Between Paris and Linköping: Applications of Integral Transforms to Planning and Finance

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Abstract

We sketch the integral transform method in a close connection with its historical development and survey newer applications to problems in production economics and finance.

1 Introduction

Integral transforms have been created for solving problems in science. Now, they will be used also in business administration and production economics. We begin to sketch the historical sources of this valuable tool to get a better understanding and to develop a feeling for further applications.

The propagation of heat is one of the first examples in physics to test this method. Jean-Baptist Fourier (1768 - 1830) proposed a mathematical model and a procedure for solving it in his book [1] (submitted to the Academy in Paris in 1807, finally published in 1822). He formulated an initial-value problem for the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \quad , \quad -\infty < x < +\infty,\ t > 0, \tag{1}$$

under the condition

$$u(x,0) = f(x) \quad , \quad -\infty < x < +\infty, \tag{2}$$

where f is a given initial temperature distribution in a linear body. Introducing so-called "Fourier series" he obtained "hand made" the temperature at the point $x \in \mathbb{R}$ at time $t > 0$ as

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2} f\left(x + 2q\sqrt{\kappa t}\right) dq. \tag{3}$$

Today, we can get solution (3) by integral transforms depending on a parameter t which are not used in [1]. Define the *Fourier transform* U of u by

$$U(\omega, t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} u(x, t) dx \tag{4}$$

under the condition

$$\int_{-\infty}^{\infty} |u(x, t)| dx < \infty, \tag{5}$$

cutting short: $u \in L^1$.
In the case $U \in L^1$ there exists u the inverse of U (see [2] Prop. 5.1.10):

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} U(\omega, t) d\omega. \tag{6}$$

Fourier has found an inverse formula only for the so-called Fourier sine transform (see [1] p. 405).
In connection with problem (1), (2) we can also consider the *two-sided Laplace transform*

$$\tilde{U}(s, t) := \int_{-\infty}^{\infty} e^{-sx} u(x, t) dx. \tag{7}$$

If the integral in (7) converges absolutely for $\mathrm{Re}(s) = c \in \mathbb{R}$, then a nice inversion theorem

$$u(x, t) = \lim_{r \to \infty} \left(\frac{1}{2\pi i} \int_{c-ir}^{c+ir} e^{sx} \tilde{U}(s, t) ds \right) \tag{8}$$

holds at a point of continuity of the function $u(\cdot, t)$ (see [3] p. 210).

Pierre Simon Laplace (1749 - 1827) did not use the special transforms (7), (8) or (4), (6). He developed a general transform method to solve problems or obtain properties of mathematical objects by "transforming" them into another space, then solving the problem there, and finally transforming the solution back. In his monograph "Théorie analytique des probabilités" (see [4] p. 7) he studied the generating function G for a discrete random variable X with $P(X = n) = y_n$, $n \in \mathbb{N}_0$, and distribution F_X

$$G(t) = \int_{-\infty}^{\infty} t^x dF_X(x) = \sum_{n=0}^{\infty} y_n t^n. \tag{9}$$

Here, Laplace used only the power series but not integral representation.
In the gambler's ruin problem, y_n is the ruin probability of the player having n counters. These probabilities satisfy a difference equation of the second order. Difference equations for the y_n may be solved by simple operations to the associated generating function G and its series expansion. Such a calculus (see [4] pp. 7 - 11) is a basis for the integral transform method

which we use now for solving the problem (1), (2).
Applying the Fourier transform (4) to equation (1) we obtain

$$\frac{\partial U(\omega,t)}{\partial t} = -\kappa\omega^2 U(\omega,t) \tag{10}$$

which may be solved by classical methods yielding

$$U(\omega,t) = A(\omega)\exp(-\kappa\omega^2 t). \tag{11}$$

According to the initial condition (2) with $f \in L^1$ we have

$$A(\omega) = U(\omega,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x} f(x)dx. \tag{12}$$

In the case $A \in L^1$ we apply (6) to (11), (12) and find

$$u(x,t) = \frac{1}{2\sqrt{\pi\kappa t}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4\kappa t}\right) f(y)dy. \tag{13}$$

With the substitution $y - x = 2q\sqrt{\kappa t}$ we get again solution (3).
This example shows that in Paris at the beginning of the 19th century the basic ideas of the integral transform method were developed (after Euler's preparatory work [5], [6]) by the experts Laplace and Fourier reinforced by Cauchy who introduced complex numbers in this calculus (see [7]). But the first satisfactory functional analytic justification was given more than 100 years later (see [8]).

The connection between the equations (11) and (13) may be interpreted as follows: the convolution product (13) of two functions is associated to the product (11) of their Fourier transforms. This fact is basic for the probability theory as it was explicitly pointed out by Paul Lévy (1886 - 1971) in his book [9]:
The distribution of the sum of independent random variables is equal to the convolution product of the variables' distributions, and therefore, the Fourier-Stieltjes transform of the distribution of such a sum is equal to the product of the Fourier-Stieltjes transforms (so called characteristic functions) of those variables.
The characteristic function φ_X of a $\mathbb{R}^n$-valued random variable X with distribution F_X is defined on $\mathbb{R}^n$ by

$$\varphi_X(v) = \int_{\mathbb{R}^n} \exp(i < v, x >) F_X(dx), \tag{14}$$

where $< v, x >$ denotes the scalar product of $v, x \in \mathbb{R}^n$.
Lévy found this way in Paris a simple proof of the Central Limit Theorem for independent and identically distributed random variables.

In the 1930s Harald Cramér (1893 - 1985) and William Feller (1906 - 1970) co-operated in Sweden and propagated the method of characteristic functions too. They gave conditions for properly normalized sums of random variables to converge in distribution to a normally distributed random variable showing the convergence of the associated characteristic functions and applying a uniqueness theorem instead of an inversion formula (see [10] - [13]). Cramér's graduated students Herman Wold and Ulf Grenander applied integral transforms to stochastic processes (see [14], [15]).

After the historical sketch providing simultaneously main properties of integral transforms, the aim of this note is to discuss the application of integral transforms to problems in engineering economics and finance, the different viewpoints and the limitation of their use. We start in Section 2 with a decision problem. Then we study cash flows under uncertainty and their use for material requirement planning in Section 3. Finally, we investigate special options on financial markets (Section 4).

2 Optimal planning in discrete time

Ronald A. Howard considered in [16] a discrete-time Markovian decision process describing an economic system with N states with the following properties. If the system is in state i and an action k is chosen, then the probability of a transition to state j during the next time interval is equal to p_{ij}^k and it will earn r_{ij}^k dollars. Let $v_i(n)$ denote the present value of the total expected reward for the system in state i with n transitions remaining before termination if an optimal policy is followed. Applying Bellman's "Principle of Optimality" (see [17]) Howard got for an infinite horizon

$$v_i(n+1) = \max_k \sum_{j=1}^{N} p_{ij}^k \left[r_{ij}^k + \beta v_j(n) \right], \quad \begin{array}{l} n = 0, 1, 2, \ldots, \\ i = 1, 2, \ldots, N \end{array} \tag{15}$$

where $\beta \in [0, 1)$ is the discount factor. He solved the planning problem by policy iteration which consists of the value-determination operation (i) and a policy-improvement routine (ii):

(i) The present values v_i for a given policy $k(i), i = 1, 2, \ldots, N$ are determined by solving the set of equations

$$\begin{aligned} v_i &= q_i^{k(i)} + \beta \sum_{j=1}^{N} p_{ij}^{k(i)} v_j \quad , \quad i = 1, 2, \ldots, N \qquad (16) \\ q_i^{k(i)} &:= \sum_{j=1}^{N} p_{ij}^{k(i)} r_{ij}^{k(i)}. \end{aligned}$$

(ii) The policy $k'(i)$ maximizing

$$q_i^k + \beta \sum_{j=1}^{N} p_{ij}^k v_j$$

improves the previous policy $k(i)$.

Howard obtained formula (16) using Laplace's method of generating functions just as A.A. Markov did in his pioneering paper [18] getting a limit theorem for stochastic processes with three and more states. However, integral transforms are not always the shortest way to approach a limit. In the case of discounting we get directly the present values for large n and with them Howard's nice procedure (see [19] p. 97).

3 The Linköping model

Robert W. Grubbström presented in [20] a single-product, single-location production-inventory model - the corner-stone for a general modelling of material requirements planning developed at the Linköping Institute of Technology during the last decade. The structure of this model is chosen in such a way that a Laplace transform approach is applicable (see [21] -[26]).

3.1 Model assumptions and transform analysis

Let $H(t)$ denote the Heaviside function with a unit step at zero. The deterministic production process may be described by the staircase function

$$P(t) = \sum_{k=1}^{\infty} Q_k \cdot H(t - t_k), \tag{17}$$

$0 < t_n < t_{n+1}, n \in \mathbb{N}$, that means, production takes place in batches of the size Q_k made up at discrete points in time t_k .
We consider the *one-sided Laplace transform* of the time function H:

$$\tilde{H}(s) = \mathfrak{L}\{H(t)\} := \int_0^{\infty} e^{-st} H(t) dt = \frac{1}{s} \tag{18}$$

and obtain

$$\tilde{P}(s) = \frac{1}{s} \sum_{k=1}^{\infty} Q_k e^{-st_k}. \tag{19}$$

The stochastic demand process is a compound renewal process. More precisely, the cumulative demand up to time t is given by

$$D(t) = \sum_{k=1}^{\infty} X_k H \left(t - \sum_{j=1}^{k} T_j \right), \tag{20}$$

where T_j is the random time between $(j-1)st$ and jth demand events, and X_k is the size of the kth demand.
Analogously to equations (17), (19), for a fixed scenario we get

$$\tilde{D}(s) = \frac{1}{s} \sum_{k=1}^{\infty} X_k \exp\left(-s \sum_{j=1}^{k} T_j \right) . \tag{21}$$

If the random variables X_k, T_j are all independent, if the $T_j, j \in \mathbb{N}$, have the same density f with $f(t) = 0$ for $t \leq 0$, and $E(X_k) = m$ for all k, then the expected transformed cumulative demand will be

$$E\tilde{D}(s) = \frac{m}{s} \sum_{k=1}^{\infty} \prod_{j=1}^{k} E(e^{-sT_j}) = \frac{m}{s} \sum_{k=1}^{\infty} \tilde{f}^k(s) = \frac{m\tilde{f}(s)}{s(1-\tilde{f}(s))} . \tag{22}$$

Here, a production plan is a static policy characterized by the batch times t_k and the batch sizes Q_k or the cumulative production levels $P_k := \sum_{j=1}^{k} Q_j$. An objective function takes into account costs of production P, inventory I, stockouts S or revenues. In a backlogging system at any time t, it holds

$$I(t) = P(t) - D(t) + S(t), \qquad S(t) := [D(t) - P(t)]^+. \tag{23}$$

In the case $X_k = 1, P(t) = P_k \in \mathbb{N}$ for $t \in [t_k, t_{k+1})$ Grubbström has found for the stockouts

$$E\tilde{S}(s) = \tilde{f}^{P_k}(s) \cdot E\tilde{D}(s), \tag{24}$$

as well as (use (8) and (22))

$$ES(t) = \lim_{r\to\infty} \left(\frac{1}{2\pi i} \int_{c-ir}^{c+ir} e^{st} \frac{\tilde{f}^{P_k+1}(s)}{s(1-\tilde{f}(s))} \, ds \right) . \tag{25}$$

The right hand side of equation (25) may be evaluated explicitely for special distributions by using the residue theorem (see [20], pp. 412 - 414, [24], pp. 34 - 39). This handling of shortages is also applicable to holding inventories. Laplace transform analysis leads in the same way to $EI(t)$.

3.2 Present value and Laplace transform

The present value V of a given continuous cash flow C for a given interest rate ρ is defined by

$$V(\rho) := \int_0^{\infty} e^{-\rho t} C(t) dt. \tag{26}$$

Comparing (26) with (18) yields

$$V(\rho) = [\mathfrak{L}\{C(t)\}]_{s=\rho} = \tilde{C}(\rho). \tag{27}$$

Grubbström has already used in [22] (see also [26]) the Laplace transform for determining the present value of economic processes.
Especially, he considered a deterministic cash flow

$$C(t) = \sum_{k=1}^{\infty} V_k \delta(t - t_k) \tag{28}$$

where the Dirac Pulse $\delta(t)$ is a generalized function in the usual sense of L. Schwartz (see [27]). We use only the property

$$\mathfrak{L}\{\delta(t - t_k)\} = s\mathfrak{L}\{H(t - t_k)\}. \tag{29}$$

From (26) and (28) with $V_k = -(K + pQ_k)$ we obtain analogously to (17) and (19) the present value of the production costs

$$V_{\text{production}} = \tilde{C}(\rho) = -\sum_{k=1}^{\infty} (K + pQ_k)e^{-\rho t_k}, \tag{30}$$

where K is a fixed and p a proportional cost factor.
Now, we consider a stochastic cash flow which is connected to (28) with a random variable X_k and $V_k = EX_k$.
The expected present value of revenues by sales of demanded units reduced through backlogging is easily derived as

$$V_{\text{revenues}} = q\rho E[\tilde{D}(\rho) - \tilde{S}(\rho)], \tag{31}$$

where q is the payment per sold unit at the time of each sale.

The sum $V_{\text{production}} + V_{\text{revenues}}$ is a special objective function which will be maximized by the optimal plan $t_k^*, P_k^*, k = 1, 2, \ldots$ as a solution of the set of equations obtained from the usual first-order optimization conditions.

3.3 Modifications and MRP

Let us consider further objective functions for a single-product, single-location model. For a given finite horizon $\tau < \infty$ and a limited number n of setups we have to modify (30) to a sum running up to n, and (31) by a lost sale term (see [24], p. 56).
For the finite horizon, Grubbström used the traditional average cost approach proposing the objective function

$$\begin{aligned} &\frac{1}{\tau}\int_0^{\tau} [hEI(t) + gES(t) + nK]dt \\ &= \frac{h}{\tau}\left[\mathfrak{L}^{-1}\left\{\frac{1}{s}E\tilde{I}(s)\right\}\right]_{t=\tau} + \frac{g}{\tau}\left[\mathfrak{L}^{-1}\left\{\frac{1}{s}E\tilde{S}(s)\right\}\right]_{t=\tau} + \frac{NK}{\tau} \end{aligned} \tag{32}$$

which is to minimize with $N = n\tau$ and $\mathcal{L}^{-1}\{\tilde{x}(s)\} = x(t)$ (see [20] p. 415).
Instead of expected cost we may consider the reliability of the system characterized by the probability that no stockouts arise in the time interval $[0, \tau)$:

$$P\left(\inf_{t\in[0,\tau)}[P(t) - D(t)] \geq 0\right). \tag{33}$$

Models with reliability constraints have been investigated by A. Prékopa, the present author and others (see [28] - [30]).
The important extension of the Linköping model to two-level and finally multi-level structure as a basic model for Material Requirements Planning (MRP) has been realized by R. Grubbström, A. Molinder, O. Tang, and others (see [31], [32], [21], [23], [24]).

4 The price of Asian options

The aim of this section on stochastic finance is to give further examples of application of integral transforms which is based on a simple trick creating the needed structure for such a transform.

4.1 The Wiener-Bachelier-process

We return to Paris where Louis Bachelier defenced his doctoral dissertation in 1900 and where the first World Congress of the Bachelier Finance Society has celebrated the centenary of this event. With his Thesis (published in [33], rediscovered in the 1950's) modern finance was born. Bachelier presented a model for the price changes on the Exchange consistent with the market at a given instant. He derived the transition density

$$p = p(x,t) = \frac{1}{2\sqrt{\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right) \tag{34}$$

under temporal and spatial homogeneity assumptions. He tried to show with a "radiation of probability" argument that p is a solution of the diffusion equation (1). This statement is true and may be easily proved using equation (13) under the special initial condition

$$p(x,0) = \delta(x), \quad -\infty < x < +\infty \tag{35}$$

where $\delta(x)$ is the Dirac Pulse (see section 3.2).
A stochastic process $W = \{W(t) : 0 \leq t < \infty\}$ with independent and homogeneous increments and the transition density (34) is a Brownian motion. It has been called by Feller (for $\kappa = 0.5\sigma^2$) a Wiener-Bachelier process.
It was N. Wiener (1923) - not A. Einstein (1906) - who gave a clearly mathematical foundation of this process (see [25] p. 99, [34], [35]).

4.2 The Geman-Yor approach

Consider a stochastic process $\{X(t) : 0 \leq t < \infty\}$ where $X(t) = X(t, \omega)$ is a real-valued, and $T = T(\omega)$ a non-negative random variable. We introduce the process stopped at the random time T as a random variable X_T defined by

$$X_T(\omega) := X(T(\omega), \omega).$$

For $Y(t) = \int\limits_0^t X(t')dt'$ we may study Y_T.

Under certain assumption it is easier to compute the distribution of Y_T and especially its expectation $E(Y_T)$ than $E[Y(t)]$ for a fixed time t. This is the case for instance for a geometric Wiener-Bachelier process $\{X^{(\nu)}(t) : 0 \leq t < \infty\}$ with

$$X^{(\nu)}(t) := \exp(W(t) + \nu t), \tag{36}$$

$\nu \in \mathbb{R}$, which yields a better modelling of the price changes of an asset than Bachelier's model (see [36]).

For pricing an Asian option, this is an exotic option depending on the history of the price process, we have to investigate

$$C^{(\nu)}(t) := E[(Y^{(\nu)}(t) - K)^+] \tag{37}$$

where

$$Y^{(\nu)}(t) := \int_0^t X^{(\nu)}(t')dt'$$

and the strike price K is a constant.

The transform approach given by H. Geman and M. Yor works as follows: if T is exponential with parameter λ and does not depend on W, then

$$E[C^{(\nu)}(T)] = \lambda \int_0^\infty C^{(\nu)}(t)e^{-\lambda t}dt = \lambda \left[\mathfrak{L}\{C^{(\nu)}(t)\}\right]_{s=\lambda} \tag{38}$$

can be determined explicitly, and inversion yields

$$C^{(\nu)}(t) = \mathfrak{L}^{-1}\left\{\frac{1}{s}E[C^{(\nu)}(T)]_{\lambda=s}\right\}. \tag{39}$$

For details and explicit price formulas see [37] - [39]. Further applications may be found in [40], [41].

5 Conclusion

We had started our trip through the story of integral transforms in Paris about 200 years ago, where Laplace and Fourier were developing a transform approach for series but not for integrals. Our well-known Laplace transform is a younger child born after Laplace's death carrying his features. Finally, we terminated today in the capital of France again, where H. Geman, the President of the Bachelier Finance Society, opened the Bachelier Congress 2000 mentioned above (see section 4.1). We have seen the origin and the development of integral transforms to a useful method for solving certain problems in physics, and now especially in production economics and finance.

Howard wrote the first comprehensive textbook on Markov models from simple Markov chains to semi-Markov decision processes which based on Laplace's method of generating functions (z-transform). He applied transform analysis also to a simple inventory system and a marketing example (see [42]).

Systematic applications of integral transforms to production economics is closely connected with the name Linköping, more precisely, with PROFIL, that is "Production-Economic Research in Linköping". Since the 1970s, in the PROFIL series interesting results of associates of Linköping Institute of Technology have been published (see [43], [44], and in particular [32], [24]).

The transform approach is limited to linear systems and by the effort for the inversion procedure. That seems to be the reason why this useful approach does not dominate other methods neither in inventory research nor in options pricing.

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