RIGIDITY OF THE QUATERNIONIC HYPERBOLIC SPACE

MARIO LISTING

Abstract. This paper proves that a strongly asymptotically quaternionic hyperbolic spin manifold is isometric to the quaternionic hyperbolic space. In order to obtain this result, we consider imaginary quaternionic Killing spinors and compute an integrated Bochner–Weitzenböck formula for quaternionic Kähler manifolds.

1. Introduction

Rigidity of symmetric spaces of non–compact type is a frequently studied problem (cf. [3, 7, 14]). For instance, M. Min–Oo proved in [14] that a strongly asymptotically hyperbolic spin manifold (M^n, g) with scalar curvature $\mathrm{scal} \geq -n(n-1)$ is isometric to the hyperbolic space. Moreover, M. Herzlich showed in [7] that a strongly asymptotically complex hyperbolic spin manifold (M^{2m}, g) of odd complex dimension m and with scalar curvature $\mathrm{scal} \geq -4m(m+1)$ must be isometric to the complex hyperbolic space.

In this paper we prove the analogous result for the quaternionic hyperbolic space. As in [7] we need the holonomy assumption $\mathrm{Hol}(M, g) \subseteq \mathrm{Hol}(M_0, g_0)$ where in the present case $\mathrm{Hol}(M_0, g_0)$ is given by $\mathrm{Sp}(1)\mathrm{Sp}(n)$. In particular since quaternionic Kähler manifolds are Einstein (cf. [4, thm. 14.39]), the scalar curvature bound $\mathrm{scal} \geq \mathrm{scal}_0$ in Min–Oo's respectively M. Herzlich's result is not necessary.

Definition 1.1. $\mathbb{H}\mathrm{H}^n$ denotes the quaternionic hyperbolic space of quaternionic dimension n and sectional curvature $K \in [-1, -\frac{1}{4}]$ as well as $B_R(q) \subset M$ is the set of all $p \in M$ with geodesic distance to q less than R.

Let (M^{4n}, g) be a quaternionic Kähler manifold. (M, g) is said to be *strongly asymptotically quaternionic hyperbolic* if there is a compact manifold $C \subset M$ and a diffeomorphism $f : E := M - C \to \mathbb{H}\mathrm{H}^n - \overline{B_R(0)}$ in such a way that the positive definite gauge transformation $A \in \Gamma(\mathrm{End}(TM_{|E}))$ given by

$$g(AX, AY) = (f^* g_0)(X, Y) \qquad g(AX, Y) = g(X, AY)$$

satisfies:

(1) A is uniformly bounded.
(2) $\left|(f^*\nabla^0)A\right| + |A - \mathrm{Id}| \in L^1(E; e^r \mathrm{vol}_g) \cap L^2(E; e^r \mathrm{vol}_g)$, where r is the f^*g_0–distance to a fixed point and $f^*\nabla^0$ is the Levi–Civita connection for f^*g_0.
(3) There are g–compatible almost complex structures $I, J, K \in \Gamma(\mathrm{End}(TM_{|E}))$ satisfying the quaternionic axioms and defining the holonomy representation of g such that

$$|AI_0 - I| + |AJ_0 - J| + |AK_0 - K| \in L^1(E; e^r \mathrm{vol}_g) \cap L^2(E; e^r \mathrm{vol}_g),$$

Date: May 8, 2003.
2000 *Mathematics Subject Classification.* Primary 53C24, Secondary 53C26.
Key words and phrases. quaternionic Kähler manifolds, rigidity, quaternionic Killing spinors.
Thanks: This result originated from the author's PhD–thesis "Rigidity of Hyperbolic Spaces" and it was supported by the Graduiertenkolleg "Analysis, Geometrie und ihre Verbindung zu den Naturwissenschaften". The author would like to express particular gratitude to the supervisor of this thesis Prof. Dr. Hans–Bert Rademacher. The author is also grateful to Frank Klinker for discussion.

where $I_0, J_0, K_0 \in \Gamma(\mathrm{End}(TM_{|E}))$ is a f^*g_0–compatible quaternionic structure defining the holonomy representation of f^*g_0.

Theorem 1.2. *Let (M^{4n}, g) be a complete quaternionic Kähler spin manifold of quaternionic dimension $n > 2$. If (M, g) is strongly asymptotically quaternionic hyperbolic, (M, g) is isometric to $\mathbb{H}\mathrm{H}^n$.*

Since $\mathbb{H}\mathrm{H}^1$ is the real hyperbolic space, Min–Oo's result is appropriate if $n = 1$. But the case $n = 2$ still seems to be open. The problem is to find a suitable integrated Bochner–Weitzenböck formula.

The proof of this rigidity theorem is as usual based on the non–compact Bochner technique which was introduced by E. Witten in [15]. We show an integrated Bochner–Weitzenböck formula which allows the usage of this technique. The connection in this formula will be a modification of the quaternionic Killing connection introduced in [11]. Moreover, we prove that the corresponding Dirac operator is an isomorphism of Hilbert spaces.

2. Preliminaries

The following arguments strictly rely on the calculations done in [11] respectively [12], so that if nothing is mentioned, the same notations will be used. In particular in this section we will only give an overview of the results in [11], i.e. in order to get the explicit definitions confer [11].

The *quaternionic hyperbolic space* $\mathbb{H}\mathrm{H}^n$ is the rank one symmetric space which is dual to the quaternionic projective space $\mathbb{H}\mathrm{P}^n$. $\mathbb{H}\mathrm{H}^n$ is diffeomorphic to $\mathbb{R}^{4n}$ and an Einstein space of negative scalar curvature. A *quaternionic Kähler manifold* is a Riemannian manifold (M^{4n}, g) with holonomy contained in $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$. Assuming (M^{4n}, g) to be quaternionic Kähler and spin, there exists a parallel and orthogonal decomposition of the spinor bundle

$$\not{S}M = \not{S}_0 \oplus \cdots \oplus \not{S}_n$$

induced from the Clifford action of the Kraines 4–form Ω on $\not{S}M$ (cf. [2], [8]). In this case $\not{S}_r$ is an eigenspace of $\Omega\cdot$ to the eigenvalue $6n - 4r(r + 2)$. Supposing H and E are the defining representations of $\mathrm{Sp}(1)$ and $\mathrm{Sp}(n)$, we obtain representations $\rho^{l,k}$ of $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ on $\mathrm{Sym}^l H \otimes \Lambda^k_0 E$. Since the $\mathrm{Spin}(4n)$ principal bundle of a quaternionic Kähler spin manifold is reducible to a $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ principal bundle $P_{\mathrm{Sp}(1)\times\mathrm{Sp}(n)}(M, g)$, this leads to bundles

$$\not{S}^{l,k} := P_{\mathrm{Sp}(1)\times\mathrm{Sp}(n)}(M, g) \times_{\rho^{l,k}} \mathrm{Sym}^l H \otimes \Lambda^k_0 E$$

which in [11] are denoted by $\mathrm{Sym}^l \mathbf{H} \otimes \Lambda^k_0 \mathbf{E}$. Thus, if M is spin, $\not{S}^{l,k}$ exists globally as well as $\not{S}_r = \not{S}^{r,n-r}$ holds (cf. [2]). These bundles will be equipped with the Levi–Civita connection arising from the spin connection and metrics $\langle ., . \rangle$ defined in [12]. Moreover, there is some kind of Clifford action given with the properties

$$\gamma^\pm_+ : TM \to \mathrm{Hom}(\not{S}^{l,k}, \not{S}^{l\pm1,k+1}) \qquad \begin{array}{rcl} (\gamma^+_+(X))^* &=& \gamma^-_-(X) \\ (\gamma^-_+(X))^* &=& -\gamma^+_-(X) \end{array}$$

$$\gamma^\pm_- : TM \to \mathrm{Hom}(\not{S}^{l,k}, \not{S}^{l\pm1,k-1}) \qquad \begin{array}{rcl} (\gamma^-_-(X))^* &=& \gamma^+_+(X) \\ (\gamma^+_-(X))^* &=& -\gamma^-_+(X), \end{array}$$

where the notation $\gamma^\pm_\pm$ is used instead of $\mu^\pm_\pm$ in [11]. The corresponding Dirac operator is denoted by $\not{D}^\pm_\pm = \sum \gamma^\pm_\pm(e_j)\nabla_{e_j}$. Furthermore, $\gamma^\pm_\pm$ is parallel with respect to ∇ and

$$\gamma^+_- + \gamma^-_+ : TM \otimes \not{S}M \to \not{S}M$$

coincides with the usual Clifford multiplication on the spinor bundle. For notational simplicity define $\not{S}_- := \not{S}^{0,n-2} = \Lambda^{n-2}_0 \mathbf{E}$ in case $n \geq 2$.

On the quaternionic projective space $\mathbb{HP}^n$ there exists so called quaternionic Killing spinors (cf. [11]). A *quaternionic Killing spinor* is a section in

$$\mathcal{V} := \not{S}_0 \oplus \not{S}_1 \oplus \not{S}_-$$

which is parallel with respect to $\nabla^{\mathrm{Killing}} = \nabla + \mathfrak{A}$ with

$$\mathfrak{A}_X := \lambda \begin{pmatrix} 0 & \frac{1}{n+3}\gamma_+^-(X) & 0 \\ \frac{1}{4n}\gamma_-^+(X) & 0 & -\frac{3}{2(n+3)}\gamma_+^+(X) \\ 0 & \frac{1}{4n}\gamma_-^-(X) & 0 \end{pmatrix} \in \Gamma(\mathrm{End}(\mathcal{V})).$$

If there is a non–trivial quaternionic Killing spinor, λ satisfies $\lambda^2 = \frac{n+3}{n+2}\frac{\mathrm{scal}}{4}$, i.e. λ has to be real or purely imaginary. This leads to the distinction of *real* and *imaginary* quaternionic Killing spinors. We know from [11, prop. 4.2] $R^{\mathrm{Killing}} = R^{\mathrm{hyper}}$, and since R^{hyper} vanishes if and only if the quaternionic Kähler manifold is locally isometric to $\mathbb{HP}^n$ or $\mathbb{HH}^n$, $\mathcal{V}$ is trivialized by quaternionic Killing spinors only if (M, g) is locally isometric to one of the quaternionic model spaces. Moreover, since $\mathbb{HP}^n$ and $\mathbb{HH}^n$ are simply connected, there is a full set of quaternionic Killing spinors on these spaces (cf. [9, Ch. II, cor. 9.2]).

Proposition 2.1. *Suppose (M^{4n}, g) is a simply connected and complete quaternionic Kähler spin manifold, then the bundle $\mathcal{V}$ is trivialized by spinor parallel with respect to $\nabla^{\mathrm{Killing}}$ with $\lambda \in \mathbf{i}\mathbb{R} - \{0\}$ if and only if (M^{4n}, g) is (up to scaling) isometric to $\mathbb{HH}^n$.*

3. Integrated Bochner–Weitzenböck formula

The non–compact Bochner technique works for bundles with a second order Bochner–Weitzenböck formula having a selfadjoint boundary operator in its integrated version. In order to obtain such a selfadjoint boundary operator, the existence of a generalized Dirac operator turns out to be necessary. Nevertheless, $\mathcal{V}$ does not seem to be part of a Dirac bundle which makes the computations more difficult. Moreover, the quaternionic Killing connection itself does not supply a selfadjoint boundary operator, i.e. we have to balance the factors in the connection.

To simplify the computations, we consider the bundles $\mathcal{V}$ and

$$\widetilde{\mathcal{V}} := \not{S}_0 \oplus \not{S}_1 \oplus \not{S}_1 \oplus \not{S}_-$$

with the usual inner products $\langle .,. \rangle_{\mathcal{V}}$ and $\langle .,. \rangle_{\widetilde{\mathcal{V}}}$ arising from the inner products on $\not{S}_0$, $\not{S}_1$ and $\not{S}_-$ (cf. [12]). Defining two linear operators between these spaces by $L : \widetilde{\mathcal{V}} \to \mathcal{V}$ and $L^* : \mathcal{V} \to \widetilde{\mathcal{V}}$ with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad L^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we can verify $LL^* = \mathrm{Id}_{\mathcal{V}}$ as well as

$$\Big\langle L\Psi, \Phi \Big\rangle_{\mathcal{V}} = \Big\langle \Psi, L^*\Phi \Big\rangle_{\widetilde{\mathcal{V}}}$$

for $\Psi \in \widetilde{\mathcal{V}}$ and $\Phi \in \mathcal{V}$. On $\widetilde{\mathcal{V}}$ there is some kind of Clifford action given by

$$\tau(X) := \begin{pmatrix} 0 & \gamma_+^-(X) & 0 & 0 \\ \gamma_-^+(X) & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_+^+(X) \\ 0 & 0 & \gamma_-^-(X) & 0 \end{pmatrix}$$

where the adjoint w.r.t. $\langle .,. \rangle_{\widetilde{\mathcal{V}}}$ is

$$\tau^*(X) := \begin{pmatrix} 0 & -\gamma_+^-(X) & 0 & 0 \\ -\gamma_-^+(X) & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_+^+(X) \\ 0 & 0 & \gamma_-^-(X) & 0 \end{pmatrix}.$$

The Dirac operator $\mathcal{D}$ on $\widetilde{\mathcal{V}}$ is defined by $\sum \tau(e_j)\nabla_{e_j}$, while the formal L^2–adjoint satisfies $-\sum \tau^*(e_j)\nabla_{e_j}$. The non–vanishing second order Dirac terms of a quaternionic Killing spinor $\Psi = \psi_0 + \psi_1 + \psi_-$ are $\mathcal{P}_+^-\mathcal{P}_-^+\psi_0$, $\mathcal{P}_-^+\mathcal{P}_+^-\psi_1$, $\mathcal{P}_+^+\mathcal{P}_-^-\psi_1$ and $\mathcal{P}_-^-\mathcal{P}_+^+\psi_-$. Moreover, compare with the proof of proposition 3.2, $-\mathcal{P}_+^+\mathcal{P}_-^-$ on $\mathcal{S}_1$ is related to $2\mathcal{P}_-^+\mathcal{P}_+^- \sim 2\nabla^*\nabla$ and $-\mathcal{P}_-^-\mathcal{P}_+^+$ on $\mathcal{S}_-$ to $2\nabla^*\nabla$. This leads to a second definition of inner products on $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ with the property

$$\Big(\nabla^*\nabla\Psi, \Phi\Big)_{\mathcal{V}} \sim \Big\langle \mathcal{D}^*\mathcal{D}L^*\Psi, L^*\Phi\Big\rangle_{\widetilde{\mathcal{V}}}.$$

Using the fact

$$\mathcal{D}^*\mathcal{D} = \mathrm{diag}\Big[\mathcal{P}_+^-\mathcal{P}_-^+\ ,\ \mathcal{P}_-^+\mathcal{P}_+^-\ ,\ -\mathcal{P}_+^+\mathcal{P}_-^-\ ,\ -\mathcal{P}_-^-\mathcal{P}_+^+\Big]$$

and the definition of L^*, we have

$$\begin{aligned} \Big([\psi_0,\psi_1,\psi_-],[\phi_0,\phi_1,\phi_-]\Big)_{\mathcal{V}} &= \langle\psi_0,\phi_0\rangle_0 + \frac{3}{2}\langle\psi_1,\phi_1\rangle_1 + 2\langle\psi_-,\phi_-\rangle_- \\ \Big([\psi_0,\psi_1,\tilde\psi_1,\psi_-],[\phi_0,\phi_1,\tilde\phi_1,\phi_-]\Big)_{\widetilde{\mathcal{V}}} &= \langle\psi_0,\phi_0\rangle_0 + \frac{3}{2}\langle\psi_1,\phi_1\rangle_1 + \frac{3}{2}\langle\tilde\psi_1,\tilde\phi_1\rangle_1 + \\ &\quad +2\langle\psi_-,\phi_-\rangle_-\,. \end{aligned}$$

In particular L^* is the adjoint of L with respect to these metrics. We consider the following Killing structure on $\widetilde{\mathcal{V}}$:

$$\mathfrak{T}_X := \lambda\sqrt{2}\begin{pmatrix} 0 & \frac{\alpha}{n+3}\gamma_+^-(X) & 0 & 0 \\ \frac{1}{4n\alpha}\gamma_-^+(X) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2(n+3)\beta}\gamma_+^+(X) \\ 0 & 0 & \frac{\beta}{4n}\gamma_-^-(X) & 0 \end{pmatrix}$$

where $\lambda \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$. This leads to

$$L\mathfrak{T}_X L^* = \lambda\begin{pmatrix} 0 & \frac{\alpha}{n+3}\gamma_+^-(X) & 0 \\ \frac{1}{4n\alpha}\gamma_-^+(X) & 0 & -\frac{3}{2(n+3)\beta}\gamma_+^+(X) \\ 0 & \frac{\beta}{4n}\gamma_-^-(X) & 0 \end{pmatrix}$$

and in the case $\alpha = \beta = 1$, $L\mathfrak{T}L^*$ is the quaternionic Killing structure $\mathfrak{A}$ defined above. If $\Psi = \psi_0+\psi_1+\psi_-$ is a quaternionic Killing spinor, the spinor $\Phi := \alpha\psi_0+\psi_1+\beta\psi_- \in \Gamma(\mathcal{V})$ is parallel w.r.t. $\nabla + L\mathfrak{T}L^*$. Note that $L\nabla_X L^* = \nabla_X$ holds on $\mathcal{V}$. The adjoint of $\mathfrak{T}_X$ w.r.t. $(.,.)_{\widetilde{\mathcal{V}}}$ is given by

$$\mathfrak{T}_X^* = \overline{\lambda}\sqrt{2}\begin{pmatrix} 0 & -\frac{3}{8n\alpha}\gamma_+^-(X) & 0 & 0 \\ -\frac{2\alpha}{3(n+3)}\gamma_-^+(X) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta}{3n}\gamma_+^+(X) \\ 0 & 0 & -\frac{9}{8(n+3)\beta}\gamma_-^-(X) & 0 \end{pmatrix}.$$

Let $\widehat{\nabla} = \nabla + L\mathfrak{T}L^*$ be the modified quaternionic Killing connection on $\mathcal{V}$. Since ∇ is a Riemannian connection and $L\mathfrak{T}L^*$ is parallel, we have for any compact $N \subset M$

with boundary ∂N:

$$\begin{aligned}(3.1)\quad \int\limits_N \Big(\widehat{\nabla}\Psi,\widehat{\nabla}\Phi\Big)_{\mathcal{V}} = \int\limits_{\partial N}\Big(\widehat{\nabla}_\nu\Psi,\Phi\Big)_{\mathcal{V}} + \int\limits_N \Big(\nabla^*\nabla\Psi,\Phi\Big)_{\mathcal{V}} + \Big(L\mathfrak{T}L^*\Psi, L\mathfrak{T}L^*\Phi\Big)_{\mathcal{V}} + \\ + \int\limits_N \sum_j \Big((\mathfrak{T}^\star_{e_j}L^* - \mathfrak{T}_{e_j}L^*)\nabla_{e_j}\Psi, L^*\Phi\Big)_{\widetilde{\mathcal{V}}},\end{aligned}$$

where integration is done with respect to the volume form and $e_1,\dots,e_m$ is an orthonormal frame. Denote by $\widehat{\mathcal{D}} := \mathcal{D} + \mathcal{T}$ the Dirac operator on $\widetilde{\mathcal{V}}$ with the property that $\widehat{\mathcal{D}}L^*\Psi$ vanishes if Ψ is parallel with respect to $\widehat{\nabla}$. Compare [11, thm. 4.2], $\mathcal{T}$ must be given by

$$\mathcal{T} := -\lambda\cdot\mathrm{diag}\left[\frac{1}{\sqrt{2}\alpha}\ ,\ \sqrt{2}\alpha\ ,\ -\sqrt{2}\frac{(n-1)\beta}{2n}\ ,\ \sqrt{8}\frac{n+4}{(n+3)\beta}\right].$$

Again if $N\subset M$ is a compact manifold with boundary ∂N, we obtain

$$\begin{aligned}(3.2)\quad \int\limits_N \Big\langle\widehat{\mathcal{D}}L^*\Psi,\widehat{\mathcal{D}}L^*\Phi\Big\rangle_{\widetilde{\mathcal{V}}} = \int\limits_{\partial N}\Big\langle\tau^*(\nu)\widehat{\mathcal{D}}L^*\Psi, L^*\Phi\Big\rangle_{\widetilde{\mathcal{V}}} + \int\limits_N \Big\langle\mathcal{T}L^*\Psi,\mathcal{T}L^*\Phi\Big\rangle_{\widetilde{\mathcal{V}}} + \\ + \int\limits_N\sum_j\Big\langle(\mathcal{T}^*\tau(e_j) - \tau^*(e_j)\mathcal{T})\nabla_{e_j}L^*\Psi, L^*\Phi\Big\rangle_{\widetilde{\mathcal{V}}} \\ - \int\limits_N\sum_j\Big\langle\tau^*(e_j)\nabla_{e_j}\mathcal{D}L^*\Psi, L^*\Phi\Big\rangle_{\widetilde{\mathcal{V}}}.\end{aligned}$$

The first order terms of the last two equations are the problematic one, so that we have to search for α and β eliminating these terms in the difference (3.1)-(3.2).

Lemma 3.1. *If λ is purely imaginary and the quaternionic dimension n is >2, there is a choice of α and β (depending on n) such that $\mathfrak{T}$ satisfies on $\widetilde{\mathcal{V}}$:*

$$(3.3)\qquad \Big((\mathfrak{T}^\star_X - \mathfrak{T}_X)\Psi,\Phi\Big) = \Big\langle(\mathcal{T}^*\tau(X) - \tau^*(X)\mathcal{T})\Psi,\Phi\Big\rangle.$$

Proof. Since λ is purely imaginary, we have $\mathcal{T}^* = -\mathcal{T}$. In particular using the definition of the inner products and computing the lhs respectively rhs of (3.3), we conclude the equivalence of (3.3) with

$$\sqrt{2}\lambda\begin{bmatrix}(\frac{3}{8n\alpha} - \frac{\alpha}{n+3})\gamma_+^-(X)\psi_1 \\ \frac{3}{2}(\frac{2\alpha}{3(n+3)} - \frac{1}{4n\alpha})\gamma_-^+(X)\psi_0 \\ \frac{3}{2}(\frac{3}{2(n+3)\beta} - \frac{\beta}{3n})\gamma_+^+(X)\psi_- \\ 2(\frac{9}{8(n+3)\beta} - \frac{\beta}{4n})\gamma_-^-(X)\tilde{\psi}_1\end{bmatrix} = \frac{\lambda}{\sqrt{2}}\begin{bmatrix}(\frac{1}{\alpha} - 2\alpha)\gamma_+^-(X)\psi_1 \\ (2\alpha - \frac{1}{\alpha})\gamma_-^+(X)\psi_0 \\ (4\frac{n+4}{(n+3)\beta} - \frac{(n-1)\beta}{n})\gamma_+^+(X)\psi_- \\ (4\frac{n+4}{(n+3)\beta} - \frac{(n-1)\beta}{n})\gamma_-^-(X)\tilde{\psi}_1\end{bmatrix}.$$

This supplies the equivalence of the the first and the second row as well as the equivalence of the third and fourth row. Moreover,

$$\frac{3}{8n\alpha} - \frac{\alpha}{n+3} = \frac{1}{2\alpha} - \alpha$$

implies $\alpha^2 = \frac{n+3}{n+2}(\frac{1}{2} - \frac{3}{8n})$ and

$$\frac{9}{4(n+3)\beta} - \frac{\beta}{2n} = \frac{1}{2}\left(4\frac{n+4}{(n+3)\beta} - \frac{(n-1)\beta}{n}\right)$$

reduces to $\beta^2 = \frac{n}{(n-2)(n+3)}(4(n+4) - \frac{9}{2})$ if $n>2$. $\square$

This important property of $\mathfrak{T}$ leads to the following integrated Bochner–Weitzenböck formula.

Proposition 3.2. *Let (M^{4n}, g), $n > 2$, be a quaternionic Kähler spin manifold of negative scalar curvature and $N \subset M$ be a compact manifold with boundary ∂N. Choose α, β like in the last lemma and $\lambda^2 = \frac{\mathrm{scal}}{4}\frac{n+3}{n+2}$, then the Bochner–Weitzenböck formula*

$$\int_{\partial N} \Big(\widehat{\nabla}_\nu \Psi, \Phi\Big) - \Big\langle \tau^*(\nu)\widehat{\mathfrak{D}}L^*\Psi, L^*\Phi\Big\rangle - \Big\langle \zeta(\nu)\mathscr{D}\Psi, \Phi\Big\rangle =$$

$$= \int_N \Big(\widehat{\nabla}\Psi, \widehat{\nabla}\Phi\Big) - \Big\langle \widehat{\mathfrak{D}}L^*\Psi, \widehat{\mathfrak{D}}L^*\Phi\Big\rangle - \Big\langle \mathscr{D}\Psi, \mathscr{D}\Phi\Big\rangle$$

holds for all $\Psi, \Phi \in \Gamma(\mathcal{V})$ with $\mathscr{D}([\psi_0, \psi_1, \psi_-]) = \sqrt{\frac{7}{6}}\slashed{D}_-^+\psi_1 + \sqrt{\frac{2}{3}}\slashed{D}_+^+\psi_1 + \frac{2}{\sqrt{3}}\slashed{D}_-^+\psi_-$ and $\zeta(X) : \slashed{S}_2 \oplus \slashed{S}^{2,n} \oplus \slashed{S}^{1,n-3} \to \slashed{S}_0 \oplus \slashed{S}_1 \oplus \slashed{S}_-$,

$$\zeta(X) := \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{\frac{7}{6}}\gamma_+^-(X) & \sqrt{\frac{2}{3}}\gamma_-^-(X) & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}}\gamma_+^-(X) \end{pmatrix}.$$

In particular the boundary operator is selfadjoint.

Proof. A straightforward calculation yields

$$\Big(L\mathfrak{T}L^*\Psi, L\mathfrak{T}L^*\Phi\Big) = |\lambda|^2 \Bigg[\frac{3}{8n\alpha^2}\langle\psi_0, \phi_0\rangle + \frac{\alpha^2}{n+3}\langle\psi_1, \phi_1\rangle + \\ + \frac{(n-1)\beta^2}{4n^2}\langle\psi_1, \phi_1\rangle + 9\frac{n+4}{(n+3)^2\beta^2}\langle\psi_-, \phi_-\rangle\Bigg]$$

as well as

$$L\mathcal{T}^*\mathcal{T}L^* = |\lambda|^2 \ \mathrm{diag}\left[\frac{1}{2\alpha^2} \ , \ \alpha^2 + \frac{(n-1)^2\beta^2}{4n^2} \ , \ \frac{(n+4)^2}{(n+3)^2\beta^2}\right].$$

Using α and β from the proof of the last lemma, we obtain

$$\Big(L\mathfrak{T}L^*\Psi, L\mathfrak{T}L^*\Phi\Big) - \Big\langle \mathcal{T}L^*\Psi, \mathcal{T}L^*\Phi\Big\rangle = -|\lambda|^2\frac{n+2}{n+3}\langle\psi_0, \phi_0\rangle + \\ + |\lambda|^2\frac{16 - 6n(n+2)}{4n(n+3)}\langle\psi_1, \phi_1\rangle - 2|\lambda|^2\frac{(n-2)(n+4)}{n(n+3)}\langle\psi_-, \phi_-\rangle. \tag{3.4}$$

The Weitzenböck formula in [11, section 3] supplies on $\slashed{S}_-$ (multiplication from the left with the row vector $[-2, 0, 2, 0, 0, 0]$):

$$2\nabla^*\nabla\psi_- + \slashed{D}_-^-\slashed{D}_+^+\psi_- = \frac{4}{3}\slashed{D}_+^-\slashed{D}_-^+\psi_- - 2\frac{(n-2)(n+4)}{n(n+2)}\frac{\mathrm{scal}}{4}\psi_-, \tag{3.5}$$

and formula 5.25 in [12] reduces on $\slashed{S}_1$ to

$$\frac{3}{n(n+2)}\frac{\mathrm{scal}}{4}\psi_1 = \frac{1}{2}\slashed{D}_-^-\slashed{D}_+^+\psi_1 + \frac{3}{4}\slashed{D}_-^+\slashed{D}_+^-\psi_1 + \frac{1}{4}\slashed{D}_+^-\slashed{D}_-^+\psi_1 + \frac{3}{8}\slashed{D}_+^+\slashed{D}_-^-\psi_1. \tag{3.6}$$

Computing (3.1)$-$(3.2), we conclude from $\frac{\mathrm{scal}}{4} = -\frac{n+2}{n+3}|\lambda|^2$ as well as the help of lemma 3.1, equations (3.4),(3.5),(3.6) and the Lichnerowicz formula on $\slashed{S}_0, \slashed{S}_1$:

$$\int_N \Big(\widehat{\nabla}\Psi, \widehat{\nabla}\Phi\Big) - \Big\langle \widehat{\mathfrak{D}}L^*\Psi, \widehat{\mathfrak{D}}L^*\Phi\Big\rangle = \int_{\partial N}\Big(\widehat{\nabla}_\nu\Psi, \Phi\Big) - \Big\langle \tau^*(\nu)\widehat{\mathfrak{D}}L^*\Psi, L^*\Phi\Big\rangle \\ + \int_N \frac{7}{6}\Big\langle \slashed{D}_+^-\slashed{D}_-^+\psi_1, \phi_1\Big\rangle - \frac{2}{3}\Big\langle \slashed{D}_-^-\slashed{D}_+^+\psi_1, \phi_1\Big\rangle + \frac{4}{3}\Big\langle \slashed{D}_+^-\slashed{D}_-^+\psi_-, \phi_-\Big\rangle, \tag{3.7}$$

where the facts $\not{\mathcal{D}}^2 = \not{\mathcal{D}}_+^- \not{\mathcal{D}}_-^+$ on $\not{S}_0$ and

$$-\sum_j \Big\langle \tau^*(e_j)\nabla_{e_j}\mathfrak{D}L^*\Psi, L^*\Phi\Big\rangle = \Big\langle \not{\mathcal{D}}_+^-\not{\mathcal{D}}_-^+\psi_0, \phi_0\Big\rangle + \frac{1}{2}\Big\langle \not{\mathcal{D}}_-^+\not{\mathcal{D}}_+^-\psi_1, \phi_1\Big\rangle$$
$$- \frac{1}{2}\Big\langle \not{\mathcal{D}}_+^+\not{\mathcal{D}}_-^-\psi_1, \phi_1\Big\rangle - \Big\langle \not{\mathcal{D}}_-^-\not{\mathcal{D}}_+^+\psi_-, \phi_-\Big\rangle$$

are used. The last line of equation (3.7) equals

$$\int_N \Big\langle \mathscr{D}\Psi, \mathscr{D}\Phi\Big\rangle - \int_{\partial N} \Big\langle \zeta(\nu)\mathscr{D}\Psi, \Phi\Big\rangle$$

which supplies the claim. The selfadjointness of the boundary operator follows immediately from the symmetry of the right hand side. $\square$

Remark 3.3. If $\Psi \in \Gamma(\mathcal{V})$ is parallel with respect to $\widehat{\nabla}$, i.e. $\frac{1}{\alpha}\psi_0 + \psi_1 + \frac{1}{\beta}\psi_-$ is a quaternionic Killing spinor, then $\widehat{\mathfrak{D}}L^*\Psi$ and $\mathscr{D}\Psi$ vanish.

4. Coercivity of the quaternionic hyperbolic Dirac operator

In order to obtain theorem 1.2, we have to prove an isomorphism property of the quaternionic hyperbolic Dirac operator $\widehat{\mathfrak{D}}L^* + \mathscr{D}$. This leads to the following analytic preliminaries. If $\mathcal{E} \to M^n$ is a Riemannian vector bundle and ω is a positively oriented nowhere vanishing n–form, $L^2(M, \mathcal{E}; \omega)$ denotes the completion of compact supported sections of $\mathcal{E}$ with respect to the L^2 norm

$$\|\varphi\|_\omega = \sqrt{\int_M \langle \varphi, \varphi\rangle\, \omega}\, .$$

Suppose that $\mathcal{E}$ is equipped with a Riemannian connection ∇, and consider the scalar product

$$(\varphi, \psi)_{1,\omega} := \int_M \Big(\langle \nabla\varphi, \nabla\psi\rangle + \langle \varphi, \psi\rangle \Big)\omega$$

on $\Gamma_{\mathrm{cpt}}(\mathcal{E})$. The Sobolev space $W^{1,2}(M, \mathcal{E}; \omega)$ is the closure of $\Gamma_{\mathrm{cpt}}(\mathcal{E})$ in $L^2(M, \mathcal{E}; \omega)$ with respect to the norm $\|.\|_{1,\omega} := \sqrt{(.,.)_{1,\omega}}$. Moreover, $W^{1,2}(M, \mathcal{E}; \omega)$ becomes a Hilbert space with respect to the extension of the product $(.,.)_{1,\omega}$. For notational simplicity set $L^2(M; \omega) := L^2(M, C^\infty(M); \omega)$, $W^{1,2}(M, \mathcal{E}) := W^{1,2}(M, \mathcal{E}; \mathrm{vol}_g)$ as well as $L^2(M, \mathcal{E}) := L^2(M, \mathcal{E}; \mathrm{vol}_g)$ where vol_g denotes the volume form of (M, g).

Proposition 4.1. *Suppose (M^{4n}, g), $n > 2$, is a complete quaternionic Kähler spin manifold of negative scalar curvature. Choose $\lambda^2 = \frac{\mathrm{scal}}{4}\frac{n+3}{n+2} \neq 0$ and α, β like in lemma 3.1, then for any $\Psi \in \Gamma(\mathcal{V})$ with*

$$\widehat{\mathfrak{D}}L^*\Psi \in L^2(M, \widetilde{\mathcal{V}}) \quad \textit{and} \quad \mathscr{D}\Psi \in L^2(M, \not{S}_2 \oplus \not{S}^{2,n} \oplus \not{S}^{1,n-3}),$$

there is a unique $\Phi \in W^{1,2}(M, \mathcal{V}) \cap \Gamma(\mathcal{V})$ satisfying $\widehat{\mathfrak{D}}L^*(\Psi - \Phi) = 0$ as well as $\mathscr{D}(\Psi - \Phi) = 0$.

Proof. Set

$$\mathcal{T}^\dagger := \lambda\ \mathrm{diag}\left[\sqrt{2}\alpha\ ,\ \frac{1}{\sqrt{2}\alpha}\ ,\ -\sqrt{8}\frac{n+4}{(n+3)\beta}\ ,\ \sqrt{2}\frac{(n-1)\beta}{2n}\right] \in \mathrm{End}(\widetilde{\mathcal{V}}),$$

then $\mathcal{T}^\dagger\tau(X) = \tau^*(X)\mathcal{T}$ holds and

$$(\mathfrak{D}^* + \mathcal{T}^\dagger)(\mathfrak{D} + \mathcal{T}) = \mathfrak{D}^*\mathfrak{D} + \mathcal{T}^\dagger\mathcal{T}$$

can be verified. First we will show that

$$L(\mathfrak{D}^* + \mathcal{T}^\dagger)\widehat{\mathfrak{D}}L^* + \mathscr{D}^*\mathscr{D} : W^{1,2}(M, \mathcal{V}) \to W^{-1,2}(M, \mathcal{V})$$

is an isomorphism of Hilbert spaces. Consider the sesquilinear form

$$B(\Psi,\Phi) := \int\limits_M \left\langle (\mathfrak{D}^* + \mathcal{T}^\dagger)\widehat{\mathfrak{D}}L^*\Psi, L^*\Phi\right\rangle + \left\langle \mathscr{D}^*\mathscr{D}\Psi,\Phi\right\rangle$$

on compact supported sections of $\mathcal{V}$. Since

$$\mathcal{T}^\dagger\mathcal{T} = |\lambda|^2 \ \mathrm{diag}\left[1\ ,\ 1\ ,\ 2\frac{(n+4)(n-1)}{n(n+3)}\ ,\ 2\frac{(n+4)(n-1)}{n(n+3)}\right],$$

we obtain from equations (3.5), (3.6) and the Lichnerowicz formula

$$\begin{aligned}
B(\Psi,\Phi) &= \int\limits_M \left\langle \mathcal{D}_+^-\mathcal{D}_-^+\psi_0,\phi_0\right\rangle + \frac{1}{2}\left\langle \mathcal{D}_-^+\mathcal{D}_+^-\psi_1,\phi_1\right\rangle - \frac{1}{2}\left\langle \mathcal{D}_+^+\mathcal{D}_-^-\psi_1,\phi_1\right\rangle \\
&\qquad - \left\langle \mathcal{D}_-^-\mathcal{D}_+^+\psi_-,\phi_-\right\rangle + \frac{7}{6}\left\langle \mathcal{D}_+^-\mathcal{D}_-^+\psi_1,\phi_1\right\rangle - \frac{2}{3}\left\langle \mathcal{D}_-^-\mathcal{D}_+^+\psi_1,\phi_1\right\rangle \\
&\qquad + \frac{4}{3}\left\langle \mathcal{D}_+^-\mathcal{D}_-^+\psi_-,\phi_-\right\rangle + \left\langle \mathcal{T}^\dagger\mathcal{T}L^*\Psi, L^*\Phi\right\rangle \\
&= \int\limits_M \langle\nabla^*\nabla\psi_0,\phi_0\rangle + \frac{3}{2}\langle\nabla^*\nabla\psi_1,\phi_1\rangle + 2\langle\nabla^*\nabla\psi_-,\phi_-\rangle + \\
&\qquad + \frac{|\lambda|^2}{n+3}\left[\langle\psi_0,\phi_0\rangle + \frac{3}{2}\langle\psi_1,\phi_1\rangle + 2\frac{n+4}{n}\langle\psi_-,\phi_-\rangle\right].
\end{aligned}$$

Thus, B can be extended to a bounded and coercive sesquilinear form on $W^{1,2}(M,\mathcal{V})$. Suppose $\Psi \in \Gamma(\mathcal{V})$ is a spinor with $\widehat{\mathfrak{D}}L^*\Psi \in L^2$ and $\mathscr{D}\Psi \in L^2$, then

$$l(\Theta) := \int\limits_M \left\langle L(\mathfrak{D} + \mathcal{T}^\dagger)\widehat{\mathfrak{D}}L^*\Psi,\Theta\right\rangle + \left\langle \mathscr{D}^*\mathscr{D}\Psi,\Theta\right\rangle$$

is a bounded linear functional on $W^{1,2}(M,\mathcal{V})$. The Lax–Milgram theorem (cf. [5, Ch. 5.8]) yields some $\Phi \in W^{1,2}(M,\mathcal{V})$ with $B(\Phi,\Theta) = l(\Theta)$ for all $\Theta \in W^{1,2}(M,\mathcal{V})$. Elliptic theory implies smoothness of Φ and

$$L(\mathfrak{D} + \mathcal{T}^\dagger)\widehat{\mathfrak{D}}L^*(\Psi - \Phi) + \mathscr{D}^*\mathscr{D}(\Psi - \Phi) = 0. \tag{4.1}$$

Set $\Theta := \Psi - \Phi$, then it remains to show $\widehat{\mathfrak{D}}L^*\Theta = 0$ and $\mathscr{D}\Theta = 0$. We conclude from $\widehat{\mathfrak{D}}L^*\Theta \in L^2$, $\mathscr{D}\Theta \in L^2$ and (4.1) that $\mathfrak{D}\widehat{\mathfrak{D}}L^*\Theta \in L^2$ as well as $\mathscr{D}^*\mathscr{D}\Theta \in L^2$ hold. In particular the Gromov–Lawson argument [6, thm. 2.8] implies $\widehat{\mathfrak{D}}L^*\Theta \in W^{1,2}$ as well as $\mathscr{D}\Theta \in W^{1,2}$. Consider the $\mathcal{S}_1$ part of (4.1):

$$\begin{aligned}
&\left(\frac{1}{2}\mathcal{D}_-^+\mathcal{D}_+^- - \frac{1}{2}\mathcal{D}_+^+\mathcal{D}_-^- + \frac{1}{2}|\lambda|^2\left[1 + 2\frac{(n+4)(n-1)}{n(n+3)}\right] + \right. \\
&\qquad\qquad \left. + \frac{7}{6}\mathcal{D}_+^-\mathcal{D}_-^+ - \frac{2}{3}\mathcal{D}_-^-\mathcal{D}_+^+\right)\Theta_1 = 0,
\end{aligned} \tag{4.2}$$

and replace $\mathcal{D}_+^+\mathcal{D}_-^-$ with the help of (3.6). This eliminates the $\mathcal{D}_-^-\mathcal{D}_+^+$ term and we obtain

$$\mathcal{D}^2\Theta_1 = \left(\mathcal{D}_-^+\mathcal{D}_+^- + \mathcal{D}_+^-\mathcal{D}_-^+\right)\Theta_1 = -|\lambda|^2\Theta_1.$$

Consider the bundle $\mathcal{S}_0 \oplus \mathcal{S}_1 \oplus \mathcal{S}_2$ and

$$\mathcal{T}_1 = -\lambda\ \mathrm{diag}\left[\frac{1}{\sqrt{2}\alpha}\ ,\ \sqrt{2}\alpha\ ,\ 0\right], \qquad \mathcal{T}_1^\dagger = \lambda\ \mathrm{diag}\left[\sqrt{2}\alpha\ ,\ \frac{1}{\sqrt{2}\alpha}\ ,\ \sqrt{2}\alpha\right].$$

Since $\gamma(X)\mathcal{T}_1 = -\mathcal{T}_1^\dagger\gamma(X)_{|S_0\oplus S_1}$ and $\overline{\Theta} = \Theta_0 + \frac{1}{\sqrt{2}}\Theta_1$ lead to

$$(\mathcal{D} + \mathcal{T}_1^\dagger)(\mathcal{D} + \mathcal{T}_1)\overline{\Theta} = (\mathcal{D}^2 + \mathcal{T}_1^\dagger\mathcal{T}_1)\overline{\Theta} = 0,$$

we conclude by defining $\mathcal{T}_1^\ddagger = |\lambda|^2(\mathcal{T}_1^\dagger)^{-1}$

$$(\not{\mathcal{D}}^2 + \mathcal{T}_1^\ddagger\mathcal{T}_1^\dagger)(\not{\mathcal{D}} + \mathcal{T}_1)\overline{\Theta} = (\not{\mathcal{D}} + \mathcal{T}_1^\ddagger)(\not{\mathcal{D}} + \mathcal{T}_1^\dagger)(\not{\mathcal{D}} + \mathcal{T}_1)\overline{\Theta} = 0.$$

Integrating over M, using the selfadjointness of the Dirac operator as well as the positive definiteness of $\mathcal{T}_1^\ddagger\mathcal{T}_1^\dagger$ supplies with the fact $(\not{\mathcal{D}} + \mathcal{T}_1)\overline{\Theta} \in W^{1,2}$:

$$(\not{\mathcal{D}} + \mathcal{T}_1)\overline{\Theta} = 0.$$

Thus, $\not{\mathcal{D}}_-^+\Theta_1 = 0$ reduces equation (4.2) to

$$\left(-\not{\mathcal{D}}_+^+\not{\mathcal{D}}_-^- - \frac{4}{3}\not{\mathcal{D}}_-^-\not{\mathcal{D}}_+^+ + 2|\lambda|^2\frac{(n+4)(n-1)}{n(n+3)}\right)\Theta_1 = 0. \tag{4.3}$$

Consider the bundle $\not{S}_1 \oplus \not{S}_- \oplus \not{S}^{2,n} \oplus \not{S}^{1,n-3}$ with the Dirac operator

$$\not{D} = \begin{pmatrix} 0 & \not{\mathcal{D}}_+^+ & \frac{2}{\sqrt{3}}\not{\mathcal{D}}_-^- & 0 \\ \not{\mathcal{D}}_-^- & 0 & 0 & -\frac{2}{\sqrt{3}}\not{\mathcal{D}}_+^- \\ \frac{2}{\sqrt{3}}\not{\mathcal{D}}_+^+ & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}}\not{\mathcal{D}}_-^+ & 0 & 0 \end{pmatrix}.$$

The formal L^2–adjoint of $\not{D}$ is $\not{D}^* = -\not{D}$ and with

$$\begin{aligned} \mathcal{T}_2 &= -\lambda\ \mathrm{diag}\left[-\sqrt{2}\frac{(n-1)\beta}{2n}\ ,\ \sqrt{8}\frac{n+4}{(n+3)\beta}\ ,\ 0\ ,\ 0\right] \\ \mathcal{T}_2^\dagger &= +\lambda\ \mathrm{diag}\left[-\sqrt{8}\frac{n+4}{(n+3)\beta}\ ,\ \sqrt{2}\frac{(n-1)\beta}{2n}\ ,\ \sqrt{2}\frac{(n-1)\beta}{2n}\ ,\ -\sqrt{8}\frac{n+4}{(n+3)\beta}\right] \end{aligned}$$

equations (4.1) and (4.3) supply for $\widetilde{\Theta} = \frac{1}{\sqrt{2}}\Theta_1 + \Theta_-$

$$(\not{D}^* + \mathcal{T}_2^\dagger)(\not{D} + \mathcal{T}_2)\widetilde{\Theta} = (\not{D}^*\not{D} + \mathcal{T}_2^\dagger\mathcal{T}_2)\widetilde{\Theta} = 0.$$

Since there is $\mathcal{T}_2^\ddagger$ with $\not{D}\mathcal{T}_2^\dagger + \mathcal{T}_2^\ddagger\not{D}^* = 0$, we conclude

$$(\not{D}\not{D}^* + \mathcal{T}_2^\ddagger\mathcal{T}_2^\dagger)(\not{D} + \mathcal{T}_2)\widetilde{\Theta} = (\not{D} + \mathcal{T}_2^\ddagger)(\not{D}^* + \mathcal{T}_2^\dagger)(\not{D} + \mathcal{T}_2)\widetilde{\Theta} = 0.$$

Thus, the same argument as above and $(\not{D} + \mathcal{T}_2)\widetilde{\Theta} \in W^{1,2}$ imply

$$(\not{D} + \mathcal{T}_2)\widetilde{\Theta} = 0.$$

But this yields the claim: $\mathscr{D}\Theta = 0$ as well as $\widehat{\mathfrak{D}}L^*\Theta = 0$. □

5. The proof

First we need a discussion about the asymptotic behavior of quaternionic Killing spinors on $\mathbb{H}\mathrm{H}^n$. The asymptotics do not depend on α respectively β and for any choice of α, β the bundle $\mathcal{V}$ is trivialized by spinors parallel with respect to $\widehat{\nabla}$. Thus, we can choose α and β in such a way that $L\mathfrak{T}_XL^*$ is a Hermitian endomorphism of $\mathcal{V}$ with respect to $\langle.,.\rangle$. Therefore, up to the sign α equals $\frac{1}{2}\sqrt{\frac{n+3}{n}}$ and β equals $\sqrt{\frac{6n}{n+3}}$. Suppose X is a vector field of length $|X| = 1$ and Ψ is parallel w.r.t. $\widehat{\nabla}$, we obtain

$$|X\langle\Psi,\Psi\rangle| = 2|\langle L\mathfrak{T}_XL^*\Psi,\Psi\rangle| \leq \kappa|\Psi|^2,$$

where $\frac{\kappa}{2}$ denotes the largest absolute eigenvalue of $L\mathfrak{T}_XL^*$. Integration along a geodesic yields

$$|\Psi|^2 \leq Ce^{\kappa r}$$

for some constant C if r is the geodesic distance to a fixed point. Since $L\mathfrak{T}_XL^*$ is Hermitian, the maximal eigenvalue of $B_X = (L\mathfrak{T}_XL^*)^2$ is $\frac{\kappa^2}{4}$. A straightforward calculation shows that B_X is a diagonal matrix with the first entry $-\frac{\lambda^2}{4n(n+3)}|X|^2$. Since the $\not{S}_0$ component of a quaternionic Killing spinor Ψ can not vanish on an open subset, continuous reasons together with $\lambda^2 = \frac{\mathrm{scal}_0}{4}\frac{n+3}{n+2}$ supply the following estimate for κ^2

$$0 < \kappa^2 \leq -\frac{\mathrm{scal}_0}{4n(n+2)}.$$

The sectional curvature of $\mathbb{H}\mathrm{H}^n$ varies between -1 and $-\frac{1}{4}$, i.e. $\mathrm{scal}_0 = -4n(n+2)$ implies $\kappa \leq 1$. Moreover, if ψ is a non–trivial quaternionic Killing spinor on $\mathbb{H}\mathrm{H}^n$, ψ is not contained in $W^{1,2}(\mathbb{H}\mathrm{H}^n, \mathcal{V})$ because of the coercivity of the quaternionic hyperbolic Dirac operator.

Remark 5.1. If (M^{4n}, g) is a strongly asymptotically quaternionic hyperbolic manifold, the Kraines form Ω (with respect to g, cf. [10]) is closely related at infinity to the Kraines form Ω_0 of g_0:

$$A(\Omega_0) - \Omega \in L^1(E, \Lambda^4(T^*M); e^r\mathrm{vol}_g) \cap L^2(E, \Lambda^4(T^*M); e^r\mathrm{vol}_g),$$

where A extends to Λ^*M in such a way that $A_{|\Lambda^0} = \mathrm{Id}$ and $A(X^* \wedge \eta) = A(X)^* \wedge A(\eta)$.

Proof. Let $\omega_1, \omega_2, \omega_3$ as well as $\omega_1^0, \omega_2^0, \omega_3^0$ be the two–forms associated to I, J, K and I_0, J_0, K_0 respectively. The Kraines forms are given by

$$\begin{aligned} \Omega &= \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3 \\ \Omega_0 &= \omega_1^0 \wedge \omega_1^0 + \omega_2^0 \wedge \omega_2^0 + \omega_3^0 \wedge \omega_3^0. \end{aligned}$$

Choose an orthonormal basis $e_1, \ldots, e_{2n}, I_0e_1, \ldots, I_0e_{2n}$ of T_pM for g_0, then ω_1^0 is given by $-\sum_{k=1}^{2n} e_k \wedge I_0e_k$ and since $Ae_1, \ldots, Ae_{2n}, I(Ae_1), \ldots, I(Ae_{2n})$ is an orthonormal basis w.r.t. g we have $\omega_1 = -\sum_{k=1}^{2n}(Ae_k) \wedge I(Ae_k)$. In particular

$$A\omega_1^0 - \omega_1 = -\sum_{k=1}^{2n}(Ae_k) \wedge (AI_0 - IA)e_k$$

together with $AI_0 - IA = AI_0 - I + I(\mathrm{Id} - A)$ and the asymptotic assumptions imply $A\omega_1^0 - \omega_1 \in L^1 \cap L^2$ w.r.t. the measure $e^r\mathrm{vol}_g$. Therefore,

$$A(\omega_1^0 \wedge \omega_1^0) - \omega_1 \wedge \omega_1 = (A\omega_1^0 - \omega_1) \wedge A\omega_1^0 + \omega_1 \wedge (A\omega_1^0 - \omega_1)$$

and the fact that the above considerations work for ω_2 as well as ω_3 too, supply the claim. $\square$

Now we are in the position to prove theorem 1.2. The essential points of the rigidity result are done in proposition 3.2 and proposition 4.1. Let (M^{4n}, g) be a strongly asymptotically quaternionic hyperbolic spin manifold of quaternionic dimension $n > 2$, and denote by E the Euclidean end of M as well as by g_0 the quaternionic hyperbolic metric on E. The scalar curvature of g respectively g_0 is constant, i.e. the asymptotic assumptions imply $\mathrm{scal} - \mathrm{scal}_0 = 0$. Since (M, g) is quaternionic Kähler, the spinor bundle admits a parallel and orthogonal decomposition

$$\not{S}M = \not{S}_0 \oplus \cdots \oplus \not{S}_n$$

and $\not{S}^{k,l}$ respectively $\not{S}_{\_}$ exist globally. The last remark implies that the extension of the gauge transformation $A : \not{S}(E, g_0) \to \not{S}(E, g)$ (cf. [1, 14]) respects the splitting of the spinor bundles for the two different metrics g and g_0 up to some $L^1(e^r\mathrm{vol}_g) \cap L^2(e^r\mathrm{vol}_g)$ term [use the fact $A(\gamma_0(\sigma)(\psi)) = (\gamma(A\sigma))A\psi$]. $\not{S}^{2,0} = \mathrm{Sym}^2\mathbf{H}$ can be seen as the rank 3 subbundle of $\mathrm{End}(TM)$ which is generated by I, J, K. In particular A is up to some $L^1 \cap L^2$ value a map $\mathrm{Sym}^2\mathbf{H}(E, g_0) \to \mathrm{Sym}^2\mathbf{H}(E, g)$ which supplies, using the identification $\not{S}_2 = \mathrm{Sym}^2\mathbf{H} \otimes \not{S}_{\_}$ and the restriction of A to $\not{S}_2(E, g_0) \to \not{S}_2(E, g)$, a map $A : \not{S}_{\_}(E, g_0) \to \not{S}_{\_}(E, g)$. Moreover, since the quaternionic Killing structures

$\mathfrak{A}$, $\mathfrak{A}^0$ corresponding to g and g_0 are obtained by a Clifford multiplication and a linear combination of projection maps $\mathrm{pr}_{j,k} : \$^{*,*} \to \$^{j,k}$, we conclude for all bounded X

$$A \circ \mathfrak{A}^0_X - \mathfrak{A}_X A \in L^1(E, \mathrm{End}(\mathcal{V}); e^r \mathrm{vol}_g) \cap L^2(E, \mathrm{End}(\mathcal{V}); e^r \mathrm{vol}_g).$$

Thus, if Ψ is a quaternionic Killing spinor for the quaternionic hyperbolic metric on E, $\widetilde{\Psi} := \alpha(A\Psi)_0 + (A\Psi)_1 + \beta(A\Psi)_-$ satisfies $\widehat{\nabla}\widetilde{\Psi} \in L^2(E, T^*M \otimes \mathcal{V})$ and

$$\left(\widehat{\nabla}_\nu \widetilde{\Psi}, \widetilde{\Psi}\right) - \left\langle \tau^*(\nu)\widehat{\mathfrak{D}} L^* \widetilde{\Psi}, L^* \widetilde{\Psi}\right\rangle - \left\langle \zeta(\nu)\mathscr{D}\widetilde{\Psi}, \widetilde{\Psi}\right\rangle \in L^1(M). \tag{5.1}$$

Proposition 4.1 gives a non–trivial harmonic spinor Φ with $\Phi - \widetilde{\Psi} \in W^{1,2}(M, \mathcal{V})$. In particular proposition 3.2 together with equation (5.1) and a non–degenerate exhaustion implies $\widehat{\nabla}\Phi = 0$. The section $\frac{1}{\alpha}\Phi_0 + \Phi_1 + \frac{1}{\beta}\Phi_-$ becomes a quaternionic Killing spinor for g. In this way we obtain a full set of quaternionic Killing spinors for g, and [11, prop. 4.2] supplies that R^{hyper} vanishes. Therefore, (M, g) is locally isometric to the quaternionic hyperbolic space. Global rigidity follows from the Hadamard–Cartan theorem and the fact $E \cong (0, 1) \times S^{4n-1}$.

References

[1] L. Andersson and M. Dahl. Scalar curvature rigidity for asymptotically locally hyperbolic manifolds. *Ann. Global Anal. Geom.*, 16(1):1–27, 1998.

[2] S. R. Barker and S. M. Salamon. Analysis on a generalized Heisenberg group. *J. London Math. Soc. (2)*, 28(1):184–192, 1983.

[3] R. Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.*, 39(5):661–693, 1986.

[4] A. L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

[5] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[6] M. Gromov and H. B. Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):83–196 (1984), 1983.

[7] M. Herzlich. Scalar curvature and rigidity of odd-dimensional complex hyperbolic spaces. *Math. Ann.*, 312(4):641–657, 1998.

[8] O. Hijazi and J.-L. Milhorat. Twistor operators and eigenvalues of the Dirac operator on compact quaternion-Kähler spin manifolds. *Ann. Global Anal. Geom.*, 15(2):117–131, 1997.

[9] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.

[10] V. Y. Kraines. Topology of quaternionic manifolds. *Trans. Amer. Math. Soc.*, 122:357–367, 1966.

[11] W. Kramer, U. Semmelmann, and G. Weingart. Quaternionic Killing spinors. *Ann. Global Anal. Geom.*, 16(1):63–87, 1998.

[12] W. Kramer, U. Semmelmann, and G. Weingart. Eigenvalue estimates for the Dirac operator on quaternionic Kähler manifolds. *Math. Z.*, 230(4):727–751, 1999.

[13] M. Listing. *Rigidity of Hyperbolic Spaces.* PhD thesis, Uni Leipzig, 2003.

[14] M. Min-Oo. Scalar curvature rigidity of asymptotically hyperbolic spin manifolds. *Math. Ann.*, 285(4):527–539, 1989.

[15] E. Witten. A new proof of the positive energy theorem. *Comm. Math. Phys.*, 80(3):381–402, 1981.

E-mail address: `listing@mathematik.uni-leipzig.de`

MATHEMATICAL INSTITUTE, UNIVERSITY OF LEIPZIG, D-04109 LEIPZIG, GERMANY