SECTIONAL CURVATURE RIGIDITY OF ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. This paper presents a rigidity result of the real hyperbolic space in analogy to M. Min–Oo's result [19], but without assuming that the strongly asymptotically hyperbolic manifold is spin. In order to prove this, we consider special Killing forms on the exterior form bundle. Moreover, we make an assumption on the sectional curvature to obtain the necessary eigenvalue estimates of the curvature endomorphism in the Bochner–Weitzenböck formula of $\Omega^k M$.

1. Introduction

M. Min–Oo proved in [19] that a strongly asymptotically hyperbolic spin manifold (M^n, g) with scalar curvature $\mathrm{scal} \geq -n(n-1)$ must be isometric to the real hyperbolic space $\mathbb{R}H^n$. In order to obtain this result, Min–Oo used Witten's idea for the proof of the positive energy theorem (cf. [4, 17, 22]) as well as the existence of imaginary Killing spinors on $\mathbb{R}H^n$.

In this paper a non–spin rigidity result of $\mathbb{R}H^n$ is given by replacing the scalar curvature bound by a sectional curvature bound.

Definition 1.1. $(\mathbb{R}H^n, g_0)$ denotes the real hyperbolic space of constant sectional curvature -1 and $B_R(q) \subset M$ is the set of all $p \in M$ with geodesic distance to q less than R. (cf. [19]) A connected Riemannian manifold (M^n, g) will be called *strongly asymptotically hyperbolic* if there is a compact set $C \subset M$ and a diffeomorphism $f : M - C \to \mathbb{R}H^n - \overline{B_R(0)}$ in such a way that the positive definite gauge transformation $A : TM_{|M-C} \to TM_{|M-C}$ induced by f and given by

$$g(AX, AY) = (f^* g_0)(X, Y) \qquad g(AX, Y) = g(X, AY)$$

satisfies:

(1) A is uniformally bounded, i.e. there is a constant $c > 0$ with
$$\frac{1}{c}\, |X|_g \leq |AX|_g \leq c\, |X|_g$$
for all $X \in TM_{|M-C}$.

(2) If r is the $f^* g_0$–distance to a fixed point, then
$$\left|(f^*\nabla^0)A\right|_g + |A - \mathrm{Id}|_g \in L^1(M - C; e^{2r}\mathrm{vol}_g) \cap L^2(M - C; e^{2r}\mathrm{vol}_g).$$

Note that this definition differs from the one given in [19] respectively [3], since the objects which are used for the proof of the following theorem grow like e^{2r} and imaginary Killing spinors grow like e^r.

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Theorem 1.2. *A complete, orientable, and strongly asymptotically hyperbolic manifold with sectional curvature $K_g \geq -1$ is isometric to the real hyperbolic space $\mathbb{R}H^n$.*

2. Preliminaries

2.1. **Special Killing forms on the hyperbolic space.** Let $(\mathbb{R}H^n(K), g)$ be the n–dimensional real hyperbolic space of constant sectional curvature $K < 0$. On this space there exists $(n+1)$–linearly independent functions $f_1, \ldots, f_{n+1}$ of the equation

$$\nabla^2 f = -Kf \cdot g\,,$$

where $\nabla^2 f$ is the Hessian of f. Under the canonical isomorphism $* : T\mathbb{R}H^n \to \Lambda^1(\mathbb{R}H^n)$, $X \mapsto g(X, .)$, this is equivalent to

$$\nabla_X(\mathrm{d}f) = -KfX^*\,.$$

In the disc model

$$\Big(D^n, -\frac{1}{K\rho^2}\sum_j \mathrm{d}x_j^2\Big)$$

with $\rho := \frac{1}{2}\big(1 - \sum x_j^2\big)$ these functions can be given as follows $(1 \leq k \leq n)$:

$$f_k := \frac{x^k}{\rho}\,, \qquad f_{n+1} := \frac{1-\rho}{\rho}\,.$$

If $\sigma \in \mathfrak{S}_{n+1}$ is a permutation, we define

$$\psi_l^\sigma := \psi_{l-1}^\sigma \wedge \mathrm{d}f_{\sigma(l+1)} + \frac{(-1)^l}{l} f_{\sigma(l+1)}\mathrm{d}\psi_{l-1}^\sigma \in \Omega^l(\mathbb{R}H^n)$$

with $\psi_0^\sigma := f_{\sigma(1)}$ in an inductive way. This leads to

$$\begin{aligned} \mathrm{d}\psi_l^\sigma &= \frac{l+1}{l}\mathrm{d}\psi_{l-1}^\sigma \wedge \mathrm{d}f_{\sigma(l+1)} \\ &= (l+1)\mathrm{d}f_{\sigma(1)} \wedge \mathrm{d}f_{\sigma(2)} \wedge \ldots \wedge \mathrm{d}f_{\sigma(l+1)} \end{aligned}$$

as well as

$$\begin{aligned} \nabla_X(\mathrm{d}\psi_l^\sigma) &= -K(l+1)\, X^* \wedge \psi_l^\sigma \\ \nabla_X \psi_l^\sigma &= \frac{1}{l+1} X \lrcorner \mathrm{d}\psi_l^\sigma\,. \end{aligned}$$

Such a l–form ψ_l^σ is called *special Killing form* (cf. [20]). The vector fields dual to ψ_1^σ are Killing fields.

More generally, let X be a Killing field on a space of constant sectional curvature K. The equation

$$\begin{aligned} L_X g(V,W) &= g(\nabla_V X, W) + g(V, \nabla_W X) \\ &= -\mathrm{d}X^*(W,V) + 2g(V, \nabla_W X) \end{aligned}$$

yields for X:

$$\nabla_Y X^* = \frac{1}{2} Y \lrcorner \mathrm{d}X^*\,.$$

A space of constant sectional curvature K satisfies

$$R(X,Y,V,W) := -K \cdot X^* \wedge Y^*(V,W)$$

which implies

$$\begin{aligned} (\nabla_Y \mathrm{d}X^*)(V,W) &= 2g(\nabla_Y\nabla_V X - \nabla_{\nabla_Y V}X, W) \\ &= 2R_{YX}(V,W) \\ &= -2K \cdot Y^* \wedge X^*(V,W) \end{aligned}$$

for the exterior derivative of a Killing 1–form. Thus, $\Psi = X^* + \frac{1}{2}\mathrm{d}X^* \in \Gamma(\mathbf{V}^1)$ is parallel with respect to $\widetilde{\nabla}$ if K equals $-\frac{1}{n-2}$.

2.2. **Bochner–Weitzenböck formula.** Let (M,g) be an oriented Riemannian manifold and $\Lambda^*(M)$ be the exterior form bundle of M. $\Lambda^*(M)$ is a Dirac bundle (cf. [14]) with respect to the relation

$$V\cdot\varphi = V^*\wedge\varphi - V\llcorner\varphi,$$

$V\in\Gamma(TM)$ and $\varphi\in\Omega^p(M) := \Gamma(\Lambda^p TM)$, if the Levi–Civita connection ∇ and the Riemannian metric induced by g are chosen. The corresponding Dirac operator is given by $\mathcal{D} = \mathrm{d}+\mathrm{d}^*$, where d^* is the codifferential of the exterior derivative d. Sections of $\Lambda^*(M)$ satisfy the so called Bochner–Weitzenböck formula (cf. [5, 14]):

$$\triangle = (\mathrm{d}+\mathrm{d}^*)^2 = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} = \nabla^*\nabla + \mathfrak{R}\,. \tag{2.1}$$

In this case $\mathfrak{R}$ is a section of $\mathrm{End}(\Lambda^*(M))$ preserving $\Omega^p(M)$ and depending on the Riemannian curvature operator. On the subbundle $\Lambda^1(M)$, $\mathfrak{R}$ is the Ricci tensor considered as endomorphism of $\Omega^1(M)$ and (2.1) is the classical Bochner equation. If the manifold has constant sectional curvature K, we obtain

$$\mathfrak{R} = p(n-p)K\cdot\mathrm{Id}_{\Omega^p}\,.$$

Conversely, if there is a p with $1<p<n-1$ and a constant K in such a way that

$$\mathfrak{R}(\varphi) = p(n-p)K\,\varphi$$

holds for all $\varphi\in\Omega^p(M)$, the manifold has to be of constant sectional curvature K (cf. [20, prop. B4]).

Let $\alpha\in\mathbb{R}$ and $l\in\{0,1,\ldots,n-1\}$. Consider the following connection on the subbundle $\mathbf{V}^l := \Lambda^l(M)\oplus\Lambda^{l+1}(M)$ of the exterior form bundle:

$$\widetilde{\nabla}_X\Phi := \nabla_X\Phi + \mathfrak{T}^l_X\Phi$$

with

$$\mathfrak{T}^l_X : \Gamma(\mathbf{V}^l)\to\Gamma(\mathbf{V}^l),\ \Phi = \phi_l+\phi_{l+1}\mapsto -\alpha X^*\wedge\phi_l - X\llcorner\phi_{l+1}\,,$$

where $\phi_l\in\Omega^l(M)$ and $\phi_{l+1}\in\Omega^{l+1}(M)$. The Dirac operator to $\widetilde{\nabla}$ is given by

$$\widetilde{\mathcal{D}}\Phi = (\mathrm{d}+\mathrm{d}^*)\Phi - (l+1)\phi_{l+1} + \alpha(n-l)\phi_l\,.$$

In particular an exterior form Φ parallel with respect to $\widetilde{\nabla}$ satisfies

$$0 = \mathrm{d}\phi_{l+1} = \mathrm{d}^*\phi_l$$

as well as

$$\mathrm{d}^*\phi_{l+1} = -\alpha(n-l)\phi_l \qquad\text{and}\qquad \mathrm{d}\phi_l = (l+1)\phi_{l+1}\,.$$

Thus, if $\psi_l+\mathrm{d}\psi_l\in\Gamma(\mathbf{V}^l)$ is a special Killing form, $\psi_l+\frac{1}{l+1}\mathrm{d}\psi_l$ is parallel with respect to $\widetilde{\nabla}$. In the preceding subsection is shown that on the real hyperbolic space of constant sectional curvature K the bundle $\mathbf{V}^l$ is trivialized by forms parallel with respect to $\widetilde{\nabla}$ if and only if α equals $-K$.

We are interested in an integrated Bochner–Weitzenböck for $\widetilde{\nabla}$ which allows the usage of the non–compact Bochner technique. Some straightforward calculations supply

$$\begin{aligned}\left|\widetilde{\nabla}\Phi\right|^2 &= |\nabla\Phi|^2 + 2\alpha\left\langle \mathrm{d}^*\phi_{l+1},\phi_l\right\rangle - 2\left\langle\phi_{l+1},\mathrm{d}\phi_l\right\rangle + \\ &\quad +\alpha^2(n-l)\left|\phi_l\right|^2 + (l+1)\left|\phi_{l+1}\right|^2\end{aligned}$$

as well as

$$\begin{aligned}\left|\widetilde{\mathcal{D}}\Phi\right|^2 &= \left|(\mathrm{d}+\mathrm{d}^*)\,\Phi\right|^2 + \alpha^2(n-l)^2\left|\phi_l\right|^2 + (l+1)^2\left|\phi_{l+1}\right|^2 + \\ &\quad +2\alpha(n-l)\left\langle \mathrm{d}^*\phi_{l+1},\phi_l\right\rangle - 2(l+1)\left\langle\phi_{l+1},\mathrm{d}\phi_l\right\rangle.\end{aligned}$$

Thus, if N is a compact and oriented manifold with boundary ∂N, Stokes theorem implies

$$\begin{aligned}\int_N \left|\widetilde{\nabla}\Phi\right|^2 - \left|\widetilde{\mathcal{D}}\Phi\right|^2 &= \int_{\partial N} \langle \nabla_\nu \Phi + \nu \cdot \mathcal{D}\Phi, \Phi\rangle + 2l\,\langle \nu \llcorner \phi_{l+1}, \phi_l\rangle \\ &\quad + \int_N \langle (\nabla^*\nabla - \triangle)\,\Phi, \Phi\rangle - l(l+1)\,|\phi_{l+1}|^2 \\ &\quad - \int_N \alpha^2 (n-l-1)(n-l)\,|\phi_l|^2 \\ &\quad + \int_N 2\,(l - \alpha(n-l-1))\,\langle \mathrm{d}^*\phi_{l+1}, \phi_l\rangle\,,\end{aligned}$$

where integration is done with respect to the volume form in the cases that no n–form is specified. Applying the Bochner–Weitzenböck formula (2.1) yields the following.

Proposition 2.1. *Let (N, g) be a compact and oriented Riemannian manifold with boundary ∂N. Then the integrated Bochner–Weitzenböck formula*

$$\int_{\partial N} \left\langle \widetilde{\nabla}_\nu \Phi + \nu \cdot \widetilde{\mathcal{D}}\Phi, \Phi \right\rangle = \int_N \left|\widetilde{\nabla}\Phi\right|^2 + \left\langle \widetilde{\mathfrak{R}}\Phi, \Phi\right\rangle - \left|\widetilde{\mathcal{D}}\Phi\right|^2$$

holds for all sections Φ of $\mathbf{V}^l = \Lambda^l \oplus \Lambda^{l+1}$ if $l < n-1$ and $\alpha = \frac{l}{n-l-1}$. In this case ν is the outward normal vector field on ∂N and

$$\widetilde{\mathfrak{R}} := \mathfrak{R} + \alpha l(n-l)\mathrm{Id}_{\Omega^l} + l(l+1)\mathrm{Id}_{\Omega^{l+1}}$$

is the curvature endomorphism on $\Gamma(\mathbf{V}^l)$. Since the curvature endomorphism $\mathfrak{R}$ is symmetric w.r.t. g, the boundary operator is symmetric too (cf. [18]*):*

$$\int_{\partial N} \left\langle \widetilde{\nabla}_\nu \Phi + \nu \cdot \widetilde{\mathcal{D}}\Phi, \Psi \right\rangle = \int_{\partial N} \left\langle \Phi, \widetilde{\nabla}_\nu \Psi + \nu \cdot \widetilde{\mathcal{D}}\Psi \right\rangle$$

for all $\Phi, \Psi \in \Gamma(\mathbf{V}^l)$.

Lemma 2.2. *If (M^n, g) has sectional curvature*

$$K_g \geq -\alpha = -\frac{l}{n-l-1}\,,$$

the eigenvalue functions of $\widetilde{\mathfrak{R}} \in \mathrm{End}(\Gamma(\mathbf{V}^l))$ are non–negative.

Proof. Let X and Y be linearly independent. The sectional curvature for the plane $\vartheta = \mathrm{span}\{X, Y\}$ satisfies:

$$K_\vartheta = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}\,.$$

Consider the Riemannian curvature tensor R as endomorphism on $\Omega^2(M)$:

$$\langle R(X^* \wedge Y^*), Z^* \wedge T^*\rangle = R(X,Y,Z,T)\,,$$

then $K \geq -\alpha$ implies that every eigenvalue function of $R : \Omega^2(M) \to \Omega^2(M)$ is less or equal to α. Confer [10, cor. 2.6], we have

$$\langle \mathfrak{R}(\varphi), \varphi\rangle \geq -p(n-p)\alpha\,|\varphi|^2$$

for every p–form φ. But this supplies the claim. ∎

2.3. **Coercivity of the Dirac operator $\widetilde{\mathcal{D}}$.** If $\mathbf{E} \to M^n$ is a Riemannian vector bundle and ω is a positively oriented nowhere vanishing n–form, $L^2(M, \mathbf{E}; \omega)$ denotes the completion of compact supported sections of $\mathbf{E}$ with respect to the L^2 norm

$$\|\varphi\|_\omega = \sqrt{\int\limits_M \langle \varphi, \varphi \rangle \, \omega} \, .$$

Suppose that $\mathbf{E}$ is equipped with a Riemannian connection ∇, and consider the scalar product

$$(\varphi, \psi)_{1,\omega} := \int\limits_M \Big(\langle \nabla\varphi, \nabla\psi \rangle + \langle \varphi, \psi \rangle \Big) \omega$$

on $\Gamma_{\rm cpt}(\mathbf{E})$. The Sobolev space $W^{1,2}(M, \mathbf{E}; \omega)$ is the closure of $\Gamma_{\rm cpt}(\mathbf{E})$ in $L^2(M, \mathbf{E}; \omega)$ with respect to the norm $\|.\|_{1,\omega} := \sqrt{(.,.)_{1,\omega}}$. Moreover, $W^{1,2}(M, \mathbf{E}; \omega)$ becomes a Hilbert space with respect to the extension of the product $(.,.)_{1,\omega}$. For notational simplicity set $W^{1,2}(M, \mathbf{E}) := W^{1,2}(M, \mathbf{E}; \mathrm{vol}_g)$ and $L^2(M, \mathbf{E}) := L^2(M, \mathbf{E}; \mathrm{vol}_g)$, where vol_g means the volume form of (M, g).

Proposition 2.3. *Suppose (M^n, g) is a complete and oriented Riemannian manifold as well as $\widetilde{\mathfrak{R}}$ is uniformly bounded and has no negative eigenvalues. Then*

$$\widetilde{\mathcal{D}} : W^{1,2}(M, \mathbf{V}^l) \to L^2(M, \mathrm{Im}(\widetilde{\mathcal{D}}))$$

is an isomorphism of Hilbert spaces if $0 < l < n-1$. $L^2(M, \mathrm{Im}(\widetilde{\mathcal{D}}))$ denotes the closure of

$$\Big\{ \widetilde{\mathcal{D}}\Psi \in L^2(M, \Lambda^* M), \Psi \in \Gamma(\mathbf{V}^l) \Big\}$$

in $L^2(M, \Lambda^*(M))$.

Proof. Since $\widetilde{\mathfrak{R}}$ is uniformly bounded, the extension of the smooth operator $\widetilde{\mathcal{D}}$ is well defined and continuous. Let $\widetilde{\mathcal{D}}^*$ be the operator

$$\widetilde{\mathcal{D}}^* : \Gamma(\Lambda^{l-1} \oplus \Lambda^l \oplus \Lambda^{l+1} \oplus \Lambda^{l+2}) \to \Gamma(\Lambda^*(M))$$

with

$$\widetilde{\mathcal{D}}^* \Phi := (\mathrm{d} + \mathrm{d}^*)\Phi + (l+1)\left(\phi_{l+2} + \phi_l\right) - \alpha(n-l)\left(\phi_{l-1} + \phi_{l+1}\right) \, .$$

Then

$$\widetilde{\mathcal{D}}^* \widetilde{\mathcal{D}} = \triangle + \alpha(n-l)(l+1)\mathrm{Id}_{\mathbf{V}^l}$$

preserves $\Gamma(\mathbf{V}^l)$ and

$$\widetilde{\mathcal{D}}^* : \Gamma(\Lambda^*(M)) \cap L^2(M, \mathrm{Im}(\widetilde{\mathcal{D}})) \to \Gamma(\mathbf{V}^l) \, . \tag{2.2}$$

Let Φ and Ψ be compact supported sections of $\mathbf{V}^l$. The symmetric bilinear form

$$B(\Phi, \Psi) := \int\limits_M \langle \triangle\Phi + \alpha(n-l)(l+1)\Phi, \Psi \rangle$$

satisfies

$$B(\Phi, \Phi) = \int\limits_M |\nabla\Phi|^2 + \left\langle \widetilde{\mathfrak{R}}\Phi, \Phi \right\rangle + \alpha(n-l)\,|\phi_l|^2 + \alpha(l+1)\,|\phi_{l+1}|^2 \, .$$

Thus, B can be extended to a bounded and coercive symmetric bilinear form on $W^{1,2}(M, \mathbf{V}^l)$ if $\alpha > 0$. In particular $W^{1,2}(M, \mathbf{V}^l)$ is a Hilbert space with respect to the scalar product B. Since $\widetilde{\mathcal{D}}\Phi = 0$ implies $B(\Phi, \Phi) = 0$, $\widetilde{\mathcal{D}}$ is injective on $W^{1,2}(M, \mathbf{V}^l)$. Therefore, the surjectivity of $\widetilde{\mathcal{D}}$ remains to show. For any $\Psi_0 \in L^2(M, \mathrm{Im}(\widetilde{\mathcal{D}}))$ the linear functional

$$l_{\Psi_0} : \Upsilon \mapsto \int\limits_M \left\langle \widetilde{\mathcal{D}}^* \Psi_0, \Upsilon \right\rangle$$

is bounded on $W^{1,2}(M,\mathbf{V}^l)$. The Riesz representation theorem supplies a form $\Phi_0 \in W^{1,2}(M,\mathbf{V}^l)$ with $B(\Phi_0,\Upsilon) = l_{\Psi_0}(\Upsilon)$ for all $\Upsilon \in W^{1,2}(M,\mathbf{V}^l)$.

Set $\Theta := \widetilde{\mathcal{D}}\Phi_0 - \Psi_0 \in L^2(M,\mathrm{Im}(\widetilde{\mathcal{D}}))$, then

$$0 = \int_M \left\langle \widetilde{\mathcal{D}}^*\Theta, \Upsilon \right\rangle \quad \forall \Upsilon \in W^{1,2}(M,\mathbf{V}^l)$$

together with equation (2.2) yields the weak solution $\widetilde{\mathcal{D}}^*\Theta = 0$. Moreover, elliptic regularity implies smoothness of Θ and the strong solution $\widetilde{\mathcal{D}}^*\Theta = 0$. Since $\mathcal{D}\Theta = (\mathrm{d}+\mathrm{d}^*)\Theta$ is contained in $L^2(M,\Lambda^*(M))$, Θ is in the maximal L^2 domain of a generalized Dirac operator in the sense of Gromov and Lawson [13], i.e. Θ is contained in $W^{1,2}(M,\Lambda^*(M))$. Set

$$\begin{aligned}\widetilde{\mathcal{D}}^\dagger \quad := \quad & \mathrm{d}^* + \mathrm{d} - (l+1)\mathrm{Id}_{\Omega^{l+3}} - (l+1)\mathrm{Id}_{\Omega^{l+1}} - (l+1)\mathrm{Id}_{\Omega^{l-1}} \\ & +\alpha(n-l)\mathrm{Id}_{\Omega^l} + \alpha(n-l)\mathrm{Id}_{\Omega^{l+2}} + \alpha(n-l)\mathrm{Id}_{\Omega^{l-2}},\end{aligned}$$

this leads to

$$\begin{aligned}\widetilde{\mathcal{D}}^\dagger\widetilde{\mathcal{D}}^* \quad = \quad & \triangle + \alpha(n-l)(l+1)\mathrm{Id}_{\Omega^{l+2}} + \alpha(n-l)(l+1)\mathrm{Id}_{\Omega^{l+1}} \\ & +\alpha(n-l)(l+1)\mathrm{Id}_{\Omega^l} + \alpha(n-l)(l+1)\mathrm{Id}_{\Omega^{l-1}}\,.\end{aligned}$$

Therefore, $\widetilde{\mathcal{D}}^*\Theta = 0$ and

$$\Theta \in W^{1,2}(M,\Lambda^*(M)) \cap \Gamma(\Lambda^{l-1}\oplus\Lambda^l\oplus\Lambda^{l+1}\oplus\Lambda^{l+2})$$

imply

$$0 = \int_M \left\langle \widetilde{\mathcal{D}}^\dagger\widetilde{\mathcal{D}}^*\Theta, \Theta \right\rangle = \int_M |(\mathrm{d}+\mathrm{d}^*)\Theta|^2 + \alpha(n-l)(l+1)|\Theta|^2.$$

Since $\alpha > 0$, this gives the claim $\Theta = 0$. □

The proof of the last proposition fails if (M,g) is not assumed to be complete, since in this case the Dirac operator $\mathcal{D} = \mathrm{d} + \mathrm{d}^*$ is not essentially selfadjoint on $W^{1,2}(M,\Lambda^*M)$.

3. Proof of the rigidity theorem

Choose $l := 1$ and $\alpha := \frac{1}{n-2}$. Let $(M,\hat{g})$ be the strongly asymptotically hyperbolic manifold in theorem 1.2, then $g := (n-2)\hat{g}$ has sectional curvature $K_g \geq -\frac{1}{n-2}$. Consider the gauge transformation $A : TM_{|E} \to TM_{|E}$ with

$$g(AX,AY) = g'(X,Y) \qquad g(AX,Y) = g(X,AY)\,,$$

where g' is the hyperbolic metric on the end $E = M - C$ with sectional curvature $-\frac{1}{n-2}$. Set $A(X^{*'}) := A(X)^*$ for the canonical isomorphisms $*', * : TM_{|E} \to \Lambda^1(M)_{|E}$, then A is an isomorphism $\Lambda^1(M)_{|E} \to \Lambda^1(M)_{|E}$ which satisfies

$$g(A\omega, A\eta) = g'(\omega,\eta)\,.$$

Extend A to the exterior algebra in such a way that $A(\omega\wedge\eta) = A(\omega)\wedge A(\eta)$ and $A(X\llcorner\omega) = A(X)\llcorner A(\omega)$ hold. Then $A : \Lambda^*(M)_{|E} \to \Lambda^*(M)_{|E}$ respects Clifford multiplication:

$$A(w\cdot'\varphi) = A(w)\cdot A(\varphi)\,,$$

where $\cdot'$ and $\cdot$ are the Clifford multiplications w.r.t. g' and g respectively. Define $\overline{\nabla}_X Y := A(\nabla'_X A^{-1}(Y))$ and extend this connection as derivation to $\Lambda^*(M)$. Let ψ' be a special Killing 1–form on (E,g'), i.e. ψ' is dual to a Killing field. Then

$$\Psi' := \psi' + \frac{1}{2}\mathrm{d}\psi' \in \Gamma(\mathbf{V}^1_{|E})$$

is parallel with respect to $\widetilde{\nabla}'$ on (E, g'). Suppose $f : M \to [0,1]$ is a smooth cut off function for E, i.e. $\mathrm{supp}(\mathrm{d}f)$ is compact, $f = 0$ outside of E and $f = 1$ at infinity of E. Set

$$\Psi_0 := f A\,(\Psi') \in \Gamma(\mathbf{V}^1) = \Omega^1(M) \oplus \Omega^2(M)\,,$$

we obtain from $A\mathfrak{T}'_X \Phi = \mathfrak{T}_{AX} A\Phi$

$$\widetilde{\nabla}_X \Psi_0 = \mathrm{d}f(X) A\Psi' + f\left(\nabla_X A\Psi' - \overline{\nabla}_X A\Psi'\right) - f\mathfrak{T}_{(A-\mathrm{Id})(X)}(A\Psi')\,.$$

Compare estimates for $\nabla - \overline{\nabla}$ in [3] respectively [9], this supplies near infinity for some constant $c > 0$:

$$\left|\widetilde{\nabla}\Psi_0\right|^2 \leq c\left(|\nabla' A|^2 + |A - \mathrm{Id}|^2\right) g'(\Psi', \Psi')\,.$$

Consider $g_0 = \frac{1}{n-2} g'$, then $(\Psi' = \psi_1' + \psi_2')$

$$\nabla^0_X \Psi' = \nabla'_X \Psi' = X^{*_0} \wedge \psi_1' + X \lrcorner \psi_2'$$

as well as

$$|X^* \wedge \psi_1' + X \lrcorner \psi_2'|^2 \leq |X|^2\, |\Psi'|^2$$

imply

$$\begin{aligned} |\nabla_X g_0(\Psi', \Psi')| &= |2g_0(\nabla'_X \Psi', \Psi')| \\ &= |2g_0(X^{*_0} \wedge \psi_1' + X \lrcorner \psi_2', \Psi')| \\ &\leq 2\,|X|_{g_0}\,|\Psi'|^2_{g_0}\,. \end{aligned}$$

Thus, $g_0(\Psi', \Psi')$ as well as $g'(\Psi', \Psi')$ can be estimated by $c \cdot e^{2r}$ for some constant $c > 0$ if r is the g_0–distance to a fixed point. Therefore, the asymptotic assumptions imply $\widetilde{\nabla}\Psi_0 \in L^2(M, T^*M \otimes \mathbf{V}^1)$ as well as $\widetilde{\mathcal{D}}\Psi_0 \in L^2(M, \mathrm{Im}(\widetilde{\mathcal{D}}))$. Lemma 2.2 and Proposition 2.3 yield a form $\Theta \in \Gamma(\mathbf{V}^1)$ with $\widetilde{\mathcal{D}}\Theta = \widetilde{\mathcal{D}}\Psi_0$. Set $\Phi := \Psi_0 - \Theta$, then Φ is non–trivial, since $\Psi_0 \notin L^2(M, \mathbf{V}^1)$, and it satisfies $\widetilde{\mathcal{D}}\Phi = 0$. The above estimates also imply

$$\left|\widetilde{\mathcal{D}}\Psi_0\right|^2 \leq c\left(|\nabla' A|^2 + |A - \mathrm{Id}|^2\right) e^{2r}\,,$$

which gives

$$\left|\left\langle \widetilde{\nabla}_\nu \Psi_0 + \nu \cdot \widetilde{\mathcal{D}}\Psi_0, \Psi_0 \right\rangle\right| \leq c e^{2r} \sqrt{|\nabla' A|^2 + |A - \mathrm{Id}|^2}\,.$$

Thus, if $\{M_r\}$ is a non–degenerate exhaustion of M by compact manifolds M_r (cf. [3, 18]), the asymptotic assumptions as well as the selfadjointness of the boundary term guarantee

$$\liminf_{r\to\infty} \int\limits_{\partial M_r} \left\langle \widetilde{\nabla}_\nu \Phi + \nu \cdot \widetilde{\mathcal{D}}\Phi, \Phi \right\rangle = 0.$$

Therefore, proposition 2.1 and $\widetilde{\mathfrak{R}} \geq 0$ imply $\widetilde{\nabla}\Phi = 0$, i.e. $\Phi = \phi_1 + \phi_2$ is a special Killing 1–form on M and the vector field dual to ϕ_1 is a Killing field. Since this method works for any g' special Killing 1–form ψ', there is a full set of Killing vector fields on (M, g) (cf. [18, thm. 5.3]):

$$\dim \mathrm{Isom}(M, g) = \frac{1}{2} n(n+1).$$

Remark 3.1. This proof works for any $l \in \{1, \ldots, n-2\}$ and special Killing form ψ_l^σ if α equals $\frac{l}{n-l-1}$, and if we consider the manifold $(M, \alpha^{-1} g)$, where g is the strongly asymptotically hyperbolic metric in theorem 1.2. The final conclusion follows in the case $l > 1$ if we compute the curvature of the Killing connection.

4. Remarks

- The proof of this rigidity result can be extended to hyperbolic quotients. Let (M_0, g_0) be a non–compact oriented hyperbolic quotient, i.e. complete and of constant sectional curvature K_{g_0}. If the manifold (M, g) is complete and strongly asymptotic to (M_0, g_0) with $K_g \geq K_{g_0}$, we obtain
$$\dim \mathrm{Isom}(M, g) \geq \dim \mathrm{Isom}(M_0, g_0).$$
Nevertheless, (M, g) may not be a hyperbolic quotient if it is not possible to conclude constant scalar curvature from this inequality.
- Does the non–spin rigidity result holds under weaker curvature assumptions, for example $\mathrm{Ric} \geq -(n-1)g$ in analogy to Bartnik's result [4]? This seems to be a very difficult question. Presuming that this is false, leads to the search for a non–spin conformally compact Einstein manifold having the standard sphere as conformal infinity. Conversely, in order to prove rigidity we could use the conformal vector fields of the hyperbolic space. For any V denote the conformal Lie–derivative of g by $\mathfrak{L}_V g := L_V g - \frac{2}{n}(\mathrm{div}\ V)g$. If $\mathcal{F} : \Omega^1(M) \to \Omega^1(M)$ is the elliptic operator
$$\frac{n-1}{n}\mathrm{d}\mathrm{d}^* + \frac{1}{2}\mathrm{d}^*\mathrm{d} + (n-1)\mathrm{Id},$$
a straightforward calculation gives for all $Y \in \Gamma(TM)$:
$$\int\limits_{\partial M} \mathfrak{L}_Y g(Y, \nu) = \int\limits_{M} \frac{1}{2}\left|\mathfrak{L}_Y g\right|^2 - \langle \mathcal{F}(Y^*), Y^* \rangle + \langle \overline{\mathrm{Ric}}(Y), Y \rangle,$$
where $\overline{\mathrm{Ric}} = \mathrm{Ric} + (n-1)g$. In the case $\overline{\mathrm{Ric}} \geq 0$, $\mathcal{F}$ is coercive on $W^{1,2}(M, \Lambda^1 M)$. In particular for any asymptotic conformal vector field V_0, there is a $W^{1,2}$ perturbation X with $\mathcal{F}(V_0^* + X^*) = 0$. But in general the boundary values at infinity of $V := V_0 + X$ and V_0 do not coincide:
$$\begin{aligned}\int\limits_{\partial M_r} \mathfrak{L}_V g(V, \nu) \quad &= \int\limits_{\partial M_r} \mathfrak{L}_{V_0} g(V_0, \nu) + \mathfrak{L}_X g(V_0, \nu) + \\ &\qquad\qquad + \mathfrak{L}_{V_0} g(X, \nu) + \mathfrak{L}_X g(X, \nu).\end{aligned}$$
The first part in the sum of the right hand side only depends on the asymptotic assumptions while the third and fourth summand are in L^1. The reason for failing of this method is the second summand on the RHS. Even if we make the full elliptic estimate for $\mathcal{F}$ on weighted Sobolev spaces of conformally compact manifolds (cf. [2, 11, 12, 15, 16]), $\mathfrak{L}_X g$ will not have the necessary decay behavior.
- There exists several mass definitions for asymptotically hyperbolic manifolds and for anti–de Sitter space times respectively (cf. [1, 6, 7, 8, 21]). But in contrast to the asymptotic flat case, the mass functional depends on the choice of the asymptotic Killing spinors.

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