APPLICATIONS OF HOFER'S GEOMETRY TO HAMILTONIAN DYNAMICS

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Abstract. We prove that for every subset A of a tame symplectic manifold (W, ω) the π_1-sensitive Hofer–Zehnder capacity of A is not greater than four times the displacement energy of A,

$$c_{\mathrm{HZ}}^{\circ}(A, W) \leq 4\, e(A, W).$$

This estimate yields almost existence of periodic orbits near displaceable energy levels of time-independent Hamiltonian systems. Our main applications are:

- The Weinstein conjecture holds true for every displaceable hypersurface of contact type in (W, ω).
- The flow describing the motion of a charge on a closed Riemannian manifold subject to a non-vanishing magnetic field and a conservative force field has contractible periodic orbits at almost all sufficiently small energies.
- Every closed Lagrangian submanifold of (W, ω) whose fundamental group injects and which admits a Riemannian metric without contractible closed geodesics has the intersection property.

The proof of the above energy-capacity inequality combines a curve shortening procedure in Hofer geometry with the following detection mechanism for periodic orbits: If the ray $\{\varphi_F^t\}$, $t \geq 0$, of Hamiltonian diffeomorphisms generated by a compactly supported time-independent Hamiltonian stops to be a minimal geodesic in its homotopy class, then a non-constant contractible periodic orbit must appear.

1. Introduction and Results

On their search for periodic orbits of autonomous Hamiltonian systems, Hofer and Zehnder [24, 25] associated to every open subset A of a symplectic manifold (V, ω) a number, the Hofer–Zehnder capacity $c_{\mathrm{HZ}}(A) \in [0, \infty]$, in such a way that $c_{\mathrm{HZ}}(A) < \infty$ implies almost existence of periodic orbits near any compact regular energy level of an autonomous Hamiltonian system on A. Showing that $c_{\mathrm{HZ}}(A)$ is finite

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is, however, often a difficult problem. Our main result states that if a subset A of a tame symplectic manifold can be displaced from itself by a Hamiltonian isotopy in a stabilized sense, then the Hofer–Zehnder capacity of A is indeed finite.

In order to set notations, we abbreviate $I = [0, 1]$ and consider an arbitrary smooth symplectic manifold (V, ω) without boundary. Denote by $\mathcal{H}_c(I \times V)$ the set of smooth functions $I \times V \to \mathbb{R}$ whose support is compact and contained in $I \times V$. The Hamiltonian vector field of $H \in \mathcal{H}_c(I \times V)$ defined by

$$\omega(X_{H_t}, \cdot) = -dH_t(\cdot)$$

generates a flow h_t. The set of time-1-maps h form the group

$$\mathrm{Ham}_c(V, \omega) := \{h \mid H \in \mathcal{H}_c(I \times V)\}$$

of compactly supported Hamiltonian diffeomorphisms of (V, ω). The set of functions in $\mathcal{H}_c(I \times V)$ which do not depend on $t \in I$ is denoted by $\mathcal{H}_c(V)$. We shall denote functions in $\mathcal{H}_c(I \times V)$ by H or K and functions in $\mathcal{H}_c(V)$ by F or G, and their flows by h_t or k_t and f_t or g_t, respectively.

The Hofer–Zehnder capacity we shall study is defined as follows. We say that $F \in \mathcal{H}_c(V)$ is *slow* if all non-constant contractible periodic orbits of f_t have period > 1. Following [24, 25] and [36, 52, 15] we define for each subset A of (V, ω) the π_1-sensitive Hofer–Zehnder capacity

$$(1) \quad c_{\mathrm{HZ}}^{\circ}(A, V, \omega) = \sup\{\max F \mid F \in \mathcal{H}_c(\mathrm{Int}(A)) \text{ is slow}\}.$$

We shall often suppress ω from the notation, and we shall write $c_{\mathrm{HZ}}^{\circ}(V)$ instead of $c_{\mathrm{HZ}}^{\circ}(V, V)$.

Remarks 1.1. 1. The Hofer–Zehnder capacity $c_{\mathrm{HZ}}(A)$ mentioned above is obtained by taking the supremum over the smaller class of functions $F \in \mathcal{H}_c(\mathrm{Int}(A))$ for which *all* non-constant periodic orbits of f_t have period > 1. Therefore, $c_{\mathrm{HZ}}(A) \leq c_{\mathrm{HZ}}^{\circ}(A, V)$.

2. The definition of $c_{\mathrm{HZ}}(A)$ in the original work [24, 25] and of $c_{\mathrm{HZ}}^{\circ}(A, V)$ in [36, 52] starts from the subset

$$\mathcal{F}(A) = \{F \in \mathcal{H}_c(\mathrm{Int}(A)) \mid F \geq 0,\ F|_U = \max F \text{ for some open } U \subset A\}$$

of $\mathcal{H}_c(\mathrm{Int}(A))$. It was only noticed in Theorem 2.8 of [15] that starting from the larger set $\mathcal{H}_c(\mathrm{Int}(A))$ yields *the same* invariants. $\diamond$

We shall compare the Hofer–Zehnder capacity $c_{\mathrm{HZ}}^{\circ}(A, V)$ with the displacement energy defined in [19, 29]. The norm $\|H\|$ of $H \in \mathcal{H}_c(I \times$

V) is defined as

$$\|H\| \;=\; \int_0^1 \left(\sup_{x \in V} H(t,x) - \inf_{x \in V} H(t,x) \right) dt,$$

and the displacement energy $e(A,V) = e(A,V,\omega) \in [0,\infty]$ of a subset A of V is defined as

$$e(A,V) \;=\; \inf \left\{ \|H\| \mid H \in \mathcal{H}_c(I \times V),\; h(A) \cap A = \emptyset \right\}$$

if A is compact and as

$$e(A,V) \;=\; \sup \left\{ e(K,V) \mid K \subset A \text{ is compact} \right\}$$

for a general subset A of V.

We were not able to compare the Hofer–Zehnder capacity c_{HZ}° with the displacement energy e on *all* symplectic manifolds, but on a large class of symplectic manifolds.

Definition 1.2. [18, 54, 1] A symplectic manifold (W,ω) is *tame* if W admits an almost complex structure J and a Riemannian metric g such that

- J is uniformly tame, i.e., there are positive constant c_1 and c_2 such that

  $$\omega(X,JX) \geq c_1 \|X\|^2 \qquad \text{and} \qquad |\omega(X,Y)| \leq c_2 \|X\| \, \|Y\|$$

  for all $X, Y \in TW$.
- The sectional curvature of (W,g) is bounded from above and the injectivity radius of (W,g) is bounded away from zero.

Examples of tame symplectic manifolds are closed symplectic manifolds, the standard cotangent bundle (T^*M,ω_0) as well as twisted cotangent bundles (T^*M,ω_σ) over a closed base M (see [6] and Paragraph 3 below) and symplectic manifolds which at infinity are isomorphic to the symplectization of a closed contact manifold, and the class of tame symplectic manifolds is closed under taking products or coverings.

Our main result is the following energy-capacity inequality.

Theorem 1.3. *Assume that A is a subset of a tame symplectic manifold (W,ω). Then*

$$c_{\mathrm{HZ}}^\circ(A,W) \;\leq\; 4\, e(A,W).$$

Remarks 1.4. **1.** It is important that in the definition of $c_{\mathrm{HZ}}^\circ(A,V)$ the periodic orbits in A looked for are assumed to be contractible in V, and not in A. We illustrate this by looking at the annulus $A = B^2(1) \setminus$

$\{0\}$ in $(\mathbb{R}^2, \omega_0)$. Then $c^\circ_{\mathrm{HZ}}(A, \mathbb{R}^2) = e(A, \mathbb{R}^2) = \pi$, while the Hofer–Zehnder capacity obtained from looking at periodic orbits *contractible in* A is infinite, whence Theorem 1.3 fails for this capacity.

2. The first energy-capacity inequality was obtained by Hofer [20], who proved that

$$c^\circ_{\mathrm{HZ}}\left(A, \mathbb{R}^{2n}\right) \;\leq\; e\left(A, \mathbb{R}^{2n}\right) \tag{2}$$

for every subset A of $(\mathbb{R}^{2n}, \omega_0)$, see also [25, Section 5.5] as well as [19] where (2) had been obtained for the first Ekeland–Hofer capacity instead of c°_{HZ}. Later on, the inequality

$$c^\circ_{\mathrm{HZ}}\,(A, V) \;\leq\; 2\,e\,(A, V) \tag{3}$$

was established for every subset A of a weakly exact symplectic manifold (V, ω) which is closed [52] or convex [10]. For the open ball $B^{2n}(r)$ of radius r in $(\mathbb{R}^{2n}, \omega_0)$ it holds that

$$c^\circ_{\mathrm{HZ}}\left(B^{2n}(r), \mathbb{R}^{2n}\right) \;=\; e\left(B^{2n}(r), \mathbb{R}^{2n}\right) \;=\; \pi,$$

see [25], and so (2) is sharp. It is conceivable that the factors 2 and 4 in (3) and in Theorem 1.3 can be omitted.[1]

3. The Gromov-width $c_G(A) = c_G(A, \omega)$ of a subset A of a symplectic manifold (V, ω) is defined as

$$c_G(A) \;=\; \sup\left\{\pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (A, \omega)\right\}.$$

According to [29], the energy-capacity inequality

$$c_G\,(A) \;\leq\; 2\,e\,(A, V) \tag{4}$$

holds for every subset A of *any* symplectic manifold (V, ω). This inequality implies that the Hofer norm on $\mathrm{Ham}_c(V, \omega)$ is non-degenerate. Since $c_G \leq c^\circ_{\mathrm{HZ}}$, Theorem 1.3 recovers inequality (4) for tame (W, ω) up to a factor 2. It is worthwhile to compare the proofs of (4) and Theorem 1.3. Inequality (4) was proved by combining an explicit and elementary (and ingenious) embedding technique (symplectic folding) with the general Non-squeezing Theorem

$$c_G\left(V \times B^2(r), \omega \oplus \omega_0\right) \;\leq\; \pi r^2$$

proved by J-holomorphic techniques. Similarly, Theorem 1.3 will be proved by combining Sikorav's explicit and elementary (and ingenious)

[1] It has been recently shown in [14] that the factor 2 in (3) can indeed be omitted, an so for weakly exact closed or convex symplectic manifolds the factor 2 in (4) can also be omitted.

curve shortening technique in Hofer's geometry with the area-capacity inequality

$$c^\circ_{\mathrm{HZ}}\left(W \times B^2(r), \omega \oplus \omega_0\right) \;\leq\; \pi r^2 \tag{5}$$

which for tame symplectic manifolds (W, ω) was essentially proved in [44] by Floer homological techniques, see also [9, 35]. An extension of inequality (5) to quasi-cylinders combined with the "gluing of monodromies" construction of [31] yields Theorem 1.5 below, whose contraposition is a detection mechanism for periodic orbits which together with the curve shortening procedure will imply Theorem 1.3. ◇

Theorem 1.5. *Assume that* (W, ω) *is a tame symplectic manifold, and that the autonomous Hamiltonian* $F \in \mathcal{H}_c(W)$ *is slow. Then the path* f_t, $t \in [0,1]$, *is length minimizing in its homotopy class.*

This theorem was discovered by Hofer ([20], see also [25, Section 5.7]) for standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ and was proved in [31, proof of Theorem 5.4] for weakly exact tame symplectic manifolds; it removes an additional assumption on F in [44, Theorem 1.4] and verifies Conjecture 1.2 in [44] for tame symplectic manifolds.

Theorem 1.3 shows that if $e(A, W)$ is finite, then so is $c^\circ_{\mathrm{HZ}}(A, W)$, and as we shall recall in Section 3, the finiteness of $c^\circ_{\mathrm{HZ}}(A, W)$ implies almost existence of periodic orbits near any compact regular energy level of an autonomous Hamiltonian system on A. Theorem 1.3 itself is nevertheless not too useful, since the hypothesis that $e(A, W)$ is finite imposes serious restrictions on the symplectic topology of $A \subset (W, \omega)$. Capitalizing on the fact that the finiteness of our Hofer–Zehnder capacity guarantees the existence of *contractible* periodic orbits, and using a stabilization trick used before by Macarini [39], we shall obtain an improvement of Theorem 1.3 which implies the finiteness of $c^\circ_{\mathrm{HZ}}(A, W)$ under a much weaker hypothesis on the symplectic topology of $A \subset (W, \omega)$. Here, we only describe a basic version of stabilization; the general version is given in Section 2.2. Endow the cotangent bundle T^*S^1 over the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ with the standard symplectic form $\omega_0 = dp \wedge dq$. For every subset A of a symplectic manifold (V, ω) we define the stable displacement energy $e_1(A, V) \in [0, \infty]$ by

$$e_1(A, V) \;=\; e\left(A \times S^1, V \times T^*S^1, \omega \oplus \omega_0\right).$$

It is not hard to see that $e_1(A, V) \leq e(A, V)$, and we shall give examples with $0 = e_1(A, V) < e(A, V) = \infty$ in Example 2.9.1. The following theorem thus improves Theorem 1.3.

Theorem 1.6. *Assume that A is a subset of a tame symplectic manifold (W, ω). Then*

$$c_{\mathrm{HZ}}^{\circ}(A, W) \leq 4\, e_1(A, W).$$

In the remainder of this introduction we describe applications of Theorem 1.6. We say that a subset A of a symplectic manifold (V, ω) is *displaceable* if there exists $h \in \mathrm{Ham}_c(V, \omega)$ which displaces the closure $\overline{A}$ of A, i.e., $h(\overline{A}) \cap \overline{A} = \emptyset$, and we say that A is *stably displaceable* if $A \times S^1$ is displaceable in $(V \times T^*S^1, \omega \oplus \omega_0)$. Thus $A \subset V$ is displaceable resp. stably displaceable if and only if A is relatively compact and $e(\overline{A}, V) < \infty$ resp. $e_1(\overline{A}, V) < \infty$. Note that if A is (stably) displaceable, then a whole neighbourhood of $\overline{A}$ is (stably) displaceable.

In order to apply Theorem 1.6, we need to understand which compact subsets of a symplectic manifold are (stably) displaceable. *Every* compact subset of a symplectic manifold of the form $(V \times \mathbb{R}^2, \omega \oplus \omega_0)$ is displaceable. Less obvious sufficient assumptions on A alone are collected in the following proposition due to Laudenbach [33] and to Polterovich [48] and Laudenbach–Sikorav [34]. Recall that a middle-dimensional submanifold L of a symplectic manifold (V, ω) is called *Lagrangian* if ω vanishes on L.

Proposition 1.7. *Suppose that a compact subset A of a symplectic manifold (V^{2n}, ω) meets one of the following assumptions.*

(i) *A is contained in an embedded finite CW-complex of dimension $< n$.*
(ii) *A is contained in an n-dimensional closed submanifold which is not Lagrangian.*
(iii) *A is strictly contained in a closed Lagrangian submanifold.*

Then A is stably displaceable.

The term "almost all" will always refer to the Lebesgue measure on $\mathbb{R}$.

1. Almost existence of closed characteristics and the Weinstein conjecture

A *hypersurface* S in a symplectic manifold (V, ω) is a smooth compact connected orientable codimension 1 submanifold of V without boundary. A closed characteristic on S is an embedded circle in S all of whose tangent lines belong to the distinguished line bundle

$$\mathcal{L}_S = \{(x, \xi) \in TS \mid \omega(\xi, \eta) = 0 \text{ for all } \eta \in T_xS\}.$$

Examples show that $\mathcal{L}_S$ might not carry any closed characteristic, see [13, 15]. We therefore follow [23] and consider parametrized neighbourhoods of S. Since S is orientable, there exists an open neighbourhood I of 0 and a smooth diffeomorphism

$$\vartheta\colon S \times I \to U \subset V$$

such that $\vartheta(x,0) = x$ for $x \in S$. We call ϑ a *thickening of* S, and we abbreviate $S_\epsilon = \vartheta(S \times \{\epsilon\})$. We denote by $\mathcal{P}^\circ(S_\epsilon)$ the set of closed characteristics on S_ϵ which are contractible in V. The refinement of the Hofer–Zehnder argument in [42] shows that if $c^\circ_{\mathrm{HZ}}(U,V)$ is finite, then $\mathcal{P}^\circ(S_\epsilon) \neq \emptyset$ for almost all $\epsilon \in I$. Together with Theorem 1.3 we obtain

Corollary 1.8. *Assume that S is a stably displaceable hypersurface in a tame symplectic manifold (W,ω). Then for any stably displaceable thickening $\vartheta\colon S \times I \to U \subset W$ it holds that $\mathcal{P}^\circ(S_\epsilon) \neq \emptyset$ for almost all $\epsilon \in I$.*

In [61], Zehnder constructed a symplectic form on the 4-torus $T^4 = (\mathbb{R}/\mathbb{Z})^4$ such that none of the hypersurfaces $\{x_4 = \mathrm{const}\}$ carries a closed characteristic. The assumption in Corollary 1.8 that S is stably displaceable thus cannot be omitted.

A hypersurface S in a symplectic manifold (V,ω) is called *of contact type* if there exists a Liouville vector field X (i.e., $\mathcal{L}_X\omega = d\iota_X\omega = \omega$) which is defined in a neighbourhood of S and is everywhere transverse to S. Weinstein conjectured in [59] that every hypersurface S of contact type with $H^1(S;\mathbb{R}) = 0$ carries a closed characteristic.

Corollary 1.9. *Assume that S is a stably displaceable hypersurface of contact type in a tame symplectic manifold (W,ω). Then $\mathcal{P}^\circ(S) \neq \emptyset$. In particular, the Weinstein conjecture holds true for S.*

Proofs of the Weinstein conjecture for *all* hypersurfaces of contact type of *special classes* of symplectic manifolds have been found in [56, 23, 21, 9, 22, 26, 38, 57, 36, 58, 5, 35, 37]. Corollary 1.9 generalizes or complements the results in [56, 23, 9, 58, 35], where the ambient symplectic manifold is of the form $(V \times \mathbb{R}^2, \omega \oplus \omega_0)$. Under the additional assumption that (W,ω) satisfies $[\omega]|_{\pi_2(W)} = 0$ and is convex, Corollary 1.9 has been proved in [10].

2. Periodic orbits of autonomous Hamiltonian systems

Given a Hamiltonian F on (V,ω) we denote by $\mathcal{P}^\circ(F^{-1}(r))$ the set of non-constant periodic orbits on $F^{-1}(r)$ which are contractible in V. A function $F\colon V \to \mathbb{R}$ is *proper* if $F^{-1}([r_0,r_1])$ is a compact subset of V

for all $-\infty < r_0 \leq r_1 < \infty$. For simplicity, we shall always assume that F is proper. We point out, however, that it is enough to assume that F is proper on the set of levels $F^{-1}([r_0, r_1])$ under consideration.

Corollary 1.10. *Consider a proper Hamiltonian F on a tame symplectic manifold (W, ω). If $[r_0, r_1] \subset F(W)$ and if $F^{-1}[r_0, r_1]$ is stably displaceable, then $\mathcal{P}^\circ\left(F^{-1}(r)\right) \neq \emptyset$ for almost all $r \in [r_0, r_1]$.*

From now on we focus on searching periodic orbits near the minimum set of F, which we can assume to be $F^{-1}(0)$. We abbreviate the sublevel set $F^{-1}\left([0,r]\right)$ by F^r. Given a proper Hamiltonian F on (V, ω) we define $d_1(F) \in [0, \infty]$ by

$$\begin{aligned} d_1(F) &= \sup\left\{r \in \mathbb{R} \mid F^r \text{ is stably displaceable}\right\} \\ &= \sup\left\{r \in \mathbb{R} \mid e_1\left(F^r, V\right) < \infty\right\}. \end{aligned}$$

Since F is proper, $d_1(F) > 0$ if and only if $F^{-1}(0)$ is stably displaceable. Corollary 1.10 yields

Corollary 1.11. *Consider a proper Hamiltonian F on a tame symplectic manifold (W, ω) with minimum 0, and assume that $d_1(F) > 0$. Then $\mathcal{P}^\circ\left(F^{-1}(r)\right) \neq \emptyset$ for almost all $r \in \,]0, d_1(F)]$.*

We recall that Corollary 1.11 becomes relevant in conjunction with Proposition 1.7 applied to $A = F^{-1}(0)$. Specific applications of Corollary 1.11 are given in the next paragraph. In the remainder of this paragraph we further discuss Corollary 1.11 and compare it with previous results of this kind.

Remarks 1.12. 1. (i) The Hamiltonian $(x,y) \mapsto x^2$ on $(\mathbb{R}^2, \omega_0)$ shows that the assumption that F is proper cannot be omitted.
(ii) Given $d > 0$ it is easy to construct a proper Hamiltonian F on (T^*S^1, ω_0) with minimum 0 such that $d_1(F) = d$ and $\mathcal{P}^\circ\left(F^{-1}(r)\right) \neq \emptyset$ for $r \in \,]0, d[$ and $\mathcal{P}^\circ\left(F^{-1}(r)\right) = \emptyset$ for $r > d$. More interesting examples showing that in general one cannot expect that $\mathcal{P}^\circ\left(F^{-1}(r)\right) \neq \emptyset$ for almost all $r > 0$ are given in Remark 1.15.2 below.
(iii) According to [15], every symplectic manifold (V, ω) of dimension $2n \geq 4$ admits a proper C^2-smooth Hamiltonian F with minimum 0 and $d_1(F) > 0$ such that $\mathcal{P}^\circ\left(F^{-1}(r)\right) = \emptyset$ for a sequence $r_k \to 0$ of regular values, and if $2n \geq 6$, then F can be chosen C^∞-smooth.
2. Assume that F is a proper Hamiltonian on any symplectic manifold (V, ω) attaining its minimum in a point. In view of Darboux's theorem we can assume that $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$. Corollary 1.11 or already Struwe's theorem [55, 25] show that $\mathcal{P}^\circ\left(F^{-1}(r)\right) \neq \emptyset$ for almost all sufficiently small $r > 0$.

3. Consider a tame symplectic manifold (W^{2n}, ω), and assume that the proper function $F\colon W \to \mathbb{R}$ attains its minimum 0 along a closed symplectic submanifold M^{2k} of (W, ω). In this situation, it was shown in [15, Corollary 2.16] and [40] that $\mathcal{P}^\circ\left(F^{-1}(r)\right) \neq \emptyset$ for almost all $r \in \,]0, b(F)]$, where

$$b(F) \;=\; \sup\left\{r \in \mathbb{R} \mid F^r \subset B(M,F)\right\} \;\in\;]0,\infty] \tag{6}$$

and $B(M,F)$ is "the F-maximal symplectic ball neighbourhood of M in (W,ω)", see [15, Section 4.1] for details. For $k \in \{0, 1, \dots, n\}$, this result is covered by Proposition 1.7 and Corollary 1.11 with $d_1(F) > 0$ instead of $b(F)$. It would be interesting to compare these two constants.

4. The main point of Corollary 1.11 is that there is no assumption on the infinitesimal behaviour of F near $F^{-1}(0)$. If $F^{-1}(0)$ is a Morse-Bott non-degenerate minimum of F, then $\mathcal{P}^\circ\left(F^{-1}(r)\right) \neq \emptyset$ *for all* sufficiently small $r > 0$ if $F^{-1}(0)$ is a point [59] or if $F^{-1}(0)$ is a symplectic submanifold and the transverse eigenvalues of D^2F along $F^{-1}(0)$ meet a global resonance condition [27].

3. Closed trajectories of a charge in a magnetic field and a potential

Consider a closed Riemannian manifold (M, g) of dimension at least 2, and let $\omega_0 = \sum_i dp_i \wedge dq_i$ be the standard symplectic form on the cotangent bundle T^*M. We fix a closed 2-form σ on M and define the twisted symplectic form ω_σ on $\pi\colon T^*M \to M$ by $\omega_\sigma = \omega_0 + \pi^*\sigma$. We also fix a function V on M with minimum 0. The flow of the Hamiltonian system

$$F_V\colon\; (T^*M, \omega_\sigma) \to \mathbb{R}, \qquad F_V(q,p) \mapsto \tfrac{1}{2}\,|p|^2 + V(q), \tag{7}$$

describes (for example) the motion of a unit charge on (M, g) subject to the magnetic field σ and the potential V, cf. [45, 28, 12]. As before, we denote by $\mathcal{P}^\circ\left(F_V^{-1}(r)\right)$ the set of periodic orbits on the level $F_V^{-1}(r)$ which are contractible in T^*M and hence project to contractible closed trajectories on M. We shall also study the possibly larger set $\mathcal{P}\left(F_V^{-1}(r)\right)$ of periodic orbits on $F_V^{-1}(r)$; these orbits might not be contractible in T^*M. For historical reasons, we first discuss our result for the case $V = 0$. We set $d_1(g,\sigma) = d_1(F_0)$ and $E_r = F_0^{-1}(r)$.

Corollary 1.13. *Consider a closed Riemannian manifold (M, g) endowed with a closed 2-form σ which does not vanish identically. Then $d_1(g,\sigma) > 0$ and $\mathcal{P}^\circ(E_r) \neq \emptyset$ for almost all $r \in \,]0, d_1(g,\sigma)]$.*

If σ is exact, $d_1(g, \sigma)$ is always finite in view of inequality (10) below. If σ is non-exact, $d_1(g, \sigma)$ can be infinite, however. Examples with infinite $d_1(g,\sigma)$ are non-exact closed 2-forms σ on tori. Using this we shall obtain

Corollary 1.14. *Assume that M is a manifold of the form $M = T^k \times M_2$, where T^k is a torus of dimension $k \geq 1$ and M_2 is any closed manifold, and assume that M is endowed with a Riemannian metric g and a non-vanishing closed 2-form σ such that*

$$[\sigma] \;=\; [\sigma_1] \oplus [\sigma_2] \;\in\; H^2\left(M_1\right) \oplus H^2\left(M_2\right) \;\subset\; H^2(M).$$

(i) *If $[\sigma_1] \neq 0$, then $\mathcal{P}^\circ\left(E_r\right) \neq \emptyset$ for almost all $r > 0$.*
(ii) *If $[\sigma_1] = 0$, then $\mathcal{P}\left(E_r\right) \neq \emptyset$ for almost all $r > 0$.*

Remarks 1.15. 1. As we shall prove in Proposition 6.2 below, "for most M and σ" the hypersurfaces E_r are not of contact type. Therefore, Corollaries 1.13 and 1.14 do not follow from existence results of closed characteristics on contact type hypersurfaces nor do they imply that $\mathcal{P}^\circ\left(E_r\right) \neq \emptyset$ for all sufficiently small $r > 0$.

2. One cannot expect that $\mathcal{P}^\circ\left(E_r\right) \neq \emptyset$ for almost all $r > 0$ in general. Indeed, let M be a closed oriented surface of genus 2, and let g and σ either be a Riemannian metric of constant curvature -1 and its area form or the Riemannian metric and the exact 2-form σ constructed in [47]. Then $\mathcal{P}^\circ\left(E_r\right) = \emptyset$ for all $r \geq \frac{1}{2}$, see [12, Example 3.7] and [47]. On the other hand, there are no examples known with $\mathcal{P}\left(E_r\right) = \emptyset$ for an infinite set of $r > 0$.

3. The state of the art. We review the known existence results for periodic orbits of a charge in a magnetic field on small energy levels; for results concerned with periodic orbits on intermediate and large energy levels we refer to [12]. As before, M is a closed manifold endowed with a Riemannian metric g and a non-vanishing closed 2-form σ.

A. Previous results. Corollary 1.13 generalizes various previous results: The existence of a sequence $r_k \to 0$ with $\mathcal{P}^\circ\left(E_{r_k}\right) \neq \emptyset$ has been proved by Polterovich [49] and Macarini [39] under the assumption that $[\sigma]|_{\pi_2(M)} = 0$ and by Ginzburg–Kerman [17] under the assumption that σ is symplectic. For exact forms, Corollary 1.13 has been proved in [10], and for rational symplectic forms in [6, 15, 39, 40] with $d_1(g,\sigma)$ replaced by the number $b(F_0)$ defined in (6). Corollary 1.14 improves Theorems 12.9 and 12.10 of [10] which extended results of Lu [36].

B. Special results. Under additional assumptions on M, g or σ, Corollary 1.13 can be improved.

B1. Denote by $\mathrm{Cl}(M)$ the cup-length of M. If M is a surface and σ is symplectic or, more generally, if σ is a symplectic form compatible with g, then

$$\#\left(\mathcal{P}^\circ\left(E_r\right)\right) \;\geq\; \mathrm{Cl}(M) + \tfrac{1}{2}\dim M \tag{8}$$

for all sufficiently small $r > 0$, see [11, 27].

B2. Assume that $M = M_1 \times M_2$, where M_1 is a simply connected manifold admitting an effective semi-free circle action. Examples are spheres S^k with $k \geq 2$ and connected sums of $\mathbb{C}\mathrm{P}^k$'s. If σ is exact, then $\mathcal{P}^\circ\left(E_r\right) \neq \emptyset$ for almost all $r > 0$, see [41].

C. Results for $\mathcal{P}(E_r)$.

C1. If M is a surface and σ is exact, then $\mathcal{P}\left(E_r\right) \neq \emptyset$ for all $r > 0$, [8].

C2. Assume that $M = M_1 \times M_2$ and that M_1 admits an effective semi-free circle action. If σ is exact, then $\mathcal{P}\left(E_r\right) \neq \emptyset$ for almost all $r > 0$, see [41].

For the 2-sphere, which is of particular interest [45, 28], the state of the art thus is

Corollary 1.16. *Assume that g is a Riemannian metric on S^2 and that $\sigma \neq 0$ is a closed 2-form on S^2. Then $\mathcal{P}^\circ(E_r) \neq \emptyset$ for almost all $r \in \left]0, d_1(g,\sigma)\right]$, for all $r > 0$ if σ is exact, and for all sufficiently small $r > 0$ if σ is symplectic.*

4. Outlook. (i) Corollary 1.13 and the results listed above suggest that $\mathcal{P}^\circ\left(E_r\right) \neq \emptyset$ *for all* sufficiently small $r > 0$. In view of Proposition 2.24 of [15] this conjecture cannot, however, be established by purely symplectic topological means, and one must make use of the convexity of the function $\frac{1}{2}|p|^2$ along the fibres. If σ is symplectic, there is further evidence for this conjecture, and one in fact expects a lower bound like (8) for all sufficiently small $r > 0$, see [16, 27].

(ii) Assume that σ is exact. We define the norm of σ as

$$\|\sigma\| \;=\; \inf\left\{\|\alpha\| \mid d\alpha = \sigma\right\} \tag{9}$$

where $\|\alpha\| = \max_{x\in M}|\alpha(x)|$. According to [10, 51],

$$d_1(g,\sigma) \;\leq\; \tfrac{1}{2}\left\|\sigma\right\|^2, \tag{10}$$

but for the example from [47] mentioned in Remark 1.15.2 we have $d_1(g,\sigma) < \frac{1}{2}\left\|\sigma\right\|^2$. In view of [10] and Corollary 1.11, $\mathcal{P}^\circ\left(E_r\right) \neq \emptyset$ for almost all $r \in \left]0, d_1(g,\sigma)\right]$, and in view of [21, 7], $\mathcal{P}\left(E_r\right) \neq \emptyset$ for all $r > \frac{1}{2}\left\|\sigma\right\|^2$. In order to close the possible gap between $d_1(g,\sigma)$ and $\frac{1}{2}\left\|\sigma\right\|^2$, it suffices to show that $c_{\mathrm{HZ}}(A)$ is finite for all bounded subsets A of (T^*M, ω_0). While this result is believed to be true, it is known only for the manifolds as in C2 above, see [41]. $\diamond$

The approach used in [6, 15, 39, 40, 41] makes explicit use of the geometry of the sphere bundle $E_r \to M$ over the symplectic manifold M. This geometry does not exist if M is not symplectic or $V \neq 0$. Since our approach only uses that the zero locus $F_V^{-1}(0)$ is stably displaceable, it also yields the following generalizations of Corollaries 1.13 and 1.14.

Corollary 1.17. *Consider a closed Riemannian manifold (M,g) endowed with a closed 2-form σ which does not vanish identically, and let V be a potential on M with minimum 0. Then $d_1(F_V) > 0$ and $\mathcal{P}^\circ\left(F_V^{-1}(r)\right) \neq \emptyset$ for almost all $r \in \,]0, d_1(F_V)]$.*

Corollary 1.18. *Let V be a potential with minimum 0 on a manifold (M,g,σ) as in Corollary 1.14.*

(i) *If $[\sigma_1] \neq 0$, then $\mathcal{P}^\circ\left(F_V^{-1}(r)\right) \neq \emptyset$ for almost all $r > 0$.*
(ii) *If $[\sigma_1] = 0$, then $\mathcal{P}\left(F_V^{-1}(r)\right) \neq \emptyset$ for almost all $r > 0$.*

Remark 1.19. For a proper potential V and under some additional assumptions on M and σ, Corollaries 1.17 and 1.18 continue to hold for open manifolds M, see [51].

4. Lagrangian intersections

According to a celebrated theorem of Gromov, [18, 2.3.B_3'], a closed Lagrangian submanifold L of a tame symplectic manifold (W,ω) with $[\omega]|_{\pi_2(W,L)} = 0$ is not displaceable. The following result generalizes Theorem 13.1 in [10] and complements Theorem 1.4.A in [32].

Corollary 1.20. *Assume that L is a closed Lagrangian submanifold of a tame symplectic manifold (W,ω) such that*

(i) *the injection $L \subset W$ induces an injection $\pi_1(L) \subset \pi_1(W)$;*
(ii) *L admits a Riemannian metric none of whose closed geodesics is contractible.*

Then L is not displaceable.

Remarks 1.21. 1. Neither condition (i) nor (ii) can be omitted: This is clear for (i) in view of the Lagrangian torus $T^n \subset \mathbb{C}^n$. For (ii) we follow [1, 3]: Let $S^{2k+1} \subset \mathbb{C}^{k+1}$ be the unit sphere, and let ω_{SF} be the Study–Fubini form on $\mathbb{C}\mathrm{P}^k$ normalized so that the pull-back of ω_{SF} by the Hopf fibration $h\colon S^{2k+1} \to \mathbb{C}\mathrm{P}^k$ is equal to the restriction of the standard form ω_0 on $\mathbb{C}^{k+1}$. Then

$$L = \left\{(\bar z, h(z)) \mid z \in S^{2k+1}\right\} \subset \mathbb{C}^{k+1} \times \mathbb{C}\mathrm{P}^k$$

is a displaceable Lagrangian sphere in $\left(\mathbb{C}^{k+1} \times \mathbb{C}\mathrm{P}^k, \omega_0 \oplus \omega_{\mathrm{SF}}\right)$. Multiplying these spheres with $S^1 \subset (T^*S^1, \omega_0)$ we obtain examples in all dimensions $n \geq 3$.

2. Condition (ii) is weaker than

(ii') L admits a Riemannian metric of non-positive curvature.

Indeed, for each $k \geq 1$ there exists a $(2k+1)$-dimensional nilmanifold meeting (ii) but not (ii'), see [10, Remark 13.2.3]. Under conditions (i) and (ii'), the conclusion of Corollary 1.20 was proved for closed Lagrangian submanifolds of *arbitrary* symplectic manifolds in [32] by using the general energy-capacity inequality (4). $\diamond$

The paper is organized as follows. In Section 2 we prove Theorems 1.3, 1.5 and 1.6, discuss various displacement energies, and prove Proposition 1.7. In Section 3 we first show that the Hofer–Zehnder capacity c_{HZ}° agrees with a modification of it needed in the proof of Theorem 1.5, and then review the almost existence theorems obtained from the finiteness of c_{HZ}°. In Sections 4, 5, 6 and 7 we prove the corollaries stated in Paragraphs 1,2,3 and 4, respectively. In Section 6 we also prove Proposition 6.2 mentioned in Remark 1.15.1. In Section 8 we notice that most of our results continue to hold for C^2-smooth hypersurfaces, Hamiltonians, Riemannian metrics, potentials and Lagrangian submanifolds.

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2. Proof of the main theorems and of Proposition 1.7

2.1. **Proof of Theorem 1.3.** We follow Polterovich's beautiful argument in [49, Section 9.A]. The proof consists of two steps.

Step 1. Curve shortening in Hofer's geometry

Curve shortening in Hofer's geometry was invented by Sikorav in [53] and further developed in [30, Proposition 2.2]. Here, we closely follow the proof of Theorem 8.3.A in [50], see also Theorem 3.3.A in [2].

We consider an arbitrary symplectic manifold (V, ω). Two Hamiltonians $H, K \in \mathcal{H}_c(I \times V)$ are *equivalent*, $H \sim K$, if $h = k$ and the paths $\{h_t\}, \{k_t\}$, $t \in [0,1]$, are homotopic in $\mathrm{Ham}_c(V,\omega)$ with fixed end points. In other words, there exists a smooth family $\{H^s\}$, $s \in [0,1]$, in $\mathcal{H}_c(I \times V)$ such that $h_t^0 = h_t$ and $h_t^1 = k_t$ for all t and $h^s = h = k$ for all s. The group of equivalence classes $\mathcal{H}_c(I \times V)/\sim$ form the universal cover $\widetilde{\mathrm{Ham}}_c(V,\omega)$ of $\mathrm{Ham}_c(V,\omega)$. We denote the lift of the Hofer norm to $\widetilde{\mathrm{Ham}}_c(V,\omega)$ by

$$\rho\,[h_t] \;\equiv\; \rho[H] \;:=\; \inf\left\{\|K\| \mid K \sim H\right\}.$$

Proposition 2.1. *Consider a displaceable subset A of an arbitrary symplectic manifold (V,ω). If $F\colon V \to \mathbb{R}$ is supported in A and $\|F\| > 4\,e(A,V)$, then $\rho\,[F] < \|F\|$.*

Proof. Choose a path $\{h_t\}$, $t \in [0,1]$, in $\mathrm{Ham}_c(V,\omega)$ such that $h(A) \cap A = \emptyset$ and

$$\rho\,[h_t] \;<\; \tfrac{1}{4}\,\|F\|\,. \tag{11}$$

For $t \in [0,1]$ we decompose the path f_t as

$$f_t \;=\; \left(f_{t/2} \circ h_t \circ f_{t/2} \circ h_t^{-1}\right) \circ \left(h_t \circ f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2}\right) \;\equiv\; b_t \circ a_t.$$

As we shall see below,

$$\rho\,[a_t] < \tfrac{1}{2}\,\|F\| \qquad \text{and} \qquad \rho\,[b_t] \le \tfrac{1}{2}\,\|F\|\,. \tag{12}$$

Since $\{b_t \circ a_t\}$ is equivalent to the juxtaposition of $\{a_t\}$ and $\{b_t \circ a_1\}$ and since ρ satisfies the triangle inequality, the estimates (12) imply Proposition 2.1. In order to prove the first estimate in (12), notice that the paths $\left\{f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2}\right\}$ and $\left\{f_{1/2}^{-1} \circ h_t^{-1} \circ f_{1/2}\right\}$ are equivalent and that

$$\rho\left[f_{1/2}^{-1} \circ h_t^{-1} \circ f_{1/2}\right] \;=\; \rho\left[h_t^{-1}\right] \;=\; \rho\,[h_t]\,.$$

Together with the triangle inequality and the estimate (11) we can estimate

$$\begin{aligned}
\rho\,[a_t] \;&=\; \rho\left[h_t \circ f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2}\right] \\
&\le\; \rho\,[h_t] + \rho\left[f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2}\right] \\
&=\; 2\,\rho\,[h_t] \\
&<\; \tfrac{1}{2}\,\|F\|\,.
\end{aligned}$$

In order to prove the second estimate in (12), notice that the path $\{b_t\} = \left\{f_{t/2} \circ h_t \circ f_{t/2} \circ h_t^{-1}\right\}$ is equivalent to the path $\left\{f_{t/2} \circ h \circ f_{t/2} \circ h^{-1}\right\}$ generated by the Hamiltonian

$$K(t,x) \;=\; \tfrac{1}{2}F(x) + \tfrac{1}{2}F\left(h^{-1}f_{t/2}^{-1}x\right), \quad t \in [0,1].$$

Since F is autonomous, $F = F \circ f_{t/2}$, and since h displaces $\operatorname{supp} F \subset A$, so does h^{-1}. Therefore,

$$\begin{aligned} \|K_t\| &= \tfrac{1}{2}\left\|F + F \circ h^{-1} \circ f_{t/2}^{-1}\right\| \\ &= \tfrac{1}{2}\left\|F \circ f_{t/2} + F \circ h^{-1}\right\| \\ &= \tfrac{1}{2}\left\|F + F \circ h^{-1}\right\| \\ &= \tfrac{1}{2}\|F\|, \end{aligned}$$

and so $\rho\,[b_t] \leq \frac{1}{2}\|F\|$. The proof of Proposition 2.1 is complete. □

Step 2. The cut point has a non-constant contractible periodic orbit

We restate Theorem 1.5 as

Proposition 2.2. *Assume that (W,ω) is a tame symplectic manifold and that $F \in \mathcal{H}_c(W)$ is an autonomous Hamiltonian such that $\rho\,[F] < \|F\|$. Then F is not slow.*

Proof. We first consider an arbitrary symplectic manifold (V,ω) and recall from the introduction that $F \in \mathcal{H}_c(V)$ is *slow* if all non-constant contractible periodic orbits of f_t have period > 1. We say that $F \in \mathcal{H}_c(V)$ is *flat* if all non-constant periodic orbits of the linearized flow of F at its critical points have period > 1. Following [44] we define for each subset A of (V,ω) the Hofer–Zehnder capacity

$$c_{\mathrm{HZ}}^{\circ,f}(A,V,\omega) \;=\; \sup\left\{\max F \mid F \in \mathcal{H}_c\left(\mathrm{Int}(A)\right) \text{ is slow and flat}\right\}. \tag{13}$$

We again omit ω from the notation and abbreviate $c_{\mathrm{HZ}}^{\circ,f}(V,V) = c_{\mathrm{HZ}}^{\circ,f}(V)$.

Let now (W,ω) and F be as in Proposition 2.2. We first assume that W is closed. According to Proposition 3.9 of [44],

$$c_{\mathrm{HZ}}^{\circ,f}\left(W \times B^2(r), \omega \oplus \omega_0\right) \;\leq\; \pi r^2 \quad \text{for all } r > 0.$$

In fact, this area-capacity inequality is shown to hold for all quasi-cylinders over (W,ω), and this implies that F cannot be slow *and* flat, see [31]. As we shall prove in Proposition 3.1, the capacities $c_{\mathrm{HZ}}^{\circ,f}$ and c_{HZ}° always agree, and so the arguments in [31] in fact yield Proposition 2.2 as stated. Assume now that (W,ω) is an arbitrary tame symplectic manifold. Then the compactness theorems in [18, 54] hold, and so the arguments in [44] establishing compactness of the relevant Floer moduli space go through. □

End of the proof of Theorem 1.3. We can assume that $e(A,W) < \infty$, and in view of the definitions of the capacity c_{HZ}° and the displacement energy e we can assume that A is compact. Hence A is displaceable. Let $F \in \mathcal{H}_c(\operatorname{Int} A)$ be such that $\max F > 4\,e(A,W)$. Then $\|F\| > 4\,e(A,W)$. According to Proposition 2.1 we have $\rho\,[F] < \|F\|$, and so Proposition 2.2 shows that F is not slow. Therefore, $c_{\mathrm{HZ}}^{\circ}(A,W) \leq 4\,e(A,W)$. □

2.2. **Stabilization.**

Definition 2.3. A *stabilizer* is a triple (A_S, V_S, ω_S) consisting of a symplectic manifold (V_S, ω_S) and a subset A_S of V_S such that there exists a Hamiltonian $F_S \colon V_S \to [0,\infty)$ with the following properties.

(i) $F_S^{-1}(0) = A_S$.
(ii) There exists $\epsilon > 0$ such that F_S^{ϵ} is compact and such that all contractible period orbits of the flow of F_S contained in F_S^{ϵ} are constant.

Denote the class of stabilizers by $\mathcal{S}$, and by $\mathcal{S}^{\tau}$ the subclass of stabilizers for which (V_S, ω_S) is tame.

Example 2.4. Given any closed manifold M admitting a Riemannian metric of non-positive curvature (or, more generally, admitting a Riemannian metric none of whose closed geodesics is contractible) the triple (M, T^*M, ω_0) belongs to $\mathcal{S}^{\tau}$. Indeed, the geodesic flow Hamiltonian $(q,p) \mapsto \frac{1}{2}|p|^2$ with respect to such a metric does the job. By the Symplectic Neighbourhood Theorem, a neighbourhood of a closed Lagrangian submanifold L in an arbitrary symplectic manifold symplectically identifies with a neighbourhood of L in (T^*L, ω_0). A triple (L, W, ω) as in Corollary 1.20 thus belongs to $\mathcal{S}^{\tau}$.

Remark 2.5. The set A_S of a stabilizer $(A_S, W_S^{2n}, \omega_S) \in \mathcal{S}^{\tau}$ cannot be contained in a finite CW-complex of dimension $< n$. Indeed, using such a stabilizer and Proposition 1.7 (i) and Corollary 5.2 we could conclude that the flow of $(q,p) \mapsto \frac{1}{2}p^2$ on (T^*S^1, ω_0) has contractible periodic orbits on almost all small enough energy levels. ◇

For every subset A of a symplectic manifold (V,ω) we define the stable displacement energy $e_S^{\tau}(A,V) \in [0,\infty]$ by

$$e_S^{\tau}(A,V) \;=\; \inf\left\{ e\left(A \times A_S, V \times W_S\right) \mid (A_S, W_S, \omega_S) \in \mathcal{S}^{\tau} \right\}.$$

Since $e_1 \geq e_S^{\tau}$, Theorem 1.6 follows from

Theorem 2.6. *Assume that A is a subset of a tame symplectic manifold (W, ω). Then*

$$c_{\mathrm{HZ}}^{\circ}(A, W) \leq 4\, e_S^{\tau}(A, W).$$

Proof. We shall derive Theorem 2.6 from Theorem 1.3. We start with

Lemma 2.7. *Let (A_S, V_S, ω_S) be a stabilizer and choose F_S and $\epsilon > 0$ as in Definition 2.3. For any subset A of a symplectic manifold (V, ω),*

$$c_{\mathrm{HZ}}^{\circ}(A, V) \;\leq\; c_{\mathrm{HZ}}^{\circ}\left(A \times F_S^{\epsilon}, V \times V_S\right).$$

Proof. We can assume that $\operatorname{Int} A \neq \emptyset$. Let $F \in \mathcal{H}_c(\operatorname{Int} A)$ be slow. We choose a smooth function $a \colon \mathbb{R} \to [0,1]$ such that

$$a(t) = 1 \;\text{ if } t \leq \tfrac{1}{3}\epsilon \quad \text{ and } \quad a(t) = 0 \;\text{ if } t \geq \tfrac{2}{3}\epsilon.$$

The function $G \colon V \times V_S \to \mathbb{R}$, $(v, v_S) \mapsto F(v)\, a(F_S(v_S))$ belongs to $\mathcal{H}_c\left(\operatorname{Int}\left(A \times F_S^{\epsilon}\right)\right)$. In order to see that G is slow, assume that $x(t)$ is a contractible periodic orbit of g_t. Then $x(t) = (x_1(t), x_2(t)) \subset V \times V_S$, where both $x_1(t)$ and $x_2(t)$ are contractible periodic orbits. Denoting the Hamiltonian vector fields of F and F_S by X_F and X_{F_S}, we find

$$\begin{aligned} \dot{x}_1(t) &= a\big(F_S(x_2(t))\big)\, X_F\left(x_1(t)\right), \\ \dot{x}_2(t) &= F(x_1(t))\, a'\big(F_S(x_2(t))\big)\, X_{F_S}\left(x_2(t)\right). \end{aligned}$$

Since F and F_S do not depend on t, the functions $a\big(F_S(x_2(t))\big)$ and $F(x_1(t))\, a'\big(F_S(x_2(t))\big)$ are constant. Since $\big|a\big(F_S(x_2)\big)\big| \in [0,1]$ and F is slow, the orbit $x_1(t)$ is constant or has period > 1, and since all contractible periodic orbits of the flow of F_S in F_S^{ϵ} are constant, the orbit $x_2(t)$ is constant. We have constructed for every slow $F \in \mathcal{H}_c(\operatorname{Int} A)$ a slow $G \in \mathcal{H}_c\left(\operatorname{Int}\left(A \times F_S^{\epsilon}\right)\right)$ with $\max F = \max G$. Lemma 2.7 thus follows. □

In order to prove Theorem 2.6 we need to show that for every stabilizer $(A_S, W_S, \omega_S) \in \mathcal{S}^{\tau}$ and every compact subset A of W,

$$c_{\mathrm{HZ}}^{\circ}(A, W) \;\leq\; 4\, e\left(A \times A_S, W \times W_S\right). \tag{14}$$

So fix $(A_S, W_S, \omega_S) \in \mathcal{S}^{\tau}$ and a compact subset A of W. We can assume that $e\left(A \times A_S, W \times W_S\right)$ is finite. Fix $\delta > 0$, and choose $H \in \mathcal{H}_c\left(I \times W \times W_S\right)$ such that h displaces $A \times A_S$ and

$$\|H\| \;\leq\; e\left(A \times A_S, W \times W_S\right) + \delta.$$

We then find F_S and $\epsilon > 0$ as in Definition 2.3 such that h displaces $A \times F_S^{\epsilon}$. It follows that

$$e\left(A \times F_S^{\epsilon}, W \times W_S\right) \;\leq\; \|H\| \;\leq\; e\left(A \times A_S, W \times W_S\right) + \delta.$$

Since both (W, ω) and (W_S, ω_S) are tame, so is their product $(W \times W_S, \omega \oplus \omega_S)$. Together with Lemma 2.7 and Theorem 1.3 we can thus estimate

$$\begin{aligned} c^\circ_{\mathrm{HZ}}(A, W) &\le c^\circ_{\mathrm{HZ}}(A \times F^\epsilon_S, W \times W_S) \\ &\le 4\, e\,(A \times F^\epsilon_S, W \times W_S) \\ &\le 4\, e\,(A \times A_S, W \times W_S) + 4\,\delta. \end{aligned}$$

Since $\delta > 0$ was arbitrary, inequality (14) follows, and so Theorem 2.6 is proved. □

In the remainder of this subsection we further discuss the invariant e^τ_S.

Lemma 2.8. *For any subset A of a symplectic manifold (V, ω) and any stabilizer $(A_S, V_S, \omega_S) \in \mathcal{S}$,*

$$e\,(A \times A_S, V \times V_S) \;\le\; e\,(A, V)\,.$$

Proof. We can assume that A is compact and that $e\,(A, V)$ is finite. Fix $\delta > 0$, and choose $H \in \mathcal{H}_c\,(I \times V)$ such that h displaces A and $\|H\| \le e(A, V) + \delta$. Choose now F_S and $\epsilon > 0$ as in Definition 2.3, and choose $a\colon \mathbb{R} \to [0, 1]$ as in the proof of Lemma 2.7. The Hamiltonian

$$K\colon I \times V \times V_S \to \mathbb{R}, \qquad (v, v_S) \mapsto H(t, v)\, a(F_S(v_S)),$$

belongs to $\mathcal{H}_c\,(I \times V \times V_S)$, and its time-1-map k restricts to $h \times \mathrm{id}$ on $V \times A_S$. Since h displaces A, we conclude that k displaces $A \times A_S$, and so

$$e\,(A \times A_S, V \times V_S) \;\le\; \|K\| \;=\; \|H\| \;\le\; e(A, V) + \delta.$$

Since $\delta > 0$ was arbitrary, Lemma 2.8 follows. □

Given two stabilizers in $\mathcal{S}^\tau$, we write $(A_S, W_S, \omega_S) \le (A'_S, W'_S, \omega'_S)$ if there exists $(A''_S, W''_S, \omega''_S) \in \mathcal{S}^\tau$ such that

$$(A'_S, W'_S, \omega'_S) \;=\; (A_S \times A'_S, W_S \times W''_S, \omega_s, \oplus \omega''_S)\,.$$

Since the 0-function on a point is a stabilizer, $(\mathcal{S}^\tau, \le)$ is a partially ordered set, which is directed. In view of Lemma 2.8, $e^\tau_S(A, V)$ is the direct limit

$$e^\tau_S(A, V) \;=\; \lim_{(\mathcal{S}^\tau, \le)} e\,(A \times A_S, V \times W_S)\,.$$

For each $k = 1, 2, \dots$ we let $T^k = \times_k S^1$ be the k-torus and set

$$e_k(A, V) \;=\; e\left(A \times T^k, V \times T^*T^k, \omega \oplus \omega_0\right).$$

By Lemma 2.8, the sequence $e_k(A, V)$ is decreasing, so that the limit

$$e_\infty(A, V) \;=\; \lim_{k \to \infty} e_k\,(A, V)$$

exists, and

$$e(A,V) \geq e_1(A,V) \geq e_\infty(A,V) \geq e_S^\tau(A,V). \tag{15}$$

At this point we must admit that we have introduced the invariants e_S^τ and e_∞ for conceptual reasons only, and in the hope that interesting examples with $\infty = e_1(A,V) > e_S^\tau(A,V)$ will be found. In fact, we do not know any such example. It would also be interesting to know an example with $\infty > e(A,V) > e_1(A,V)$ or $\infty = e_1(A,V) > e_2(A,V)$.

Examples 2.9. **1.** Let σ be a non-vanishing closed 2-form on a closed manifold M and let ω_σ be the twisted symplectic form on T^*M as in Paragraph 3 of the introduction. If the Euler characteristic $\chi(M)$ vanishes, then $e(M,T^*M) = 0$, and if $\chi(M)$ does not vanish, then $e(M,T^*M) = \infty$ for topological reasons, while $e_1(M,T^*M) = 0$, see Proposition 2.10 (ii).

2. Consider the unit circle $S^1 \subset (\mathbb{R}^2, \omega_0)$. If follows from [4] that $e_\infty(S^1,\mathbb{R}^2) \geq e(S^1,\mathbb{R}^2)$, and so $\pi = e(S^1,\mathbb{R}^2) = e_\infty(S^1,\mathbb{R}^2)$.

3. For every stabilizer $(A_S, W_S, \omega_S) \in \mathcal{S}^\tau$ we have $e_S^\tau(A_S, W_S) = \infty$, see Corollary 7.1.

2.3. **Proof of Proposition 1.7.** Recall that a compact subset A of a symplectic manifold (V,ω) is stably displaceable if and only if $e_1(A,V) < \infty$. The following proposition thus refines Proposition 1.7.

Proposition 2.10. *Let A be a compact subset of a symplectic manifold (V^{2n},ω).*

(i) *Assume that A is contained in an embedded finite CW-complex X of dimension $< n$. Then $e_1(A,V) < \infty$.*
(ii) *Assume that A is contained in an n-dimensional closed submanifold M which is not Lagrangian. Then $e_1(A,V) = 0$.*
(iii) *Assume that A is strictly contained in a closed Lagrangian submanifold L. Then $e_1(A,V) = 0$.*

Proof. (i) By assumption, the set $A \times S^1$ is contained in the finite CW-complex $X \times S^1$ of dimension $< n+1$ in the $(2n+2)$-dimensional symplectic manifold $(V \times T^*S^1, \omega \oplus \omega_0)$. Since $X \times S^1$ can be displaced from itself in $V \times T^*S^1$ by a smooth isotopy, a result of Laudenbach [33] implies that $X \times S^1$ is displaceable in $(V \times T^*S^1, \omega \oplus \omega_0)$. It follows that X and hence A are stably displaceable, and so $e_1(A,V) < \infty$.

(ii) Consider the closed submanifold $M \times S^1$ of $V \times T^*S^1$. Since $\omega|_M \neq 0$ we have $\omega \oplus \omega_0|_{M\times S^1} \neq 0$. Moreover, the Euler characteristic of $M \times S^1$ vanishes. A result of Polterovich [48] and Laudenbach–Sikorav [34] thus implies that $e(M \times S^1, V \times T^*S^1) = 0$, and so $e_1(A,V) = 0$.

(iii) The proof of the case $n = 1$ is elementary and omitted. So assume that $n \geq 2$. Since A is compact, $L \setminus A$ is open. Using the Lagrangian Neighbourhood Theorem we easily find a closed submanifold L' of V which is not Lagrangian and such that $A \subset L'$. By assertion (ii) we have $e_1(L', V) = 0$, and so $e_1(A, V) = 0$. □

The example $S^1 \subset (T^*S^1, \omega_0)$ shows that neither the dimension assumption in (i) nor the assumption $\omega|_M \neq 0$ in (ii) nor the assumption $A \subsetneq L$ in (iii) can be omitted.

3. Hofer–Zehnder capacities and almost existence

In this section we first show that the Hofer–Zehnder capacities c_{HZ}° and $c_{\text{HZ}}^{\circ,f}$ defined in (1) and (13) agree, and then review the almost existence theorems obtained from the finiteness of c_{HZ}°.

The following proposition, which was pointed out to me by Viktor Ginzburg, answers a question in [44].

Proposition 3.1. *For any subset A of a symplectic manifold (V, ω),*

$$c_{\text{HZ}}^{\circ,f}(A, V) \;=\; c_{\text{HZ}}^\circ(A, V).$$

Proof. The inequality $c_{\text{HZ}}^{\circ,f}(A, V) \leq c_{\text{HZ}}^\circ(A, V)$ follows from definitions. In order to show the reverse inequality, it suffices to construct for any slow $F \in \mathcal{H}_c(A)$ and any $\epsilon > 0$ a slow and flat $G \in \mathcal{H}_c(A)$ such that $\max G \geq \max F - \epsilon$. Let $F \in \mathcal{H}_c(A)$ be slow and fix $\epsilon > 0$. Since F is smooth and compactly supported and by Sard's theorem, the set C of critical values of F is compact and has zero Lebesgue measure. If $F(A) = [a, b]$, we thus find finitely many intervals $[a_i, b_i] \subset [a, b] \setminus C$ such that $\sum_i (b_i - a_i) \geq (b - a) - \epsilon$. Choose a smooth function $r \colon [a, b] \to \mathbb{R}$ such that $r(a) = a$ and such that $0 \leq r'(t) \leq 1$ for all t and

$$r'(t) = 1 \;\text{ if }\; t \in \bigcup_i [a_i, b_i] \qquad \text{and} \qquad r'(t) = 0 \;\text{ if }\; t \in C.$$

The function $G = r \circ F$ belongs to $\mathcal{H}_c(A)$ and is both slow and flat. Moreover,

$$\max G \;=\; r(b) \;\geq\; r(a) + (b - a) - \epsilon \;=\; \max F - \epsilon,$$

as we set out to prove. □

We now come to the almost existence theorems for closed characteristics near hypersurfaces and for periodic orbits of autonomous Hamiltonians.

Theorem 3.2. *Consider a hypersurface S in a symplectic manifold (V, ω). For any thickening $\vartheta \colon S \times I \to U \subset V$ with $c^{\circ}_{\mathrm{HZ}}(U, V) < \infty$ it holds that $\mathcal{P}^{\circ}(S_\epsilon) \neq \emptyset$ for almost all $\epsilon \in I$.*

We refer to [25, Sections 4.1 and 4.2] and [42] for a proof. Using that the set of critical values of a compactly supported smooth function is compact and, by Sard's theorem, of Lebesgue measure zero, we obtain

Corollary 3.3. *Consider a smooth function F on a symplectic manifold (V, ω), and assume that $[r_0, r_1] \subset F(V)$ and that $F^{-1}([r_0, r_1])$ is compact. If $c^{\circ}_{\mathrm{HZ}}(F^{-1}([r_0, r_1]), V) < \infty$, then $\mathcal{P}^{\circ}(F^{-1}(r)) \neq \emptyset$ for almost all $r \in [r_0, r_1]$.*

4. Proof of Corollaries 1.8 and 1.9

Corollary 1.8 is a special case of the following corollary which follows from Theorem 2.6 and Theorem 3.2.

Corollary 4.1. *Consider a thickening $\vartheta \colon S \times I \to U \subset W$ of a hypersurface S in a tame symplectic manifold (W, ω). If $e^{\tau}_{S}(U, W) < \infty$, then $\mathcal{P}^{\circ}(S_\epsilon) \neq \emptyset$ for almost all $\epsilon \in I$.*

A hypersurface S in a symplectic manifold (V, ω) is *stable* if there exists a thickening $\vartheta \colon S \times I \to U \subset V$ of S such that the local flow ϑ_t near S induced by ϑ induces bundle isomorphisms

$$T\vartheta_\epsilon \colon \mathcal{L}_S \to \mathcal{L}_{S_\epsilon}$$

for every $\epsilon \in I$. It then follows that $\vartheta_{-\epsilon}(x) \in \mathcal{P}^{\circ}(S)$ for every $x \in \mathcal{P}^{\circ}(S_\epsilon)$. Corollary 4.1 yields

Corollary 4.2. *Assume that S is a stable hypersurface of a tame symplectic manifold (W, ω). If $e^{\tau}_{S}(S, W) < \infty$, then $\mathcal{P}^{\circ}(S) \neq \emptyset$.*

It is well known that every hypersurface of contact type is stable, see [25, page 122], and so Corollary 1.9 follows from Corollary 4.2.

5. Proof of Corollaries 1.10 and 1.11

Corollary 1.10 follows from the following corollary, which is a consequence of Theorem 2.6 and Corollary 3.3.

Corollary 5.1. *Consider a proper Hamiltonian F on a tame symplectic manifold (W, ω). If $[r_0, r_1] \subset F(W)$ and $e^{\tau}_{S}(F^{-1}([r_0, r_1]), W) < \infty$, then $\mathcal{P}^{\circ}(F^{-1}(r)) \neq \emptyset$ for almost all $r \in [r_0, r_1]$.*

Assume for the remainder of this section that F is a proper function on a symplectic manifold (V,ω) attaining its minimum along $F^{-1}(0)$. We define $d_S^\tau(F) \in [0,\infty]$ by

$$d_S^\tau(F) \;=\; \sup \left\{ r \in \mathbb{R} \,\middle|\, e_S^\tau\left(F^r, V\right) < \infty \right\}.$$

Since $d_1(F) \le d_S^\tau(F)$, Corollary 1.11 follows from

Corollary 5.2. *Consider a proper Hamiltonian F on a tame symplectic manifold (W,ω) with minimum 0, and assume that $d_S^\tau(F) > 0$. Then $\mathcal{P}^\circ\left(F^{-1}(r)\right) \neq \emptyset$ for almost all $r \in \left]0, d_S^\tau(F)\right]$.*

6. Proof of Corollaries 1.13, 1.14, 1.17 and 1.18.

Recall that Corollaries 1.13 and 1.14 are special cases of Corollaries 1.17 and 1.18. It is shown in [6] that for any closed 2-form σ on a closed manifold M the symplectic manifold (T^*M, ω_σ) is tame. Moreover, $F_V^{-1}(0) \subset M$ for any function V on M with minimum 0, and since σ does not vanish, M is not Lagrangian. Proposition 1.7 (ii) thus yields $d_1(F_V) > 0$. Since $d_S^\tau(F_V) \ge d_1(F_V)$, Corollary 1.17 follows from the following corollary, which is now a consequence of Corollary 5.2.

Corollary 6.1. *Consider a closed Riemannian manifold (M,g) endowed with a closed 2-form σ which does not vanish identically. Then $d_S^\tau(F_V) > 0$ and $\mathcal{P}^\circ\left(F_V^{-1}(r)\right) \neq \emptyset$ for almost all $r \in \left]0, d_S^\tau(F_V)\right]$.*

Proof of Corollary 1.18: We follow [10, Sections 12.3 and 12.4]. Let σ_T be the unique translation-invariant 2-form on T^k cohomologous to σ_1. By assumption on σ there exists a 1-form α on $M = T^k \times M_2$ such that $\sigma = \sigma_T \oplus \sigma_2 + d\alpha$.

(i) Since $[\sigma_T] \neq 0$, the proof of Theorem 3.1 in [16] guarantees a symplectomorphism

$$\varphi\colon \left(T^*T^k, \omega_T\right) \;\to\; \left(\mathbb{R}^{2l} \times W, \Omega_{\text{can}} \oplus \Omega_T\right),$$

where $2l > 0$ and Ω_T is a translation-invariant symplectic form on $W = \mathbb{R}^{k-2l} \times T^k$. Composing the shift

$$(T^*M, \omega_\sigma) \to (T^*M, \omega_{\sigma_T \oplus \sigma_2}), \qquad (q,p) \mapsto (q, p + \alpha(q))$$

with $\varphi \times id$ we thus obtain a symplectomorphism

$$(T^*M, \omega_\sigma) \;\to\; \left(\mathbb{R}^{2l} \times W \times T^*M_2, \Omega_{\text{can}} \oplus \Omega_T \oplus \omega_{\sigma_2}\right).$$

Since every compact subset of this space is displaceable, we see that $d_1(F_V) = \infty$, and so assertion (i) follows.

(ii) The proof is similar, but replaces the symplectomorphism φ by symplectic embeddings

$$\varphi_R\colon \left\{(q,p) \in \left(T^*T^k, \omega_0\right) \,\middle|\, |p| \le R\right\} \;\hookrightarrow\; \left(\mathbb{R}^{2k}, \Omega_{\text{can}}\right).$$

Since these embeddings are not injective on π_1, the periodic orbits found might not be contractible in T^*M. $\square$

We consider again a closed Riemannian manifold (M, g) endowed with a non-vanishing closed 2-form σ. For the sake of clarity, we assume $V = 0$. If the hypersurfaces E_r, $r \in \,]0, d_S^\tau(g,\sigma)[$, are of contact type, then Corollary 4.2 shows that $\mathcal{P}^\circ(E_r) \neq \emptyset$ *for all* $r \in \,]0, d_S^\tau(g,\sigma)[$. The following proposition, which was explained to me by Viktor Ginzburg and Sasha Dranishnikov, shows that "for most M and σ", E_r is not of contact type.

Proposition 6.2. *Assume that M is neither a 2-sphere nor an orientable surface of genus greater than or equal to 2. If E_r is a contact type hypersurface of (T^*M, ω_σ), then σ is exact.*

Proof. Fix $r > 0$. Since E_r is of contact type, we find a 1-form α on E_r such that $d\alpha = \omega_\sigma|_{E_r}$. Let $\pi \colon E_r \to M$ be the projection. In $H^2(E_r;\mathbb{R})$ we then have

$$\pi^*[\sigma] = [\pi^*\sigma] = [\omega_\sigma|_{E_r}] = [d\alpha] = 0. \tag{16}$$

Assume first that M is orientable. We then consider the part

$$H^{1-k}(M;\mathbb{R}) \xrightarrow{\wedge e} H^2(M;\mathbb{R}) \xrightarrow{\pi^*} H^2(E_r;\mathbb{R})$$

of the Gysin sequence of the k-sphere bundle $\pi \colon E_r \to M$. Here, $\wedge e$ is multiplication by the Euler class. If M is the 2-torus, then $e = 0$, and if $\dim M \geq 3$, then $H^{1-k}(M;\mathbb{R}) = 0$, and so π^* is injective in both cases. Together with (16) we conclude that $[\sigma] = 0$, as claimed.

Assume now that M is not orientable. If $\dim M = 2$, then $H^2(M;\mathbb{R}) = 0$, and so there is nothing to prove. If $\dim M \geq 3$, let $p \colon \widehat{M} \to M$ be the orientable double cover of M and $p_r \colon E_r(\widehat{M}) \to E_r(M)$ its lift. Their induced maps fit into the commutative diagram

$$\begin{tikzcd}
H^2(E_r(M);\mathbb{R}) \arrow[r, "p_r^*"] & H^2\big(E_r(\widehat{M});\mathbb{R}\big) \\
H^2(M;\mathbb{R}) \arrow[u, "\pi^*"] \arrow[r, "p^*"'] & H^2\big(\widehat{M};\mathbb{R}\big) \arrow[u, "\hat{\pi}^*"']
\end{tikzcd}$$

We need to show that π^* is injective. Since we already know that $\hat{\pi}^*$ is injective, we are left with proving

Lemma 6.3. *The map p^* is injective.*

Proof. Let H_* denote simplicial homology. We choose a triangulation $\mathcal{T}$ of M so fine that it lifts to a triangulation $\widehat{\mathcal{T}}$ of $\widehat{M}$. For each simplex σ of $\mathcal{T}$ we fix an orientation, and we endow both simplices $\hat{\sigma}_1$ and $\hat{\sigma}_2$ in $p^{-1}(\sigma)$ with the orientation compatible with p. The map $\sigma \mapsto \hat{\sigma}_1 + \hat{\sigma}_2$ is a chain map between the chain complexes $C_*(M;\mathbb{R})$ and $C_*\big(\widehat{M},\mathbb{R}\big)$ freely generated by $\mathcal{T}$ and $\widehat{\mathcal{T}}$, and thus induces the "transfer homomorphism"

$$p_! \colon H_*(M;\mathbb{R}) \,\to\, H_*\big(\widehat{M};\mathbb{R}\big).$$

The composition $p_* \circ p_! \colon H_*(M;\mathbb{R}) \to H_*(M;\mathbb{R})$ is multiplication by 2, and so $p_* \colon H_*\big(\widehat{M};\mathbb{R}\big) \to H_*(M;\mathbb{R})$ is surjective. Passing to cohomology we find that $p^* \colon H^*(M;\mathbb{R}) \to H^*\big(\widehat{M};\mathbb{R}\big)$ is injective, and so the lemma is proved. □

Remarks 6.4. 1. Define the norm $\|\sigma\|$ of an exact 2-form σ as in (9). It is shown in [8] that if M is an orientable surface different from a torus, then E_r is a contact type hypersurface of (T^*M, ω_σ) if and only if $r > \frac{1}{2}\|\sigma\|^2$.

2. For any closed 2-form σ on an orientable surface different from a torus, E_r is of contact type if c is large enough, see [12] and [10, Lemma 12.6].

3. Proposition 6.2 has a partial converse: If $\sigma = d\alpha$ is exact, then E_r is of contact type whenever $c > \frac{1}{2}\|\alpha\|^2$, see [10, Lemma 12.1].

7. Proof of Corollary 1.20

Recall from Example 2.4 that a triple (L, W, ω) as in Corollary 1.20 is a stabilizer in $\mathcal{S}^\tau$. Corollary 1.20 is thus a special case of

Corollary 7.1. *For every stabilizer $(A_S, W_S, \omega_S) \in \mathcal{S}^\tau$ it holds that*

$$e_S^\tau(A_S, W_S) \,=\, \infty.$$

Proof. Choose F_S and $\epsilon > 0$ as in Definition 2.3. Property (ii) shows that $\mathcal{P}^\circ\left(F_S^{-1}(r)\right) = \emptyset$ for all $r \in [0,\epsilon]$, and so Corollary 5.2 yields $d_S^\tau(F_S) = 0$. In other words, $e_S^\tau(A_S, W_S) = \infty$. □

8. Remarks on smoothness

In this section we show that most of our results continue to hold for C^2-smooth hypersurfaces, Hamiltonians, Riemannian metrics, potentials and Lagrangian submanifolds. We still assume that the manifold V and the symplectic form ω are C^∞-smooth. For each subset A of V we define the π_1-sensitive Hofer–Zehnder capacity $c^{\circ,2}_{\mathrm{HZ}}(A,V)$ by replacing the set $\mathcal{H}_c\,(\mathrm{Int}\,A)$ in the definition (1) of $c^{\circ}_{\mathrm{HZ}}(A,V)$ by the set $\mathcal{H}^2_c\,(\mathrm{Int}\,A)$ of C^2-smooth compactly supported functions on $\mathrm{Int}\,A$.

Lemma 8.1. *For any subset A of a symplectic manifold (V,ω),*

$$c^{\circ,2}_{\mathrm{HZ}}\,(A,V) \;=\; c^{\circ}_{\mathrm{HZ}}\,(A,V)\,.$$

Proof. In view of definitions it suffices to prove this identity for each compact subset A of V. Since $\mathcal{H}^2_c\,(\mathrm{Int}\,A) \supset \mathcal{H}_c\,(\mathrm{Int}\,A)$ we have $c^{\circ,2}_{\mathrm{HZ}}(A,V) \geq c^{\circ}_{\mathrm{HZ}}(A,V)$. In order to prove the reverse inequality, we can assume that $c^{\circ}_{\mathrm{HZ}}(A,V)$ is finite. Given $F \in \mathcal{H}^2_c\,(\mathrm{Int}\,A)$ with $\max F > c^{\circ}_{\mathrm{HZ}}(A,V)$ we need to show that the flow f_t has a non-trivial contractible periodic orbit of period ≤ 1. Since $\mathcal{H}_c\,(\mathrm{Int}\,A)$ is dense in $\mathcal{H}^2_c\,(\mathrm{Int}\,A)$ in the C^2-topology, we find a sequence F_k, $k \geq 1$, in $\mathcal{H}_c\,(\mathrm{Int}\,A)$ converging to F in C^2. We can assume that $\max F_k > c^{\circ}_{\mathrm{HZ}}(A,V)$ for all k, and so we find for each k a contractible periodic orbit x_k of the flow of F_k of period $T_k \in \,]0,1]$. Since $F_k \to F$ in C^2, the C^2-norms of F_k with respect to some Riemannian metric on V are uniformly bounded, and so there exists $\epsilon > 0$ such that $T_k \in [\epsilon,1]$ for all k, see the proof of Proposition 17 in Section 5.7 of [25]. Using that A is compact we find a subsequence k_j, $j \geq 1$, such that $x_{k_j}(0) \to x_0 \in A$ and $T_{k_j} \to T \in [\epsilon,1]$ as $j \to \infty$. Since $F_{k_j} \to F$ in C^2, it follows that $x(t) := f_t(x_0)$ is a contractible T-periodic orbit of f_t. □

Theorem 1.6 and Lemma 8.1 yield

Corollary 8.2. *Assume that A is a subset of a tame symplectic manifold (W,ω). Then*

$$c^{\circ,2}_{\mathrm{HZ}}\,(A,W) \;\leq\; 4\,e_1\,(A,W)\,.$$

The proof of Theorem 3.2 goes through for C^2-smooth hypersurfaces and under the assumption $c^{\circ,2}_{\mathrm{HZ}}(U,V) < \infty$. It follows that Corollaries 1.8 and 1.9 continue to hold for C^2-smooth hypersurfaces. While the set of regular values of a C^2-smooth proper function on V^{2n} is still open, it might not be of full measure since Sard's theorem only applies to C^{2n}-smooth functions. For C^2-smooth proper Hamiltonians F, Corollaries 1.10 and 1.11 thus only hold for almost all *regular* $r \in [r_0, r_1]$ and almost all *regular* $r \in \,]0, d_1(F)]$. Since 0 is the only critical value of the function $F_V(q,p) = \frac{1}{2}|p|^2 + V(q)$ on T^*M, Corollaries 1.13, 1.14, 1.16, 1.17 and 1.18 hold for C^2-smooth Riemannian metrics g and C^∞-smooth magnetic fields σ. Finally, Corollary 1.20 extends to C^2-smooth Lagrangian submanifolds.

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