \boldsymbol{\gamma}(\delta\pi_j)\mathcal{P}_j.$$

In order to see this, use the Lichnerowicz formula (2.1), the fact that $\boldsymbol{\nabla}$ is Riemannian, $(\mu_j \mathcal{P}_j)^* = \mu_j \mathcal{P}_j$, as well as the definition of the Dirac operator $\widehat{\boldsymbol{D}}$:

$$\widehat{\boldsymbol{D}} = \sum_{i=1}^{n} \boldsymbol{\gamma}(e_i)\widehat{\boldsymbol{\nabla}}_{e_i} = \boldsymbol{D} + \sum_{j=1}^{k} \mu_j m_j \mathcal{P}_j.$$

Furthermore, since $\widehat{\mathfrak{R}}$ is symmetric w.r.t. g, the boundary operator in (3.1) satisfies

$$\int\limits_{\partial N} \left\langle \widehat{\boldsymbol{\nabla}}_\nu \psi + \nu \cdot \widehat{\boldsymbol{D}}\psi, \varphi \right\rangle = \int\limits_{\partial N} \left\langle \psi, \widehat{\boldsymbol{\nabla}}_\nu \varphi + \nu \cdot \widehat{\boldsymbol{D}}\varphi \right\rangle. \tag{3.2}$$

To obtain theorem 1.2, we have to prove an isomorphism property of the Dirac operator $\widehat{\boldsymbol{D}}$. This leads to the following analytic preliminaries. If $\mathcal{E} \to M^n$ is a Riemannian vector bundle and ω is a positively oriented nowhere vanishing n–form, $L^2(M, \mathcal{E}; \omega)$ denotes the completion of compact supported sections of $\mathcal{E}$ with respect to the L^2 norm

$$\|\varphi\|_\omega = \sqrt{\int\limits_M \langle \varphi, \varphi \rangle \, \omega}\,.$$

Suppose that $\mathcal{E}$ is equipped with a Riemannian connection ∇, and consider the scalar product

$$(\varphi, \psi)_{1,\omega} := \int\limits_M \Big(\langle \nabla\varphi, \nabla\psi \rangle + \langle \varphi, \psi \rangle \Big)\omega$$

on $\Gamma_{\mathrm{cpt}}(\mathcal{E})$. The Sobolev space $W^{1,2}(M, \mathcal{E}; \omega)$ is the closure of $\Gamma_{\mathrm{cpt}}(\mathcal{E})$ in $L^2(M, \mathcal{E}; \omega)$ with respect to the norm $\|.\|_{1,\omega} := \sqrt{(.,.)_{1,\omega}}$. Moreover, $W^{1,2}(M, \mathcal{E}; \omega)$ becomes a Hilbert space with respect to the extension of the product $(.,.)_{1,\omega}$. For notational simplicity set $W^{1,2}(M, \mathcal{E}) := W^{1,2}(M, \mathcal{E}; \mathrm{vol}_g)$ and $L^2(M, \mathcal{E}) := L^2(M, \mathcal{E}; \mathrm{vol}_g)$, where vol_g denotes the volume form of (M,g).

Proposition 3.1. *Suppose μ_j is purely imaginary for all j and there is some j with $\mu_j \neq 0$. If (M,g) is complete with uniformly bounded scalar curvature such that*

$$\overline{\mathrm{scal}} := \mathrm{scal} - 4\sum_{j=1}^{k} \mu_j^2 m_j(m_j - 1) \geq 0,$$

then the Dirac operator

$$\widehat{\mathbf{D}} : W^{1,2}(M, \mathbf{S}^q M) \to L^2(M, \mathbf{S}^q M)$$

is an isomorphism of Hilbert spaces.

Proof. The operator $\widehat{\mathbf{D}}$ is well defined and bounded. The essential L^2 adjoint of the Dirac operator $\widehat{\mathbf{D}}$ satisfies $\widehat{\mathbf{D}}^* = \widehat{\mathbf{D}}$. Moreover, the parallelism of $\mathcal{P}_j$ as well as $\boldsymbol{\gamma}(X)\mathcal{P}_j = -\mathcal{P}_j\boldsymbol{\gamma}(X)$ imply

$$\widehat{\mathbf{D}}^*\widehat{\mathbf{D}} = \mathbf{D}^2 - \sum_{j=1}^{k} \mu_j^2 m_j^2.$$

Thus, the sesquilinear form

$$B(\psi, \varphi) := \int\limits_M \left\langle \widehat{\mathbf{D}}^*\widehat{\mathbf{D}}\psi, \varphi \right\rangle = \int\limits_M \langle \boldsymbol{\nabla}^*\boldsymbol{\nabla}\psi, \varphi\rangle - \sum_{j=1}^{k} \mu_j^2 m_j \langle \psi, \varphi\rangle + \frac{\overline{\mathrm{scal}}}{4} \langle \psi, \varphi\rangle$$

is bounded and coercive on $\Gamma_{\mathrm{cpt}}(\mathbf{S}^q M)$ with respect to the $W^{1,2}$ norm. In particular B can be extended to a scalar product on $W^{1,2}(M, \mathbf{S}^q M)$. This gives the injectivity of $\widehat{\mathbf{D}}$. Suppose $\zeta \in L^2(M, \mathbf{S}^q M)$, then the linear functional

$$l(\psi) := \int\limits_M \left\langle \zeta, \widehat{\mathbf{D}}\psi \right\rangle$$

is bounded on $W^{1,2}(M, \mathbf{S}^q M)$. Therefore, the Riesz representation theorem supplies some $\varphi \in W^{1,2}(M, \mathbf{S}^q M)$ with $B(\varphi, \psi) = l(\psi)$ for all $\psi \in W^{1,2}(M, \mathbf{S}^q M)$. Set $\theta := \widehat{\mathbf{D}}\varphi - \zeta$, then elliptic theory implies smoothness of θ and $\widehat{\mathbf{D}}\theta = 0$ in the strong sense. Moreover, $\theta \in L^2(M, \mathbf{S}^q M)$ together with $\widehat{\mathbf{D}}\theta = 0$ imply $\mathbf{D}\theta \in L^2(M, \mathbf{S}^q M)$, so that [5, thm. 2.8] supplies $\theta \in W^{1,2}(M, \mathbf{S}^q M)$. But $\widehat{\mathbf{D}}$ is injective on $W^{1,2}(M, \mathbf{S}^q M)$, i.e. $\theta = 0$. □

4. Proof of the rigidity result

In order to obtain theorem 1.2, it will be necessary to conclude parallelism of the projection maps $\pi_j : TM \to \mathcal{U}_j$. Thus, the following lemma is a useful tool to get parallelism of π_j from the Riemannian curvature tensor.

Lemma 4.1. *Suppose $TM = \mathcal{U} \oplus \mathcal{V}$ is an orthogonal splitting of the tangent bundle of (M, g). Then the following statements are equivalent:*

(1) *The Riemannian curvature tensor preserves sections of $\mathcal{U}$ respectively sections of $\mathcal{V}$:*

$$R_{X,Y} : \Gamma(\mathcal{U}) \to \Gamma(\mathcal{U}), \qquad R_{X,Y} : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}).$$

(2) $\mathrm{Hol}^0(M, g)$ *leaves $\mathcal{U}$ and $\mathcal{V}$ invariant.*

(3) *The projection maps $\pi_{\mathcal{U}}$ and $\pi_{\mathcal{V}}$ are parallel.*

Proof. The equivalence of the last two statements follows from [4, Ch. 10]. Moreover, that (2) respectively (3) imply (1) is obvious. Suppose the rank of $\mathcal{U}$ is l and consider $\Lambda^l(\mathcal{U}^*)$ which is at least locally defined and of rank one. Let $\omega = e_1^* \wedge \cdots \wedge e_l^*$ be a non–trivial section in this bundle, we obtain from (1) the essential fact: $R_{X,Y}\omega \in \Lambda^l(\mathcal{U}^*)$. Since $R_{X,Y}f$ vanishes for all functions f,

$$\langle R_{X,Y}\omega, \omega\rangle = \frac{1}{2} R_{X,Y}|\omega|^2 = 0$$

implies $R_{X,Y}\omega = 0$ (note that this is wrong if $R_{X,Y}$ does not leave $\mathcal{U}$ invariant). Thus, $\Lambda^l(\mathcal{U}^*)$ is flat and admits locally a non–trivial parallel section ω'. The same thing can be done for $\mathcal{V}$ to get locally another parallel exterior form (locally means in simply connected neighborhoods). Thus, [4, remark 10.22] implies (2). □

Let (M_0,g_0) as well as (M,g) be the Riemannian manifolds in the rigidity theorem 1.2, where $E\subset M$ is supposed to be the Euclidean end of M. In order to avoid any problems with notation, consider the Dirac bundle $\boldsymbol{\$}^q M=\bigoplus_{j=1}^{q}\$M$ with $q=2^{[(l+1)/2]}$ and the parallel endomorphism $\mathcal{P}_j$ from the previous section as well as the Clifford multiplication $\boldsymbol{\gamma}(X)=\mathbf{i}\gamma(X)\mathcal{P}_{l+1}$. Set $\mu_1=0$ and $\mu_j:=\frac{\mathbf{i}}{2}\sqrt{K_j}$ for $2\le j\le l$. Define the connection

$$\widehat{\boldsymbol{\nabla}}_X:=\boldsymbol{\nabla}_X-\sum_{j=1}^{l}\mu_j\boldsymbol{\gamma}(\pi_jX)\mathcal{P}_j$$

on $\boldsymbol{\$}^qM$ as well as the connection

$$\widehat{\boldsymbol{\nabla}}^0_X:=\boldsymbol{\nabla}^0_X-\sum_{j=1}^{l}\mu_j\boldsymbol{\gamma}^0(\pi^0_jX)\mathcal{P}_j$$

on $\boldsymbol{\$}^qM_{|E}$, where $\boldsymbol{\nabla}^0$ means the diagonal extension of the Levi–Civita connection with respect to f^*g_0 on E and $\boldsymbol{\gamma}^0(X)$ is given by $\mathbf{i}\gamma^0(X)\mathcal{P}_{l+1}$. We conclude from section 2 that $\boldsymbol{\$}^qM_{|E}$ is trivialized by solutions parallel with respect to $\widehat{\boldsymbol{\nabla}}^0$.

The gauge transformation A extends to a bundle morphism $A:\$M_{|E}\to\$M_{|E}$ with (cf. [1])

$$\left|\overline{\nabla}\varphi-\nabla\varphi\right|\le C\left|A^{-1}\right|\left|\nabla^0A\right||\varphi|\ ,$$

where ∇ is the usual spin connection for g and $\overline{\nabla}$ is a connection on $\$M_{|E}$ obtained from the connection $\overline{\nabla}$ on $TM_{|E}$ and given by $\overline{\nabla}Y=A(\nabla^0(A^{-1}Y))$. Moreover, the same estimates hold for the diagonal extension of ∇ respectively $\overline{\nabla}$ (denoted by a bold symbol) to the bundle $\boldsymbol{\$}^qM$.

Let ψ_0 be a spinor on $E\subset M$ which is parallel with respect to $\widehat{\boldsymbol{\nabla}}^0$. Set $\psi:=h(A\psi_0)$ for some cut off function h, i.e. $h=1$ at infinity, $h=0$ in $M-E$ and $\operatorname{supp}(\mathrm{d}h)$ compact. The following computations show that $\widehat{\boldsymbol{\nabla}}\psi\in L^2(M,T^*M\otimes\boldsymbol{\$}^qM)$. Set $\mathfrak{T}:=\widehat{\boldsymbol{\nabla}}-\boldsymbol{\nabla}$ as well as $\mathfrak{T}^0:=\widehat{\boldsymbol{\nabla}}^0-\boldsymbol{\nabla}^0$, then the facts $A\circ\gamma^0(X)=\gamma(AX)\circ A$ and $A\mathcal{P}_j=\mathcal{P}_jA$ imply

$$\begin{aligned}-A\mathfrak{T}^0_X\psi_0+\mathfrak{T}_X(A\psi_0) &= \sum_{j=2}^{l}\mu_j\Big(A(\boldsymbol{\gamma}^0(\pi^0_jX)\mathcal{P}_j\psi_0)-\boldsymbol{\gamma}(\pi_jX)\mathcal{P}_jA\psi_0\Big)\\ &= \sum_{j=2}^{l}\mu_j\boldsymbol{\gamma}\big(A\pi^0_j(X)-\pi_j(X)\big)\mathcal{P}_jA\psi_0.\end{aligned}$$

In particular

$$\begin{aligned}\widehat{\boldsymbol{\nabla}}_X\psi &= (Xh)A\psi_0+h(\boldsymbol{\nabla}_XA\psi_0+\mathfrak{T}_X(A\psi_0))\\ &= (Xh)A\psi_0+h(\boldsymbol{\nabla}_X-\overline{\boldsymbol{\nabla}}_X)A\psi_0+hA(\boldsymbol{\nabla}^0_X\psi_0)+h\mathfrak{T}_X(A\psi_0)\\ &= (Xh)A\psi_0+h(\boldsymbol{\nabla}_X-\overline{\boldsymbol{\nabla}}_X)A\psi_0-hA\mathfrak{T}^0_X\psi_0+h\mathfrak{T}_XA\psi_0\end{aligned}$$

supplies for uniformly bounded A (near infinity for some $c>0$)

$$\left|\widehat{\boldsymbol{\nabla}}\psi\right|^2\le c\bigg(\left|\nabla^0A\right|^2+\sum_{j=2}^{l}\left|A\circ\pi^0_j-\pi_j\right|^2\bigg)\left|A\psi_0\right|^2_g\ .$$

Moreover, $|A\psi_0|_g^2 = |\psi_0|_{g_0}^2$ is of order $e^{\alpha r}$ with $\alpha = \sqrt{\sum K_j}$, $|X| = 1$:

$$\begin{aligned} \left| X \, |\psi_0|_{g_0}^2 \right| &= 2 \left| \langle \boldsymbol{\nabla}_X^0 \psi_0, \psi_0 \rangle \right| \\ &= \left| \mathbf{i} \sum_{j=2}^{l} \sqrt{K_j} \, \langle \boldsymbol{\gamma}^0(\pi_j^0 X) \mathcal{P}_j \psi_0, \psi_0 \rangle \right| \\ &\leq |\psi_0|^2 \sum_{j=2}^{l} \sqrt{K_j} \, |\pi_j^0 X|_{g_0} \leq |\psi_0|_{g_0}^2 \sqrt{\sum_{j=2}^{l} K_j}\,, \end{aligned}$$

where the Cauchy–Schwarz inequality is applied to the vectors $\sum \sqrt{K_j} e_j$ and $\sum |X_j| e_j$. Thus, the estimate

$$\left\langle \widehat{\boldsymbol{\nabla}}_\nu \psi + \nu \cdot \widehat{\boldsymbol{\mathfrak{D}}}\psi, \psi \right\rangle \leq c \, |A\psi_0|^2 \sqrt{\left| \nabla^0 A \right|^2 + \sum_{j=2}^{l} \left| A \circ \pi_j^0 - \pi_j \right|^2}$$

and the assumptions on being strongly asymptotic to (M_0, g_0) imply

$$\left\langle \widehat{\boldsymbol{\nabla}}_\nu \psi + \nu \cdot \widehat{\boldsymbol{\mathfrak{D}}}\psi, \psi \right\rangle \in L^1(M). \tag{4.1}$$

A straightforward computation shows that inequality (1.1) yields $\overline{\mathrm{scal}} \geq 0$ in proposition 3.1 as well as $\widehat{\mathfrak{R}} \geq 0$ in the integrated Bochner–Weitzenböck formula (3.1). Therefore, since $\widehat{\boldsymbol{\mathfrak{D}}}\psi$ is a L^2–section, proposition 3.1 supplies some $\widetilde{\psi} \in W^{1,2}(M, \boldsymbol{\$}^q M)$ with $\widehat{\boldsymbol{\mathfrak{D}}}\widetilde{\psi} = \widehat{\boldsymbol{\mathfrak{D}}}\psi$. Set $\varphi := \psi - \widetilde{\psi}$, then φ is non–trivial and $\widehat{\boldsymbol{\mathfrak{D}}}\varphi = 0$. Moreover, we obtain in the usual way from the selfadjointness of the boundary operator (3.2), equation (4.1) as well as [1, prop. 4.1]

$$\liminf_{r \to \infty} \int_{\partial M_r} \left\langle \widehat{\boldsymbol{\nabla}}_\nu \varphi + \nu \cdot \widehat{\boldsymbol{\mathfrak{D}}}\varphi, \varphi \right\rangle = 0$$

if $\{M_r\}$ is an exhaustion of M. Since $\widehat{\mathfrak{R}}$ is non–negative and $\widehat{\boldsymbol{\mathfrak{D}}}\varphi$ vanishes, the integrated Bochner–Weitzenböck formula (3.1) supplies $\widehat{\boldsymbol{\nabla}}\varphi = 0$.

In particular φ is given by $(\mathrm{Id} - \widehat{\boldsymbol{\mathfrak{D}}}^{-1} \widehat{\boldsymbol{\mathfrak{D}}}) hA\psi_0$, where $\psi_0 \notin W^{1,2}(E, \boldsymbol{\$}^q(M)_{|E})$ is parallel with respect to $\widehat{\boldsymbol{\nabla}}^0$. Thus, the bundle $\boldsymbol{\$}^q M$ is trivialized by spinors parallel with respect to $\widehat{\boldsymbol{\nabla}}$, hence $\widehat{\boldsymbol{\nabla}}$ is a flat connection.

Computing the curvature of $\widehat{\boldsymbol{\nabla}}$, we obtain for every $\widehat{\boldsymbol{\nabla}}$ parallel section ζ:

$$\begin{aligned} \mathbf{R}^s_{X,Y}\zeta = \sum_{j=2}^{l} \mu_j \boldsymbol{\gamma}((\mathrm{d}\pi_j)(X,Y)) \mathcal{P}_j \zeta + \\ + \sum_{j=2}^{l} \mu_j^2 \left(\boldsymbol{\gamma}(\pi_j Y) \boldsymbol{\gamma}(\pi_j X) - \boldsymbol{\gamma}(\pi_j X) \boldsymbol{\gamma}(\pi_j Y) \right) \zeta, \end{aligned} \tag{4.2}$$

where $\mathrm{d}\pi_j(X,Y)$ is the vector field given by $(\nabla_X \pi_j)(Y) - (\nabla_Y \pi_j)(X)$. Moreover,

$$(\mu_j \boldsymbol{\gamma}(X) \mathcal{P}_j)^* = \overline{\mu_j} \mathcal{P}_j^* \boldsymbol{\gamma}(X)^* = -\mu_j \mathcal{P}_j \boldsymbol{\gamma}(X) = \mu_j \boldsymbol{\gamma}(X) \mathcal{P}_j$$

implies hermiticity of

$$\sum_{j=2}^{l} \mu_j \boldsymbol{\gamma}((\mathrm{d}\pi_j)(X,Y)) \mathcal{P}_j. \tag{4.3}$$

Since the other parts in (4.2) are skew Hermitian and (4.2) holds for all $\zeta \in \boldsymbol{\$}^q M$, equation (4.2) reduces to

$$\mathbf{R}^s_{X,Y}\zeta = -2\sum_{j=2}^{l} \mu_j^2 \boldsymbol{\gamma}(X_j \wedge Y_j)\zeta.$$

Clifford multiplication with two forms is effective on $\$M$ (cf. [9]) which supplies the Riemannian curvature tensor of (M, g):

$$R(X \wedge Y) = -4\sum_{j=2}^{l} \mu_j^2 (X_j \wedge Y_j).$$

Therefore, if Z_j is a section in $\mathcal{U}_j$, $R_{X,Y}Z_j$ is a section in $\mathcal{U}_j$. In particular applying lemma 4.1, gives a holonomy restriction of (M, g), and thus, (M, g) must be locally isometric to (M_0, g_0). But (M, g) is complete and has sectional curvature $K \leq 0$, so that the Hadamard–Cartan theorem together with the fact $E \cong (0,1) \times S^{n-1}$ imply global symmetry of (M, g).

5. Remarks

The method of concluding a holonomy restrictions seems to work only in the imaginary Killing case. If there is a spherical factor, the part in (4.3) is skew Hermitian too, i.e. this part in (4.2) does not vanish from algebraic reasons.

What can be said about rigidity of the Riemannian product

$$(M_0, g_0) = N \times \mathbb{R}\mathrm{H}^m$$

if (N, h) is a compact simply connected spin manifold with a non–trivial parallel spinor, in particular h is Ricci flat? In this case a Riemannian manifold is said to be *asymptotic* to (M_0, g_0) if M has an end E which is diffeomorphic to $N \times (0, \infty) \times S^{m-1}$ and the gauge transformation A given on E satisfies the usual assumptions for Riemannian products. The spinor bundle of M_0 admits non–trivial solutions of (cf. [9])

$$\nabla^0_X \psi = \lambda \pi^0_2(X) \cdot \psi$$

with $\lambda \in \mathbf{i}\mathbb{R} - \{0\}$, but $\$M_0$ is not trivialized by them. In particular if the scalar curvature satisfies

$$|\lambda||\delta\pi_2|_g \leq \frac{\text{scal}}{4} + m(m-1)|\lambda|^2,$$

the same methods like in the previous section provide $\$M$ with non–trivial solutions ψ of

$$\nabla_X \psi = \lambda \pi_2(X) \cdot \psi.$$

Nevertheless, the spinor bundle $\$M$ is not trivialized by these sections, so that the curvature of $\nabla - \lambda\pi_2(.)\cdot$ does not supply the full Riemannian curvature tensor and it seems to be much more complicated to get a holonomy restriction on (M, g). But if one could conclude the reducibility of the holonomy group, (M, g) will be isometric to (M_0, g_0).

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